Random Transfer Matrices for the Overlap in Disordered Systems

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(Received 1 February 1993)

A generating function is introduced to determine the probability $P(q)$ of the overlap q in disordered systems via a product of random transfer matrices. In one-dimensional models, the overlap is obtained by the Lyapunov exponent λ of the product. Replica symmetry breaking at zero temperature corresponds to a discontinuity of the derivative of λ with respect to an appropriate coupling variable in the replica space. The method is illustrated in a frustrated magnetic model where $q\neq 0$.

PACS numbers: 75.10.Nr, 05.50.+q

Disordered magnetic models are important because they give a simple but nontrivial example of systems where the interactions of many components produce highly complex and organized structures. Their main feature is the competing effect of interactions and disorder which implies that they are not able to satisfy all external constraints and become frustrated, following the imaginative Toulouse term. Consequently, many equilibrium states can coexist in a very rich structure which cannot be characterized only in terms of free energy as in standard problems of statistical mechanics. A deeper understanding is achieved by introducing the notion of distance between equilibrium states through the overlap q and its probability distribution. Our Letter proposes a method to analyze the overlap probability in terms of transfer matrices. As far as we know, there have been no attempts to use transfer matrix methods to study the overlap structure though these techniques are extremely powerful and commonly employed in Ising models to compute the free energy and the correlation decay. Our plan is the following: (1) A generating function for the overlap distribution is introduced and expressed in terms of a product of random matrices. (2) We show that important information on the overlap structure can be obtained by the maximum Lyapunov exponent λ of the product. In particular, a replica symmetry breaking reflects itself into a discontinuity of the derivative of λ with respect to a replica coupling parameter. (3) As an example, we use our technique to compute the overlap in a one-dimensional model, with frustration and nonvanishing entropy at zero temperature.

In order to explain differences and similarities between the calculation of the free energy and that of the overlap probability via transfer matrices in disordered systems, we consider the classical example of a spin glass with Hamiltonian $H_N=-\sum_{i,j}J_{i,j}\sigma_i\sigma_j$, where $J_{i,j}$ are random couplings among the N spin variables $\sigma_i=\pm 1$. The typical free energy is given by the quenched average over the disorder variables $J_{i,j}$:

$$f_N=-\frac{1}{\beta N}\overline{\ln Z_N}\,, \tag{1}$$

where $Z_N=\sum_\sigma \exp[-\beta H(\sigma)]$ is the partition function for a given set of $\{J_{i,j}\}$. In the thermodynamic limit almost all the disorder realizations of $\{J_{i,j}\}$ have the same free energy,

$$f=\lim_{N\to\infty} f_N=\lim_{N\to\infty} -\frac{1}{\beta N}\ln Z_N\,. \tag{2}$$

This property is called self-averaging since $(\ln Z_N)/N$ becomes a nonrandom quantity for $N\to\infty$. The most celebrated method for calculating f is the replica trick which allows one to find f by a continuation to $n=0$ of the annealed averages of the moments $\overline{Z^n}$ [1]. A discussion of the physical meaning of $\overline{Z^n}$ and its relation with the finite volume fluctuations of the free energy among different disorder realizations can be found in [2–4]. For formally one-dimensional systems (such as strips and bars) with short range interactions, the free energy can be computed by means of suitable transfer matrices $\mathbf{A}_i$ since the partition function can be written as

$$Z_N=\mathrm{Tr}\prod_i^N \mathbf{A}_i\,. \tag{3}$$

Moreover, considering the direct products of $\mathbf{A}$ times itself, one has [4,5] the disorder averages of the partition function of n noninteracting replicas as

$$\overline{Z^n}=\mathrm{Tr}(\overline{\mathbf{A}_i^{\otimes n}})^N\,,$$

with $\mathbf{A}^{\otimes n}\equiv\mathbf{A}\otimes\cdots\otimes\mathbf{A}$, n times.

Given two spin configurations (indicated by $\{\sigma_i\}$ and $\{\tau_i\}$) their overlap is

$$q^{(\{\sigma\},\{\tau\})}=\frac{1}{N}\sum_i^N\sigma_i\tau_i\,,$$

with $-1\leq q\leq 1$. The overlaps between equilibrium states are distributed according to a probability distribution

$$P_J(q;N)=Z_N^{-2}(\beta)\sum_{\sigma,\tau}\delta\left(q-\frac{1}{N}\sum_i\sigma_i\tau_i\right)e^{-\beta H(\sigma)}e^{-\beta H(\tau)}\,, \tag{4}$$

where the subscript J denotes that in general P_J depends on the particular realization of disorder, even for $N\to\infty$. In other words, the structure of the equilibrium states varies with the realizations of disorder in the thermo-

0031-9007/93/71(5)/789(4)$06.00

dynamic limit. The disorder average $\overline{P_J}$ has been computed by the replica trick; see, e.g., [6]. The question arises whether it can be computed by transfer matrices. Towards this goal, let us introduce the generating function of the overlap probability

$$G_N(\omega)=\int_{-1}^{1}P_J(q;N)e^{\omega q}dq\,. \tag{5}$$

Inserting (4) into (5), one has

$$G_N(\omega)=Z_N^{-2}(\beta)\sum_{\sigma,\tau}\exp[-\beta H(\sigma)]\times\exp[-\beta H(\tau)]\exp\left(\frac{\omega}{N}\sum\sigma_i\tau_i\right). \tag{6}$$

Noticing that the numerator can be regarded as the partition function of two replicas coupled by a coupling ω/N, it is convenient to introduce the corresponding thermodynamic potential $\Gamma_N(\omega)=\ln G_N(\omega)$. It is evident that $\Gamma_N(\omega)$ contains all the information on the systems given by the overlap probability $P_J(q;N)$ and, in general, unlike the free energy, is not a self-averaging quantity. In particular, the average overlap is given by the derivative of the generating function:

$$\int_{-1}^{1}qP_J(q;N)dq=\left.\frac{dT_N}{d\omega}\right|_{\omega=0}. \tag{7}$$

To be more explicit, consider a one-dimensional Ising model with N spins and Hamiltonian

$$H=-\sum_i^N J\sigma_i\sigma_{i+1}+h_i\sigma_i\,, \tag{8}$$

where the magnetic fields h_i are independent identically distributed random variables. In this case, the partition function is given by (3) with random transfer matrices of the form $\mathbf{A}_i=\exp\beta(J\sigma_i\sigma_{i+1}+h_i\sigma_i)$, that is,

$$\mathbf{A}_i=\begin{pmatrix} e^{\beta(J+h_i)} & e^{\beta(-J+h_i)} \\ e^{\beta(-J-h_i)} & e^{\beta(J-h_i)} \end{pmatrix}.$$

In the same way, it is straightforward to verify that the generating function (6) is given by the following product of 4×4 transfer matrices:

$$G_N(\omega)=Z_N^{-2}(\beta)\,\mathrm{Tr}\prod_i^N[\mathbf{T}_i\mathbf{L}(\omega)]\,, \tag{9}$$

where $Z_N^2(\beta)=\mathrm{Tr}\prod_i^N\mathbf{T}_i$, and the matrices are $\mathbf{T}=\mathbf{A}\otimes\mathbf{A}$, i.e.,

$$\mathbf{T}_i=\exp[\beta(J\sigma_i\sigma_{i+1}+h_i\sigma_i)]\exp[\beta(J\tau_i\tau_{i+1}+h_i\tau_i)] \tag{10}$$

and

$$\mathbf{L}(\omega)=\exp[(\omega/N)\sigma_i\tau_i]\,. \tag{11}$$

The matrix elements of (10) and (11) are obtained by choosing the values ±1 for the spins σ_i,σ_{i+1}, and τ_i,τ_{i+1}, as usual. The matrix $\mathbf{T}_i$ has random elements distributed according to the distribution of the fields h_i while the matrix $\mathbf{L}$ is diagonal:

$$\mathbf{L}(\omega)=\begin{pmatrix} e^{\omega/N} & 0 & 0 & 0 \\ 0 & e^{-\omega/N} & 0 & 0 \\ 0 & 0 & e^{-\omega/N} & 0 \\ 0 & 0 & 0 & e^{\omega/N} \end{pmatrix}. \tag{12}$$

This concludes the first part of the Letter: We have shown that it is possible to get the information contained in P_J by a product of random matrices since all our arguments can be trivially extended to any one-dimensional disordered system with short range interactions. Nevertheless, our result remains rather academic if not expressed in terms of accessible quantities characterizing an infinite product of random matrices $\mathbf{M}_i$. In analytical and numerical calculations a quantity of this type is provided by the maximum Lyapunov exponent

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\ln\left\|\prod_i^N\mathbf{M}_i\right\|, \tag{13}$$

where $\|\prod_i^N\mathbf{M}_i\|=|\mathrm{Tr}\prod_i^N\mathbf{M}_i|$ for $N\to\infty$. For instance, the maximum Lyapunov exponent of the product of matrices $\mathbf{M}_i=\mathbf{A}_i$ is equal to $-\beta f$, where f is the free energy of the random field Ising model (8).

However, the Lyapunov exponent, unlike $G_N(\omega)$, is self-averaging. In fact, for finite ω, $G_N(\omega)$ is a quantity $O(1)$ even for $N\to\infty$, since in (6) the diverging part of the numerator is balanced by the denominator $Z_N^2(\beta)\sim e^{-2\beta fN}$. To overcome this difficulty, we take the ratio $\Omega=\omega/N$ finite, for $N\to\infty$. Notice that the matrix $\mathbf{L}$ should be independent of N in order to compute the Lyapunov exponent, and, if ω were fixed, it would be impossible to use $\mathbf{L}(\omega)$ since for each N, a different matrix $\mathbf{L}(\omega)$ enters in (13). From definitions (9) and (13), the Lyapunov exponent of the product of matrices $\mathbf{M}_i=\mathbf{T}_i\mathbf{L}(\omega=\Omega N)$ is

$$\lambda(\Omega)=\lim_{N\to\infty}\frac{1}{N}\Gamma_N(\omega=\Omega N)-2\beta f\,, \tag{14}$$

and $-\lambda(\Omega)/\beta$ is the free energy of a system of two replicas coupled by a macroscopic coupling Ω.

If we take $\omega/N=\Omega$ finite for $N\to\infty$ in Eq. (5) then we lose important information. Indeed, suppose that $P_J(q;N)$ becomes a smooth distribution with a bounded support $[q_{\min},q_{\max}]$, when $N\to\infty$. As previously discussed,

$$\Gamma_N(\omega)=\ln\int_{-1}^{1}e^{\omega q}P_J(q;N)\,dq \tag{15}$$

is fully equivalent to $P_J(q;N)$. On the other hand, the Lyapunov exponent is related to the thermodynamic limit of the rescaled potential

$$\frac{1}{N}\Gamma_N(\omega=\Omega N)=\frac{1}{N}\ln\int_{-1}^{1}e^{\Omega qN}P_J(q;N)\,dq\,. \tag{16}$$

When $N\to\infty$, inserting the saddle point estimate of (16) into (14), one obtains for small Ω

$$\lambda(\Omega)-\lambda(0)=\begin{cases}\Omega q_{\max} & \text{if} \quad \Omega>0,\\ \Omega q_{\min} & \text{if} \quad \Omega<0,\end{cases} \tag{17}$$

with $\lambda(0)=-2\beta f$. It is worth stressing that there exists a rigorous proof that the Lyapunov exponent is a self-averaging quantity (Oseledec theorem) [7]. It follows that, in the thermodynamic limit, the extrema of the support of P_J are the same for almost all disorder realizations. $\lambda(\Omega)$ is in general a nonlinear function of Ω. On the basis of the large deviation theory [8] we write the finite N corrections to the asymptotic form of P_J as

$$P_J(q;N)\sim P_J(q;\infty)+A_J e^{-S_J(q)N}.$$

In this case, the saddle point estimate of (16) gives the Lyapunov exponent as the Legendre transform of the convex envelope $s(q)$ of $S_J(q)$:

$$\lambda(\Omega)-\lambda(0)=\max_q [q\Omega-s(q)]. \tag{18}$$

A similar relation linking overlap to replica coupling has been recently proposed in the context of mean field models where the transfer matrix formalism cannot be used [9].

The Oseledec theorem ensures that $\lambda(\Omega)$ takes the same value for almost all realizations of disorder, i.e., it is a self-averaging quantity. As a consequence, (18) shows that also the envelope $s(q)$ is self-averaging in systems with short range interactions. By definition $s(q)=0$ for $q\in[q_{\min},q_{\max}]$ and $s(q)>0$ for $1\geq q>q_{\max}$ and $-1\leq q<q_{\min}$. Moreover, for large $|\Omega|$ the saddle point is given by $q=\pm 1$ so that the asymptotic behavior is $\lambda(\Omega)\simeq C_{(\pm)}\pm\Omega$, where $C_{(\pm)}$ are constants. No information is lost if $P(q;\infty)$ is a delta function, implying $q_{\min}=q_{\max}=q^*$, and $s(q)$ has only one zero at $q=q^*$.

In conclusion, from the maximum Lyapunov exponent of the product of 4×4 random matrices, we have got the limits of the support of the overlap distribution

$$q_{(\max,\min)}=\left.\frac{d\lambda(\Omega)}{d\Omega}\right|_{\Omega=0^\pm}. \tag{19}$$

This result could seem rather limited but it assumes a great importance when considered as a mark of a replica symmetry breaking which can appear even in one dimension at temperature $T=0$. Indeed, if $\lim_{N\to\infty}P_J(q,N)=\delta(q-q^*)$, the derivative $d\lambda(\Omega)/d\Omega$ at $\Omega=0$ does exist and is equal to q^*. This is the case in one-dimensional systems when $T\neq 0$. On the other hand, $P_J(q)$ can differ from a delta function at $T=0$, implying a nondifferentiable $\lambda(\Omega)$, i.e., a first-order phase transition in the potential given by $\lim_{N\to\infty}(1/N)\Gamma_N(\Omega N)$.

The theoretical relevance of our result follows from the possibility to estimate the Lyapunov exponent either by a direct numerical calculation or by different analytic methods [10–13].

To illustrate our ideas, we study a particular random field Ising model given by the Hamiltonian (8) with $J=1$ and field

$$h_i=\eta_i-\eta_{i+1}, \tag{20}$$

where the variables η are independent identically distributed random variables ($\eta_i=\pm 1$ with probability $\frac{1}{2}$). As a consequence of the site correlation of the fields h_i, frustration plays an important role. In fact, one can show that the zero temperature entropy does not vanish. As usual, we compute the partition function as

$$Z_N=\sum_{\sigma_i=\pm 1}\prod_i^N \exp\{\beta[\sigma_i\sigma_{i+1}+\eta_{i+1}(\sigma_{i+1}-\sigma_i)]\} = \mathrm{Tr}\prod_i^N \mathbf{A}_i, \tag{21}$$

where

$$\mathbf{A}_i=\begin{pmatrix} e^{\beta} & e^{\beta}\\ e^{-3\beta} & e^{\beta}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\beta} & e^{-3\beta}\\ e^{\beta} & e^{\beta}\end{pmatrix}, \tag{22}$$

with respective probability $\frac{1}{2}$. We can also consider the zero temperature limit by extracting the diverging part. For $\beta\to\infty$ one has $\mathbf{A}_i=e^{\beta}\mathbf{R}_i$ with

$$\mathbf{R}_i=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}, \tag{23}$$

with respective probability $\frac{1}{2}$. The corresponding 4×4 random matrices for the generating function are $\mathbf{T}_i=e^{2\beta}\mathbf{R}_i\otimes\mathbf{R}_i$. It is thus easy to obtain the thermodynamic limit of $G_N(\omega=\Omega N)$ via a numerical calculation of the maximum Lyapunov exponent of the product of the random matrices $(\mathbf{R}_i\otimes\mathbf{R}_i)\mathbf{L}(\Omega N)$.

The result for zero temperature is shown in Fig. 1. It indicates that the overlap probability is a delta function, since the $\lambda(\Omega)$ is a smooth differentiable function. From

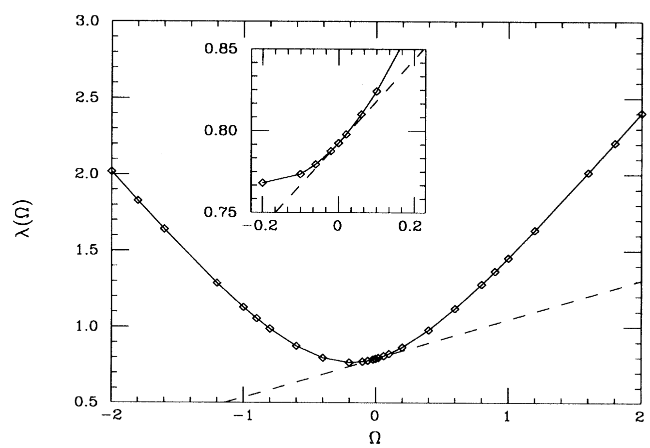

FIG. 1. Lyapunov exponent $\lambda(\Omega)$ versus Ω (squares) in the one-dimensional Ising model with random field given by (20) at zero temperature. The full line is drawn as a guide for the eye. The angular coefficient of the tangent at $\Omega=0$ (dashed straight line) is the overlap $q(T=0)\simeq 0.256$.

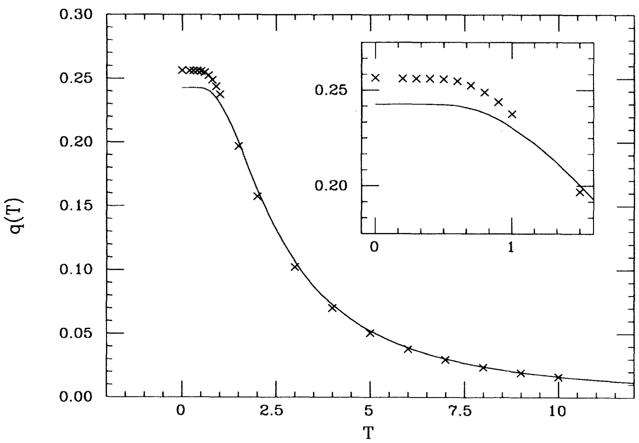

FIG. 2. Overlap $q(T)$ for the random field Ising model (8) as a function of the temperature T. The crosses are obtained from a numerical calculation of $\lambda(\Omega)$. The line is the annealed estimate which for high temperature gives $q(T)=2/T^2$.

the derivative at $\Omega=0$, the overlap is found to be $q(T=0)\simeq 0.256$. Its nonzero value confirms the importance of the frustration in the model, though there is no replica symmetry breaking. Following the Landau argument it is possible to show that in one-dimensional systems, with short range interactions and uncorrelated disorder, the $P(q)$ is a delta function at any nonzero temperature.

In Fig. 2, we report the overlap q as a function of the temperature $T=\beta^{-1}$. q vanishes as $\beta\rightarrow 0$ since at high temperature the frustration disappears and the system becomes "fully" disordered. In this limit, it is sensible to estimate the Lyapunov exponent by the so-called annealed approximation (the average of the log is approximated by the log of the average). In fact, the error of such a crude estimate is proved to vanish as β^8 [14]. By this method, we get $q(T)=2/T^2+O(T^{-3})$ in good agreement with the numerical calculation.

In conclusion, we have found a general tool to determine the overlap probability $P(q)$ via the product of random transfer matrices. This method gives a simple way to compute the overlap q in one-dimensional disordered systems with short range interactions when $P(q)$ is a delta function. Moreover, the replica symmetry breaking is marked by a phase transition in the maximum Lyapunov exponent of the product of appropriate random matrices. More sophisticated ideas are necessary to generalize our arguments to two dimensions via finite size scaling since the size of the transfer matrices is $4^L\times 4^L$ for a strip of size L.

We thank S. Franz for interesting discussions. G.P., M.S., and A.V. acknowledge the financial support of the INFN through Iniziativa Specifica FI3.

[1] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).
[2] J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).
[3] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 141 (1987).
[4] A. Crisanti, G. Paladin, H. J. Sommers, and A. Vulpiani, J. Phys. I (France) **2**, 1325 (1992).
[5] J. B. Pendry, J. Phys. C **15**, 4821 (1982); J.-P. Bouchaud, A. Georges, D. Hansel, P. Le Doussal, and J.-M. Maillard, J. Phys. A **19**, L1145 (1986).
[6] M. Mèzard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1988); K. H. Fischer and J. A. Herz, *Spin Glasses* (Cambridge Univ. Press, Cambridge, 1991).
[7] V. I. Oseledec, Trans. Moscow Math. Soc. **19**, 197 (1988).
[8] S. R. S. Varadhan, *Large Deviations and Applications* (Society of Industrial and Applied Mathematics, Belfast, 1984).
[9] S. Franz. G. Parisi, and M. Virasoro, J. Phys. I (France) **2**, 1869 (1992).
[10] B. Derrida, K. Mecheri, and J. L. Pichard, J. Phys. (Paris) **48**, 733 (1987).
[11] R. Mainieri, Phys. Rev. Lett. **68**, 1965 (1992).
[12] J. Deutsch and G. Paladin, Phys. Rev. Lett. **62**, 695 (1988).
[13] M. Serva and G. Paladin, Phys. Rev. Lett. **70**, 105 (1993).
[14] P. Contucci, Lett. Math. Phys. (to be published).