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Ferrimagnetism in a disordered Ising model

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Abstract. — We introduce a one dimensional Ising model with two competing interactions : nearest neighbor random couplings $\pm J$ with equal probability and a positive infinite range coupling Λ. At low temperature T the model exhibits a first order phase transition between a ferromagnetic state (with magnetization $m_1 = 1$ at $T = 0$) and a « ferrimagnetic » state (with $m_2 = 2/3$ at $T = 0$), when the disorder strength J/Λ is increased. For $5/12 < J/\Lambda < 1$, a whole spectrum of ferrimagnetic ground states with magnetization $m_n = 2/(n + 1)$ ($n = 2, \ldots, \infty$) is present while for $J/\lambda > 1$ the ground state is given by a trivial one dimensional spin glass with $m = 0$. The main qualitative features of the model can be described by a simplified annealed model where the random couplings can arrange themselves to minimize free energy with the constraint that the number of positive couplings is fixed by the law of large numbers in the thermodynamic limit. This model is exactly solved at all temperatures and the diagram of phase is calculated.

1. Introduction.

One dimensional Ising models with appropriate distributions of the disorder exhibit many interesting features of spin glasses [1], such as non self-averaging overlap probability, frustration, existence of many degenerate equilibrium states [2]. It is thus interesting to look for simple models which might capture at least some aspects of the high dimensional problems. In this paper we consider a one dimensional Ising model with nearest neighbor (n.n.) random couplings $\pm J$ which has a ferromagnetic infinite range interactions. The two limit cases are both trivially solvable : the one dimensional disordered Ising model by a gauge transformation and the mean field model by the Curie-Weiss approach. However, the competition of the two types of interactions leads to a rather reach behavior. Beside the paramagnetic and ferromagnetic phases (which are present in the mean field system), and spin glass phase (which is present at zero temperature in the one dimensional system), our model exhibits a new type of ferrimagnetic order in a region of low temperatures and intermediate disorder strength. It is a non-trivial consequence of the disorder, and could appear in more complicated systems which share the same ingredients of our toy model.

The paper is organized as follows. In section 2 we describe the model and show that it can be numerically studied by products of random transfer matrices even if there is an infinite range interaction. In section 3, we give the exact solution of the model at $T = 0$ and show that it

exhibits a ferrimagnetic order, which survives at $T \neq 0$. The ferrimagnetic phases are frustrated with zero temperature entropy which does not vanish and with a non-trivial value of the overlap between the degenerate ground states. In section 4 we study a simplified version of the model (a constrained annealed model [3]) which retains the ferrimagnetic order without taking into account the frustration. The advantage is that in this case the phase diagram can be computed in analytic way for all T, without Montecarlo simulations. In section 5 the reader can find a summary and a critical discussion on our results.

2. The model.

The Hamiltonian of our model is

$$H = - \sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1} - \frac{\Lambda}{N} \sum_{i>j} \sigma_i \sigma_j \tag{2.1}$$

where Λ is a positive infinite range coupling and J_i are independent identically distributed random variables which assume the two values $J_i = \pm J$ with equal probability. The partition function is a random variable depending on the $\{J_i\}$ realization of the system,

$$Z_N(\beta, \{J_i\}) = \mathrm{Tr}\, \mathrm{e}^{-\beta H} = \sum_{\{\sigma_i = \pm 1\}} \exp\left[\beta \sum_i J_i \sigma_i \sigma_{i+1} + \frac{\beta \Lambda}{2N} \left(\sum_i \sigma_i\right)^2 - \frac{\beta \Lambda}{2}\right] \tag{2.2}$$

and the quenched free energy is defined as

$$f = \lim_{N \to \infty} - \frac{1}{\beta N} \overline{\ln Z_N} \tag{2.3}$$

where $\bar{A}$ is the average of an observable A over the disorder probability distribution.

In the thermodynamic limit $N \to \infty$, all the disorder realizations (with the exception of a set of zero J-probability measure) have the same free energy, i.e. they self-average to the quenched average [1, 4],

$$\lim_{N \to \infty} - \frac{1}{\beta N} \ln Z_N = f \,. \tag{2.4}$$

The free energy can be numerically computed through the product of random transfer matrices [5], since (2.2) can be transformed in an integral over an auxiliary variable Φ. Neglecting the factors which vanish in the thermodynamic limit, it reads :

$$Z_N = \int_{-\infty}^{+\infty} \mathrm{d}\Phi \, Z_\Phi \, \mathrm{e}^{-\frac{1}{2}\Phi^2 N} \tag{2.5}$$

with

$$Z_\Phi = \sum_{\{\sigma_i = \pm 1\}} \exp\left(\sum_i \beta J_i \sigma_i \sigma_{i+1} + (\beta \Lambda)^{1/2} \Phi \sigma_i\right) . \tag{2.6}$$

The quenched free energy is given by the saddle point estimate of the integral (2.5),

$$-\beta f = \max_{\Phi} \left[\lambda(\Phi) - \frac{\Phi^2}{2}\right] \tag{2.7}$$

where

$$\lambda(\Phi) = \lim_{N \to \infty} \frac{1}{N} \ln Z_\Phi = \lim_{N \to \infty} \frac{1}{N} \ln \left\| \prod_{i=1}^{N} \mathbf{T}_i \right\| \tag{2.8}$$

is the maximum Lyapunov exponent of the product of random transfer matrices

$$\mathbf{T}_i = \begin{pmatrix} \exp(\beta J_i + (\beta \Lambda)^{1/2} \Phi) & \exp(-\beta J_i + (\beta \Lambda)^{1/2} \Phi) \\ \exp(-\beta J_i - (\beta \Lambda)^{1/2} \Phi) & \exp(\beta J_i - (\beta \Lambda)^{1/2} \Phi) \end{pmatrix} \tag{2.9}$$

corresponding to the random Ising model (2.6). However, we can describe the main features of the phase diagram without a numerical calculation by some qualitative arguments at zero temperature, and by the analysis of an exactly solvable annealed model, as discussed in the next sections.

3. The ferrimagnetic phases.

Our model obviously has some trivial phases : at high temperature T, the paramagnetic phase of magnetization $m = 0$ and, at low temperature and weak disorder (J small enough), the ferromagnetic phase. On the other hand, when $\Lambda = 0$ and $T = 0$ the system is in a spin glass phase with $m = 0$ where the up or down position of a spin σ_i is determined by the nature (ferro or antiferro) of the coupling J_i. This phase is of antiferromagnetic type, since one can define an analogous of the staggered magnetization which is equal to unity at $T = 0$, even if the true magnetization m remains zero. In particular, the overlap

$$q^{\alpha, \xi} \equiv \frac{1}{N} \sum_i \sigma_i^\alpha \sigma_i^\xi$$

between two equilibrium states α and ξ is $q = \pm 1$ in both ferromagnetic and glassy phases.

It is natural to expect that a third kind of phases might appear at intermediate values of $\omega \equiv J/\Lambda$ and low temperature with a non trivial magnetic order, neither ferromagnetic nor glassy. In other terms, as a consequence of the disorder, the ground states can have a magnetization $m \neq 0, \neq 1$. Using a language borrowed from solid state physics, such a phase of a disordered system can be called ferrimagnetic. Notice that in the following, we limit ourselves to consider states with magnetization $m \geq 0$, since the system is invariant for spin inversion, $\sigma_i \to -\sigma_i$ for all i's.

In this section, we show that at $T = 0$, the model exhibits three type of behaviours at varying the disorder strength :

1) ferromagnetic phase for $0 \leq \omega < 5/12$
2) ferrimagnetic phases for $5/12 < \omega < 1$
3) spin glass phase for $\omega > 1$.

In particular, there is a first order phase transition at $\omega_1 = 5/12$ between ferromagnetism ($m_1 = 1$) and ferrimagnetism ($m_2 = 2/3$). Moreover, there exists a whole spectrum of ferrimagnetic phases with magnetization

$$m_n = \frac{2}{n+1} \qquad n = 2, 3, \ldots, \infty \tag{3.1}$$

for disorder strength :

$$\omega \in]\omega_n, \omega_{n+1}[\quad \text{with} \quad \omega_n = \frac{n(n+3/2)}{(n+1)(n+2)} \tag{3.2}$$

as shown in figure 1. At zero temperature, the calculation is possible because one has to consider only the energy of the different configurations to determine the equilibrium states. The key idea is that the system has a ferrimagnetic state with magnetization $m \neq 1$ which is energetically convenient if the disorder is not too small, as a result of the competition between the antiferromagnetic nearest neighbor coupling $J_i = -J$ and the mean field couplings $\Lambda > 0$.

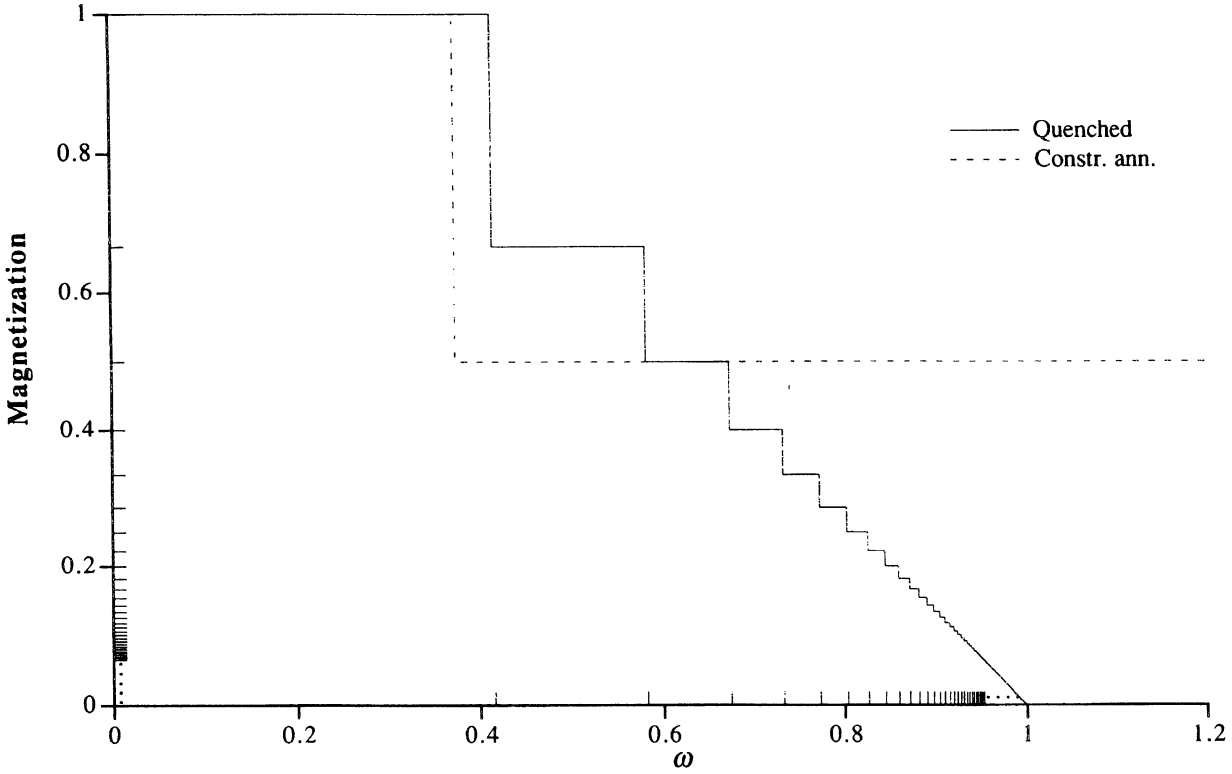

Fig. 1. — Magnetization $m(\omega)$ at zero temperature. The jumps at ω_n given by (3.2) correspond to the first order phase transitions between the n-phase and the $n+1$-phase. The transition at $\omega_\infty = 1$ toward the spin glass phase is the only continuous one. The dashed line is the constrained annealed approximation.

Here we limit the discussion to the arising of the first ferrimagnetic state m_2 *via* a phase transition at ω_1. In Appendix 1, the interested reader can find the full proof that an infinity of ferrimagnetic phases exists as anticipated by (3.1) and (3.2).

For sake of simplicity, suppose that initially $\omega \equiv J/\Lambda$ is close to zero. In this case at $T = 0$ the infinite range interaction prevails and the equilibrium state is given by all spins up. Inserting $\sigma_i = 1$ for all i's in (2.1), we see that the energy density of the ferromagnetic state is

$$E_F = \lim_{N \to \infty} \frac{1}{N} \left\{ -\sum_i J_i - \frac{\Lambda}{2} N \right\} = -\frac{\Lambda}{2} \tag{3.3}$$

since in the thermodynamic limit

$$\lim_{N \to \infty} \frac{1}{N} \sum J_i = 0 . \tag{3.4}$$

At increasing ω, some of the spins will flip in order to lower the energy due to the n.n. interaction. To illustrate the mechanism, figure 2a shows a ferromagnetic state, where the first spins to flip are σ_3 and σ_{10} since they have two lateral negative J_i, so that their jumps produce the maximal energy loss. This consideration is general, as the first spins to flip are always the spins with two lateral negative n.n. couplings. However, the situation is more complicated for the island of negative J_i's of size larger than two. For instance, as illustrated in figure 2b, in an island of 3 negative J_i's, only one of the two spins with two negative lateral n.n. couplings flips ; in an island of size 4, only two non-consecutive spins flip to lower the energy.

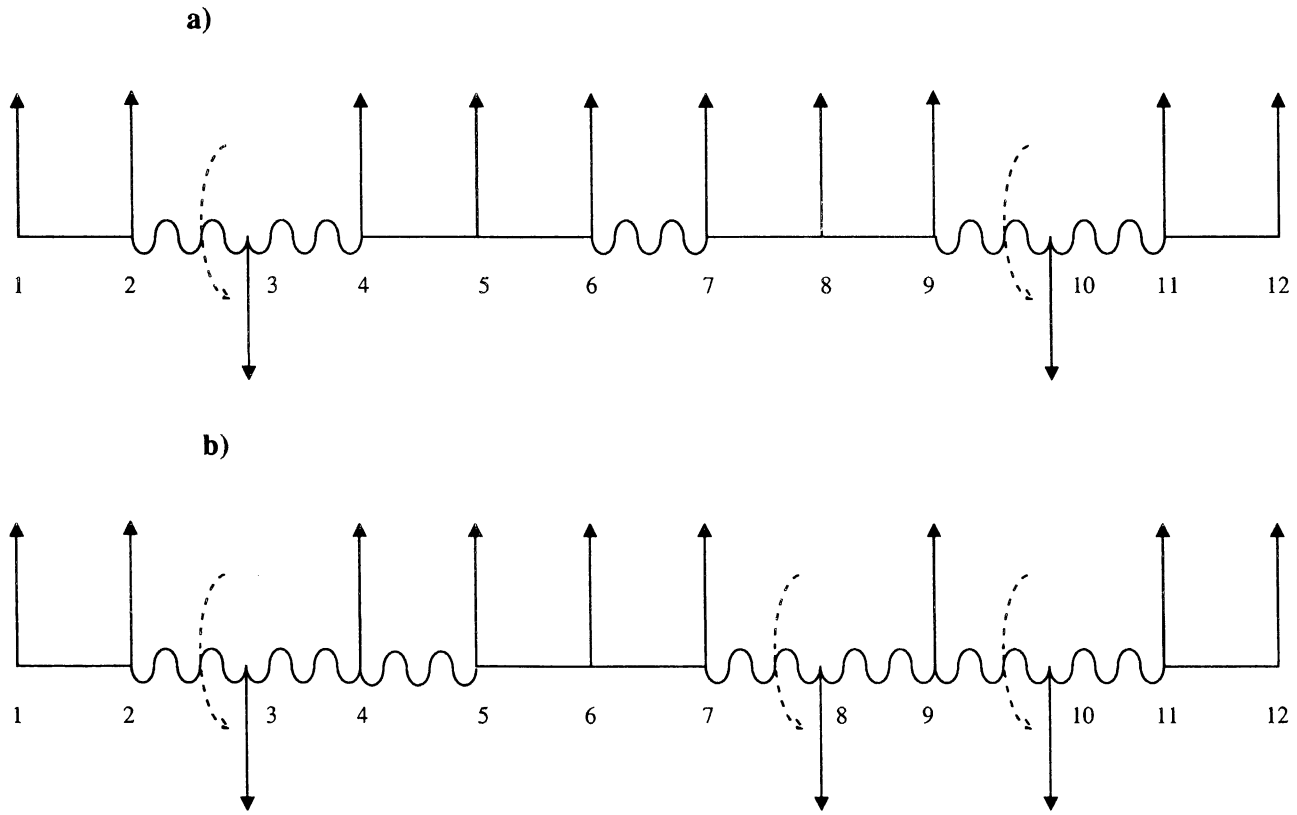

Fig. 2. — Description of the transition from the ferromagnetic to the first ferrimagnetic state. The full line are the positive n.n. couplings $J_i = +J$ and the wrinkled lines are the negative ones $J_i = -J$. The arrows represent the spins. 2a) Disorder realization with islands of $k \leq 2$ negative J_i's. At increasing disorder strength ω and for $T = 0$, the first spins to flip are located in the middle of the $k = 2$ islands. 2b) Disorder realization with some long islands of negative J_i's. The flipping spins are distributed in an alternated way among the islands of negative couplings.

In general, immediately above a critical value ω_1, the antiferromagnetic order will predominate in each island of k negative couplings $J_i = -J$, so that a number $S_k = k/2$ of spins are be down if k is even, and a number $S_k = (k-1)/2$ if k is odd. Let us stress that the « even » island has only one spin configuration of minimal energy, unlike the « odd » island which has $(k+1)/2$ equilibrium configurations of minimal energy since one negative n.n. coupling is unsatisfied in the r-th site, with odd $1 \leq r \leq k$.

Calling $\mathcal{N}$ the total number of spins which flip in all the islands, the resulting configuration has magnetization

$$m = 1 - 2\frac{\mathcal{N}}{N}. \tag{3.5}$$

A moment of reflection shows that the number of islands of negative n.n. couplings is $N/4 + O(N^{1/2})$. Since the probability that an island is made of k negative bonds is $(1/2)^k$, the number of islands of k n.n. coupling is

$$N_k = (1/2)^k \frac{N}{4} \tag{3.6}$$

and, if the antiferromagnetic order is present in all the islands, the total fraction of spins down can be easily estimated by

$$\lim_{N \to \infty} \frac{\mathcal{N}}{N} \leq \sum_{k=1}^{\infty} \frac{1}{4} (1/2)^k S_k = 1/6 . \tag{3.7}$$

inserting (3.7) into (3.5), one has $2/3 \leq m \leq 1$. At $T = 0$, to determine whether a ferrimagnetic state m_2 or the ferromagnetic one $m_1 = 1$ is of equilibrium, we should compare their respective energy densities which are given by

$$E(m) = -4J \frac{(1-m)}{2} - \frac{\Lambda}{2} m^2 \quad \text{for} \quad 2/3 \leq m \leq 1 \tag{3.8}$$

as shown in figure 3. $E(m)$ is a convex function of m (a parabola) so that its minimum is given by one of its two extrema (either at $m = 1$ or at $m = 2/3$) separated by a energy gap $\Delta E \equiv E_{\max} - \max [E(2/3), E(1)]$ where $E_{\max}$ is the maximum of $E(m)$ for $m \in [2/3, 1]$, reached at $m = 2\omega$.

The two state have the same energy at $\omega_1 = 5/12$ where there is a first order phase transition, since the magnetization has a discontinuous jump from $m_1 = 1$ to $m_2 = 2/3$.

However the first ferrimagnetic phase ($n = 2$) cannot exist when the disorder is too strong. Indeed, its energy is obtained by inserting $m = 2/3$ into (3.8), so that

$$E_{\text{ferri}} = -\frac{2}{3} J - \frac{2}{9} \Lambda . \tag{3.9}$$

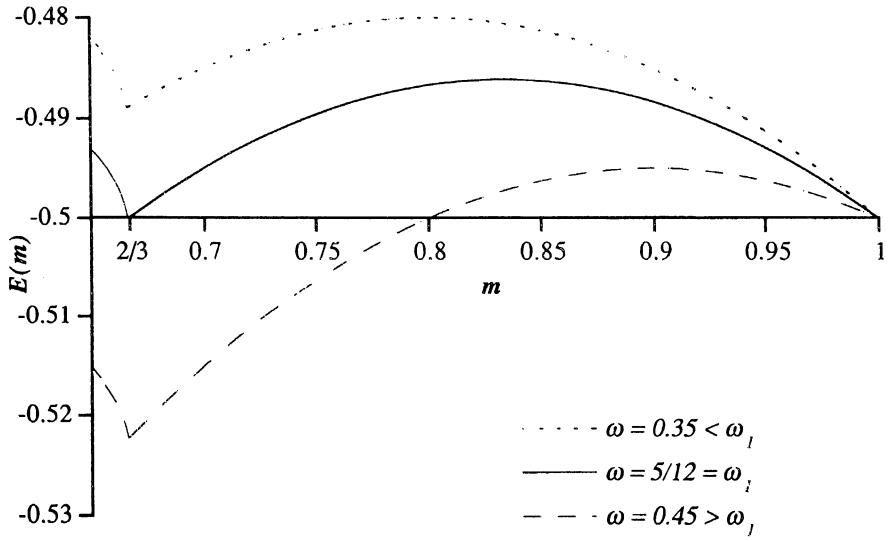

Fig. 3. — Energy density $E(m)$ as function of the magnetization m for the quenched model at $\Lambda = 1$ for different values of ω. In the first case ($\omega = 0.4 < \omega_1 = 5/12$) the energy minimum is reached for $m = 1$, in the second case ($\omega = \omega_1$) the minimum is obtained both for $m = 2/3$ and for $m = 1$, and in third case ($\omega = 0.45 > \omega_1$) for $m = 2/3$.

On the other hand the spin glass phase with $m = 0$ has energy $E_g = -J$, because all the spins are aligned with the n.n. random couplings. It follows that for $\omega > 2/3$, the first ferrimagnetic phase has higher energy than the spin glass phase. Actually, as shown in appendix 1, the first ferrimagnetic phase does not minimize the energy already for $\omega > \omega_2 = 7/12$ (where a second ferrimagnetic phase with $m_3 = 1/2$ appears), while the spin glass phase prevails only for $\omega > 1$. In fact, a numerable infinity of different ferrimagnetic phases are exhibited by the quenched model for $5/12 < \omega < 1$ with magnetization given by (3.1). This result is obtained by a renormalization procedure which generalizes the argument used for the first ferrimagnetic phase. In the latter case, the flipping spins can be regarded as islands of length one, and in appendix 1 we prove that after the first transition the system is energetically equivalent to a new one made only of islands with a number of spins $n \geq 2$, corresponding to $n - 1$ positive couplings. Note that we define the length of the island by the number of its spins and not by the number of its couplings. The renormalized system will have a transition which is due to the spin flip of the islands of length $n = 2$. The mechanism can be repeated at every step, transforming the n-system to a $(n + 1)$-system made of $k \geq n + 1$ islands and the transition involves only the shortest islands, that flip as the single spins do in the first transition.

The transition from a n-phase to the $(n + 1)$-phase is of a first order with an energy gap decreasing with n. The last transition toward the spin glass phase thus is continuous and one has a critical point at $\omega = 1$, $T = 0$.

We can also estimate the entropy and the overlap of a ferrimagnetic state at $T = 0$. These two quantities are indicative for the presence of many different minima of the energy as consequence of the frustration. In the ferromagnetic phase the system is completely ordered and there are only two minima so that the entropy $S(T = 0) = 0$ and the overlap $q(T = 0) = \pm 1$. This is also true for the spin glass phase. On the contrary, in the first ferrimagnetic state there are islands of an odd number k of n.n. couplings where the spin have $(k + 1)/2$ possible configurations of minimal energy. Typically, in an odd island, the spins follow an antiferromagnetic order except two neighbors which are parallel.

Using the number N_k of islands of k n.n. couplings (3.6), and considering only the configurations with positive magnetization $m = 2/3$, the zero temperature entropy and overlap can be obtained as sums which run on k', the odd values of k. The entropy is

$$S(T = 0) = \lim_{N \to \infty} \frac{1}{N} \sum_{k' \geq 3} N_{k'} \ln \left(\frac{k' + 1}{2} \right) = 0.034 \ldots \tag{3.10}$$

The thermal average of the overlap is defined as

$$\langle q(T) \rangle \equiv \lim_{N \to \infty} \frac{1}{Z_N^2} \sum_{\{\sigma^\alpha, \sigma^\xi\}} e^{-\beta (H^\alpha + H^\xi)} q^{\alpha, \xi}(T) .$$

In the first ferrimagnetic phase at $T = 0$, the calculation is simpler since one has to consider only the configurations of minimal energy. The overlap has the same value for almost all the disorder realizations, so that we get

$$\langle q(T = 0) \rangle = 1 - \lim_{N \to \infty} \frac{1}{N} \sum_{k' \geq 3} N_{k'} \frac{4}{((k' + 1)/2)^2} \sum_{i, i' = 1}^{\frac{k' + 1}{2}} |i - i'| =$$

$$= 1 - \frac{1}{6} \sum_{k' \geq 3} \frac{(k' - 1)(k' + 3)}{(k' + 1)} \frac{1}{2^{k'}} = \frac{19}{27} - \frac{2}{3} \ln \frac{3}{4} = 0.895 \ldots \tag{3.11}$$

The non-zero value of the entropy and the non-trivial value of the overlap indicate that a ferrimagnetic phase share an important aspect of high dimensional disordered systems, namely the existence of many degenerate ground states. Moreover ferrimagnetism and first order phase transitions survive at non-zero temperature, in contrast with the spin glass phase which disappears at $T > 0$, because the system for $\omega > 1$ is basically a one dimensional system. A qualitative justification of this fact can be given for the first ferrimagnetic phase by considering again the interval $2/3 \leq m \leq 1$ and the transition value of the parameter $\omega_1 = 5/12$. At $T = 0$ and $\omega = \omega_1$, the energy in (3.8) has two equal minima at $m = 1$ and $m = 2/3$, separated by an energy gap $\Delta E = E_{max} - \Lambda/2$ since $E(1) = E(2/3) = \Lambda/2$. At $T \neq 0$, the equilibrium states minimize free energy instead of energy and one has to take into account the entropic factor. The free energy gap ΔF between the two phases can be estimated by

$$\Delta F = \Delta E - T \Delta S . \tag{3.12}$$

where ΔS is the entropy of the state of maximal energy E_{max}.

However, the entropy gap ΔS between the two phases is a quantity independent of the temperature so that the free energy has a trivial explicit dependence on T. As a consequence, the free energy gap ΔF between ferro and ferrimagnetic states disappears only at a critical temperature estimated by

$$T_c = \frac{\Delta E}{\Delta S} . \tag{3.13}$$

We thus expect that the diagram of phase $\tau \equiv T/\Lambda$ *versus* $\omega = J/\Lambda$, there is a line of coexistence of the two phases which ends in a critical point ω_c, τ_c where $\Delta F = 0$ and the phase transition is of second order. In order to check this conjecture, we introduce an annealed version of our model which can be exactly solved at all temperatures and where the three magnetic orders (ferromagnetic, ferrimagnetic and paramagnetic) are present. Let us conclude by stressing that the ferrimagnetism is a truly high dimensional effect of the disorder since is present at non-zero temperature at difference with what happens in one dimensional systems with short range interactions.

4. The constrained annealed model.

In disordered systems, it is very easy to compute the annealed average of the partition function $\ln(\bar{Z})$. However, it is often a very poor approximation of (2.3) and, more important, can fail to describe even the qualitative aspects of the quenched system. This is due to the fact that in an annealed model the disorder variables (for us the J_i's) can arrange themselves to minimize the free energy while in a « quenched » model they are frozen in some given realizations. The annealed version of our model has no ferrimagnetic phase, since it treats the J_i's and the spin variables in the same way. In fact, it is easy to check that the annealed model is equivalent to the ferromagnetic mean field model since the averages over the J_i destroy short range interaction between spins.

In order to reproduce the ferrimagnetism in a solvable model, we should consider random couplings J_i which are only partially frozen, in the sense that they must satisfy appropriate constraints although minimizing the (annealed) free energy.

In our case, the minimal requirement is that the system has the right number of negative n.n. couplings in the thermodynamic limit, in order to reproduce the fundamental aspects of the quenched system. Let us stress again that we do not pretend to obtain a quantitative agreement between the quenched and the constrained annealed model. The purpose is to understand what are the important effects from a qualitative point of view. In a certain sense, a realistic

annealed system is a more convenient model of the physics of a disordered system than a quenched system, since it contains the essential ingredients with the advantage of being exactly solvable.

In general, a constraint can be imposed in an annealed model by a Lagrange multiplier μ [2], allowing one to obtain an upper bound L of the Lyapunov exponent (2.8) (and a lower bound for the free energy) by a minimization over annealed averages :

$$L = \min_{\mu} \lim_{N \to \infty} \frac{1}{N} \ln \overline{[e^{-N\beta\mu(\alpha - \bar{\alpha})} Z_N]} \geqslant \lambda \ . \tag{4.1}$$

This method fixes, in the thermodynamic limit, the self-averaging quantity α to its mean value $\bar{\alpha}$. As discussed, in our case we have to impose the correct number of negative n.n. couplings, using in (4.1)

$$\alpha \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{J_i}{J} \tag{4.2}$$

which is a self-averaging quantity to the value $\bar{\alpha} = 0$, as a consequence of the large number law. The free energy of the corresponding constrained annealed model is $f_A = -(1/\beta) L$. In order to calculate f_A, it is thus convenient to introduce a Grandpartition function which depends on the Lagrange multipliers μ,

$$\Omega_N(\mu) = \overline{e^{-N\beta J\mu\alpha} Z_N} \tag{4.3}$$

where Z_N is the canonical partition function defined in (2.2). Since the random variables are independent, we can easily perform the disorder average which reads

$$\Omega_N(\mu) = \sum_{\{\sigma_i = \pm 1\}} \prod_{i=1}^{N} \cosh(\beta J(\sigma_i \sigma_{i+1} - \mu)) \exp\left\{ \frac{\beta\Lambda}{2N} \left(\sum_i \sigma_i \right)^2 \right\} \tag{4.4}$$

where we have not written the constant factor $\exp(-\beta\Lambda/2)$ in the partition function (2.2) which gives a negligible contribution to the thermodynamic potentials when $N \to \infty$. This formula indicates that the system has two relevant thermodynamic parameters, the magnetization $m = (1/N) \Sigma_i \sigma_i$ and the one-step correlation

$$n = \frac{1}{N} \sum_i \sigma_i \sigma_{i+1} \ .$$

The Grandpartition function (4.3) can be written in terms of n, m as

$$\sum_{nN=-N}^{N} \sum_{mN=-N}^{N} P(n, m) [\cosh \beta J(1 - \mu)]^{N(1+n)/2} \times [\cosh \beta J(1 + \mu)]^{N(1-n)/2} e^{\beta\Lambda m^2 N/2} \tag{4.5}$$

where $(1 \pm n)/2$ is the number of times that the variable $\sigma_i \sigma_{i+1} = \pm 1$ and $P(n, m)$ is the joint probability that a spin configuration has a given value of n and m, and is non-zero for a given m only when $n \in [2m - 1, 1]$. Let us notice that the two sums in (4.5) are limited to those values of n and m compatible with their definitions. $P(n, m)$ is obviously symmetric with respect to m, so that we can limit the discussion to the case $m \geqslant 0$. The calculation of $P(n, m)$ is given in appendix 2, where one sees that, for large N,

$$P(n, m) \sim e^{-S(n, m) N} \tag{4.6}$$

with

$$-S(n, m) = \frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\ln\left(\frac{1-m}{2}\right) +$$
$$-\frac{1-2m+n}{4}\ln\left(\frac{1-2m+n}{4}\right) - \frac{1+2m+n}{4}\ln\left(\frac{1+2m+n}{4}\right) +$$
$$-\ln(2) - \frac{1-n}{2}\ln\left(\frac{1-n}{4}\right). \tag{4.7}$$

Inserting (4.6) into (4.5) the Grandpartition function becomes

$$\Omega_N(\mu) = \sum_{nN=-N}^{N} \sum_{mN=-N}^{N} e^{g(m, n, \mu)N} \tag{4.8}$$

where we have introduced the Gibbs thermodynamic potential

$$g(m, n, \mu) = -S(n, m) + \frac{\beta A m^2}{2} + \frac{1+n}{2}\ln\cosh\beta J(1-\mu) +$$
$$+\frac{1-n}{2}\ln\cosh\beta J(1+\mu). \tag{4.9}$$

In the limit $N \to \infty$, the sum (4.8) can be estimated by an integral so that the saddle point method gives

$$\Omega_N(\mu) \sim \exp\left[N \max_{m \in [0, 1]} \max_{n \in [2m-1, 1]} g(\mu, n, m)\right]. \tag{4.10}$$

This expression inserted in (4.1) gives the formal solution of the constrained annealed system :

$$f_A(\beta) = -\max_{m \in [0, 1]} \max_{n \in [2m-1, 1]} \min_{\mu \in R} \frac{1}{\beta} g(\mu, n, m). \tag{4.11}$$

The discussion of the phase diagram of (4.11) becomes particularly simple at zero temperature, since the entropic factor $S(n, m)$ disappears in $-g(\mu, n, m)/\beta$ when $\beta \to \infty$. As a consequence, one has that

$$\lim_{\beta \to \infty} \frac{g(\mu, n, m)}{\beta} = \frac{(1+n)}{2} J\,|1-\mu| + \frac{(1-n)}{2} J\,|1+\mu| + \frac{A}{2} m^2. \tag{4.12}$$

The minimization with respect to μ and the maximization with respect to n and m gives for $\omega < 3/8$ the ferromagnetic state

$$m = 1 \quad n = 1 \tag{4.13a}$$

while for $\omega > 3/8$ one obtains a ferrimagnetic state with

$$m = 1/2 \quad n = 0. \tag{4.13b}$$

Let us stress that in the constrained annealed model the spin glass phase is not present and only one ferrimagnetic phase exists for $3/8 < \omega$. This is the price we must pay for obtaining an exactly solvable model at any temperature. However, the ferrimagnetic order survives and that is our main goal.

As in the previous section, we also use combinatorial arguments to determine the phase diagram at zero temperature, instead of using (4.11), in order to make transparent the nature of our approximations. The constrained annealed system can organize the random couplings under the constraint that one half of them are negative. The ferromagnetic part of the energy is independent of the spatial organization of the couplings and also of the organization of the spins, depending only on m. Therefore, for any given value of m, the J_i's will organize themselves in order to minimize the contribution to the energy due to the n.n. interaction. It is easy to convince oneself that the energetically favourite configuration is that one which has only islands of an even number of negative n.n. couplings. In this case, the spins can organize in order to be all positive except in *some* of the islands of antiferromagnetic J_i's where they alternate up and down. This is due to the fact that islands of an even number of negative couplings are not frustrated as discussed in section 3 and a ground state without frustration has a lower energy than a frustrated one. The annealed system with the constraint (4.2) is thus able to escape frustration at difference with the quenched model. As a consequence at $T = 0$ the entropy vanishes and the overlap is $q = \pm 1$ even in the ferrimagnetic state. This is the main qualitative failure of our annealed approximation. It is worth noting that it would be possible to introduce further constraints (see Ref. [6]) to take into account the frustration effects.

In order to determine te ground states, we can limit ourselves to the configurations with magnetization $m \geq 1/2$. Indeed, at $m = 1/2$ one fourth of the spin must be down and in *all* the (even) negative islands the spin alternates up and down (see Fig. 4). The contribution to the energy due to the n.n. coupling is now $-JN$ and cannot be any reduced. On the other hand, the reduction of m to values lower than 1/2 would rise the energy because of the infinite range positive interaction. As a consequence, $m = 1/2$ is the lowest possible value for the magnetization at $T = 0$ for all finite ω's.

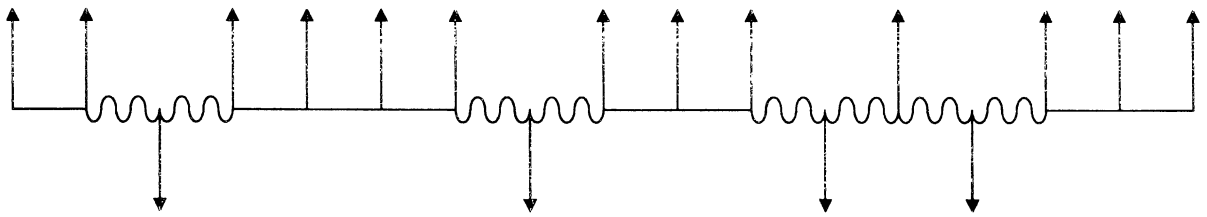

Fig. 4. — Disorder realization and spin configuration for the constrained annealed model at $T = 0$ and $\omega > 3/8$. The full line are the positive n.n. couplings $J_i = +J$ and the wrinkled lines are the negatives ones $J_i = -J$. The arrows represent the spins. The alternated flipped spins are distributed only in the (even) islands of negative couplings.

The energy as function of m has the same qualitative shape as in (3.8) with the difference that the interval for m is [1/2, 1] instead of [2/3, 1]. $E(m)$ is a convex function and its minima are given by the extrema. We thus have to compare the two energy densities at the interval limits, which are

$$E(m = 1) = -\Lambda/2 \tag{4.14a}$$

and

$$E(m = 1/2) = -\Lambda/8 - J\,. \tag{4.14b}$$

They coincide at $\omega = 3/8$ where a first order phase transition occurs.

A qualitative argument identical to the one used for the quenched model indicates that this phase transition survive at non-zero temperature until a critical value T_c estimated by (3.13). The advantage is now that this hypothesis can be checked without a Montecarlo simulation. In fact, the free energy (4.11) is an extremum of the Gibbs potential g, which can be computed in two steps. First, we determine

$$f(n, m) = - \min_{\mu \in R} \beta^{-1} g(\mu, n, m) \tag{4.15}$$

This can be easily done by solving the equation

$$\frac{\partial g}{\partial \mu} \Big|_{\mu^*} = 0 \tag{4.16}$$

which gives the minimization value of the Lagrange multiplier as a function of n,

$$\mu^* = \frac{\tau}{2\omega} \operatorname{arcsinh} \left(n \sinh \frac{2\omega}{\tau} \right) \tag{4.17}$$

where we have introduced the reduced temperature $\tau \equiv T/\Lambda$. The constrained annealed free energy is now the maximum with respect to n and m of

$$f(n, m) = - \frac{1}{\beta} g(\mu^*(n), n, m) . \tag{4.18}$$

To find it, we should solve the two equations

$$\frac{\partial g(\mu^*(n), n, m)}{\partial n} = 0 \quad \text{and} \quad \frac{\partial g(\mu^*(n), n, m)}{\partial m} = 0 . \tag{4.19}$$

By the first one, m^* is obtained as a function of n, that is

$$m^* = n^{1/2} \cosh \beta J (1 - \mu^*) . \tag{4.20}$$

In conclusion, it is left only one implicit equation in the variable n :

$$\tanh \frac{m^*}{\tau} = m^* \left(1 + \tanh \frac{2\omega\mu^*}{\tau} \tanh \frac{\omega(1 - \mu^*)}{\tau} \right)$$

which can be easily solved numerically. When the solutions are more than one we have to choose the solution which minimizes $f(n, m(n))$. This procedure permits us to obtain the magnetization as function of the temperature at fixed ω. In figure 5 one sees that the magnetization as function of the temperature has a jump for ω in the interval [3/8, ω_c], while at $\omega_c = 0.4$, $m(T)$ is a continuous function with a discontinuous derivative at $\tau = 0.18$. On the other hand, for $\omega > \omega_c$ and $\omega < \omega_1$, the magnetization as function of the temperature is an analytic function for $\tau < 1$.

Figure 6 shows the phase diagram in the ω-τ plane where there is a line of first order phase transitions which ends at the critical point ω_c, τ_c where the transition is continuous. Moreover, for $\tau = 1$, independently of the value of ω we have a second order phase transition to the paramagnetic phase, the same as in the mean field model given by $\omega = 0$. This can be easily proved by expanding $f(n, m)$ in power of n and m and finding its minimum. Finally from (4.17) and (4.19), one can show that in the limit $\omega \to \infty$ for non zero Λ the magnetization is a monotonous decreasing function of τ from $m(\tau = 0) = 1/2$ to $m(\tau = 1) = 0$.

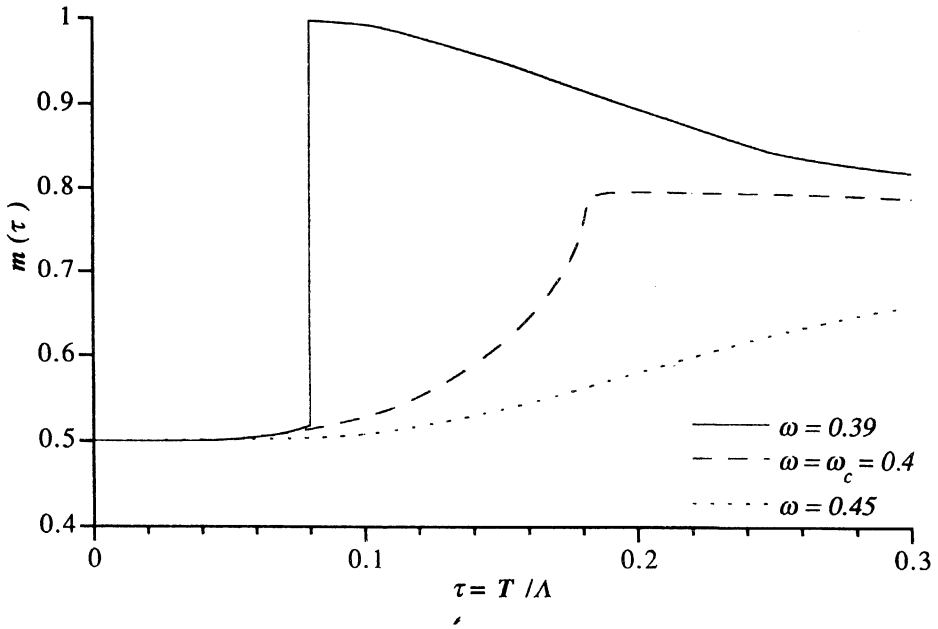

Fig. 5. — Magnetization $m(\tau)$ for the constrained annealed model as function of the reduced temperature $\tau = T/\Lambda$, for different disorder strength $\omega = J/\Lambda$: at $\omega = 0.39$ (discontinuous jump at $\tau = 0.08$) ; at $\omega = \omega_c = 0.4$ (discontinuous m-derivative at $\tau = \tau_c = 0.18$) ; at $\omega = 0.45$ (m is a analytic function for $\tau < 1$).

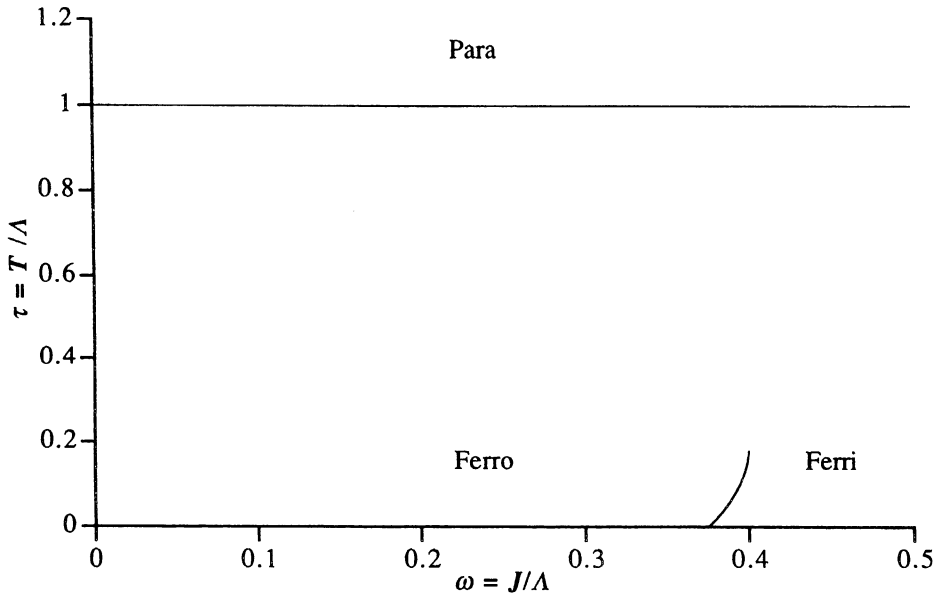

Fig. 6. — Phase diagram $\sim = T/\Lambda$ *versus* $\omega = J/\Lambda$ for the constrained annealed model. The dashed line is the line of the second order phase transition between ferromagnetic and paramagnetic phases. The full line is the line of first order phase transition between ferromagnetic and ferrimagnetic phases ending with a critical point.

5. Conclusion.

We have studied a one dimensional Ising model with nearest neighbor random couplings where some aspects of high dimensional systems are introduced by considering an infinite range

positive coupling. As a consequence of the competition between the two interactions, our model at low temperature and in a range of intermediate disorder exhibits an infinity of disordered ferrimagnetic phases with magnetization $m_n = 2/(1 + n)$, $(n = 2, 3, \ldots \infty)$ which are separated by lines of first order phase transitions. These phases are a real high dimensional effect since they are frustrated and have many degenerate ground states (the zero temperature entropy does not vanish). They could appear in more complicated systems which share the same ingredients of our toy model. For instance, long period commensurate and incommensurate phases are observed in $X - Y$ models with competing interactions and anisotropy without disorder [7-8]. It would be interesting to understand whether the same type of behaviors might arise in spin glass models.

Although our model can be studied in a lengthy numerical way at non-zero temperature *via* products of random matrices we have decided to test the method of the constrained annealed averages to reproduce its behavior. The constraint is the simplest one (the number of positive couplings is fixed by the law of large numbers) and thus is not able to retain the frustration of the quenched disorder system. However, the ferrimagnetic order survives and the main qualitative feature of the model is described by the first rough approximation. The introduction of a further constraint such as $(1/N)\,\Sigma J_i J_{i+1} = 0$ can also take into account the gross effects of frustrations [6], although the analytic calculations become rather tedious, and is far beyond the purpose of this paper. More important in our opinion is the fact that our results pave the way to the analysis of new disordered phases of ferrimagnetic type in annealed models of spin glasses, without recurring to numerical simulations.

Acknowledgements.

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Appendix 1.

At varying the disorder strength ω, the quenched model (2.1) exhibits a numerable infinity of ground states with different absolute magnetization m_n given by (3.1) separated by first order phase transitions. In this appendix we determine these phases, looking for the spin configurations which minimize the energy in the limit of large number N of spins. To simplify the exposition, in this context a k *long island* means k consecutive spins with $k - 1$ positive couplings.

Let us briefly underline the key-points of the appendix : we compute the various magnetizations and transition points by an iterative method, reducing the system at the n-th step to a new one which is made up of negatively coupled k long islands of spins up, with $k \geqslant n$ for a given value of ω. In this scenario, at increasing ω, a phase transition is related to the spin flip of some determinable n long islands.

From this point of view, the system for $\omega = 0$ can be regarded as a set of negatively coupled k long islands, with $k = 1$, and, let us say, with all the spins up (the system is symmetric along the spin direction). As seen in chapter 3, a first order phase transition occurs for $\omega = 5/12$, when some spins with both negative couplings (i.e. 1 long islands) can flip. The number of flipping spins is computable noting that for these spins is energetically convenient to flip alternately if they are consecutive : in other terms, if one has r consecutive spins with negative n.n. couplings, for $\omega = 5/12$ there are $(r + 1)/2$ spin flips if r is odd or $r/2$ spin flips if r is even, with $r/2 + 1$ spin configurations with the same energy because of the

presence of one unsatisfied negative n.n. coupling. Let us indicate the region $0 \leqslant \omega \leqslant 5/12$ as the phase $n = 1$, with magnetization $m_1 = 1$ and transition point $\omega_1 = 5/12$ between m_1 and $m_2 = 2/3$.

Now we want to show that the system, after the first phase transition, is equivalent to a new one with all spins up distributed among negatively coupled k long islands, with $k \geqslant 2$. Let us consider r consecutive spins with negative n.n. couplings, which may thought as a r long macro-island of 1 long islands. If r is even it is evident that, being satisfied all the couplings except one, any partial or global flip of the macro-island is not able to lower the system energy at increasing ω. Moreover the macro-island had zero magnetization and we may wipe out it by inserting a negative coupling between the two surviving n.n. islands.

In the other case we may have a r long macro-island with odd r. Repeating the previous considerations, one obtains that the macro-island is energetically equivalent to a single 1 long island (i.e. a single spin with both negative couplings), which has satisfied the two couplings with its neighbor islands of lengths k_1 and k_2 with $k_1, k_2 \geqslant 2$ (the nearest neighbors of the original macro-island). In an analogous way, these neighbor islands of spins up can not partially flip at increasing ω, and are forced to be both anti-aligned with respect to the central 1 long island (originated from the macro-island). In conclusion and odd macro-island and its nearest neighbors islands will flip only together, and therefore can be substituted for a single island of length $k_1 + k_2 - 1 \geqslant 3$.

These considerations suggests the following renormalization picture : after the first phase transition, the system can be reduced to a new system (similar to the one present before the transition) with all spins up, distributed among k long islands ($k \geqslant 2$) with negative couplings between them. This transformation can be iterated, allowing us to compute the features of the different phases.

To be explicit, consider a system made of $\mathcal{N}'_n$ negatively coupled islands of lengths $k \geqslant n$, at a well defined value of ω where the energetically favourite configuration has all the spins up. Let us suppose that the probability $\mathcal{P}_n(k)$ of a k long island is of the form :

$$\mathcal{P}_n(k) = \alpha_n \gamma_n^{k-1} \tag{A1}$$

with $\alpha_n \gamma_n^{n-1} = 1 - \gamma_n$ in order to get a normalized probability. For $n = 1$ γ_1 represents the probability to have a positive coupling in the original system. In our case one has $\gamma_1 = 1/2$, $\alpha_1 = 1/2$ and $\mathcal{N}'_1 = (1 - \gamma_1) N = N/2$. A moment of reflection indicates that, at increasing ω, it is not possible to observe a partial spin flip of an island, while a global flip could be energetically convenient. In this context one can give a rigorous proof that the next phase transition is related to the spin flip of a fraction of the shortest islands (i.e. $k = n$), in such a manner that it generalizes the transition between m_1 and m_2 previously described. To be more precise, in a macro-island of $r = 2s - 1$ or $r = 2s$ consecutive islands of lengths n there are s alternated flipped islands after the transition occurs.

To compute the transition point ω_n we have to compare the energy densities E_n and E_{n+1} of the two phases. In the former case we have :

$$E_n = -\frac{1}{N} \sum_{i=1}^{N} J_i \sigma_i^{(n)} \sigma_{i+1}^{(n)} - \frac{\Lambda}{2} m_n^2$$

where $\{\sigma_i^{(n)}\}_{i=1,..,N}$ is the spin configuration in the n-th phase. In the latter the expression of the energy density is :

$$E_{n+1} = -4Jx_n - \frac{1}{N} \sum_{i=1}^{N} J_i \sigma_i^{(n)} \sigma_{i+1}^{(n)} - \frac{\Lambda}{2} (m_n - 2nx_n)^2$$

where Nx_n represents the effective number of flipped n long islands according to the rule previously described, $-4JNx_n$ is the energy loss deriving from the spin flip and $-2nm_n$ is the magnetization decrease. Calling $\Delta E_n \equiv E_{n+1} - E_n$ the energy difference between the two phases, it results that :

$$-\frac{\Delta E_n}{4\Lambda} = \omega x_n + \frac{1}{2} nx_n(nx_n - m_n) .$$

The phase transition occurs when $\Delta E_n = 0$, that is for :

$$\omega_n = \frac{n}{2}(m_n - nx_n) \tag{A2}$$

and the system magnetization becomes :

$$m_{n+1} = m_n - 2nx_n . \tag{A3}$$

Our renormalization procedure reduces the system after the n-th transition, to a new one with all spins up among $k \geq n+1$ long islands. Moreover to implement the procedure, we must also show that the probability distribution keeps the form (A1). In this order of things, let us consider a macro-island of $r = 2s$ or $r = 2s-1$ consecutive islands of lengths n, which presents s alternated flipped islands after the transition. Let us stress that for even r, unlike the odd r case, there are $s+1$ configurations with the same energy, which differ for the position of the one unsatisfied n.n. negative coupling.

For even r, generalizing what said for $n = 1$, one can easily deduce that the macro-island is no longer able to vary the system energy at increasing ω and can be wiped out, by inserting a negative coupling between its two nearest neighbors islands, see figure 7a.

In the other case (odd r) the macro-island (which is energetically equivalent to a single n long island) and its nearest neighbors islands of lengths k_1 and k_2 (with $k_1, k_2 \geq n+1$) are forced to flip, if necessary, as a whole, namely as a single island of length $k_1 + k_2 - n \geq n+2$, as illustrated in figure 7b. This concludes the proof that the system is energetically equivalent to a new one with a lower number of total islands $\mathcal{N}_{n+1} \leq \mathcal{N}_n$ after the n-th phase transition.

We have now to compute the probability distribution $\mathcal{P}_{n+1}(k)$ of the islands. The new scenario may be obtained from the previous n-phase, by considering only the $\gamma_n \mathcal{N}_n$ islands of lengths $k \geq n+1$, and melting a number of them with their following neighbor (with a loss of n sites per melting). This number is equal to the number of odd macro-islands of n long islands in the phase n, that is $\gamma_n \frac{1-\gamma_n}{2-\gamma_n} \mathcal{N}_n$. This can be understood from the fact that in the n-th phase the $(1-\gamma_n)\mathcal{N}_n$ islands of length n are organized in $\gamma_n(1-\gamma_n)\mathcal{N}_n$ macro-islands, and a r long macro-island has probability $\gamma_n(1-\gamma_n)^{r-1}$. In other terms each of the $\gamma_n \mathcal{N}_n$ islands of the renormalized system has a probability

$$p_m = \frac{1-\gamma_n}{2-\gamma_n}$$

to melt with its following neighbor, so that :

$$\mathcal{N}_{n+1} = \gamma_n \mathcal{N}_n - \gamma_n \frac{1-\gamma_n}{2-\gamma_n} \mathcal{N}_n = \frac{\gamma_n}{2-\gamma_n} \mathcal{N}_n . \tag{A4}$$

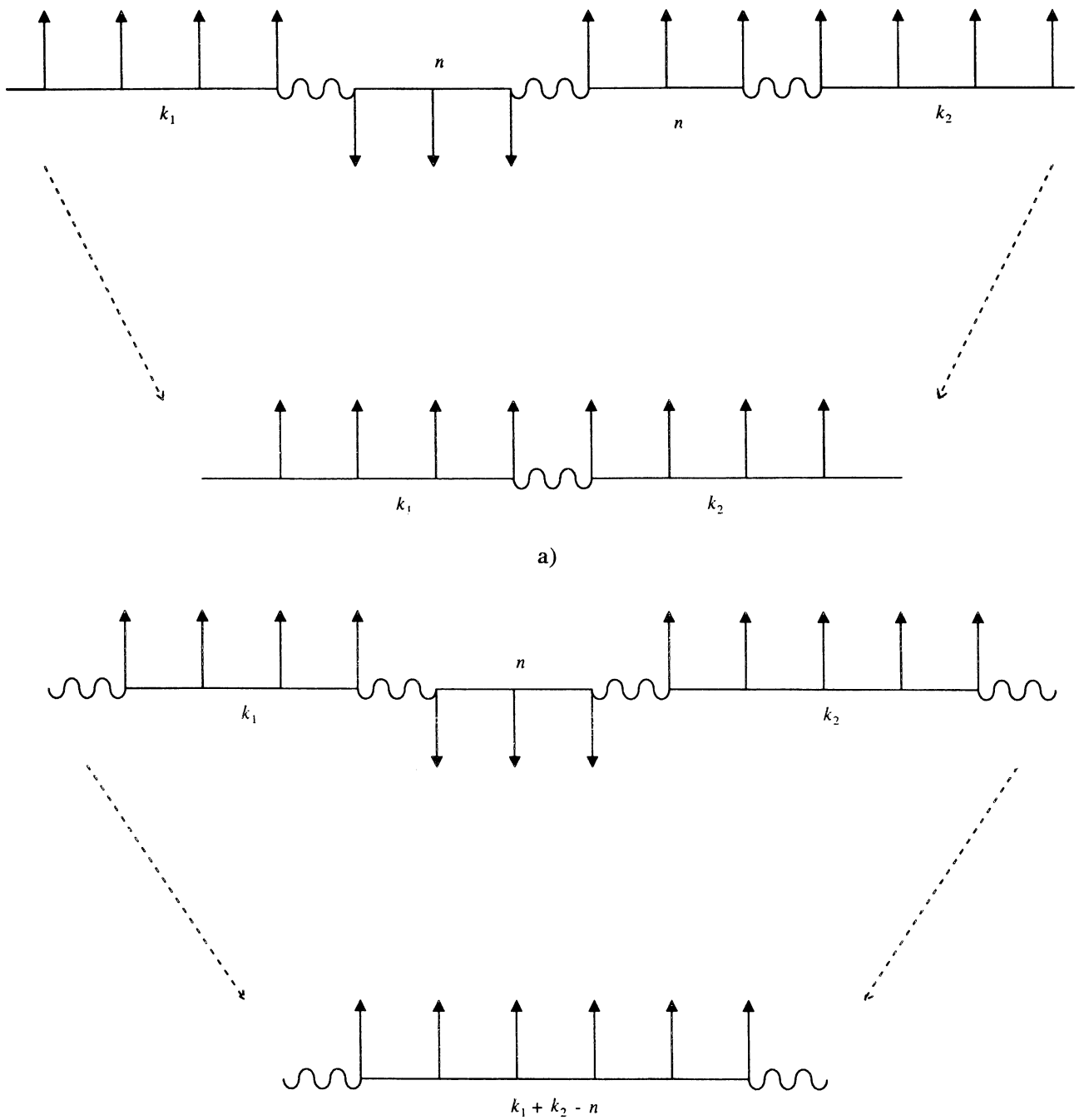

Fig. 7. — Example of the renormalization procedure of the n-th system. The reductions to the $n + 1$-th system consist in the disappearing of macro-islands made of n long islands (n spins and $n - 1$ positive J_i's). The number of islands in the macro-island can be either even (in Fig. 7a there is a macro-island made of two $n = 3$ long islands) or odd (in Fig. 7b there is a macro-island made of one $n = 3$ long island). After the renormalization, the nearest neighbors k_1 and k_2 of the macro-island are coupled by a negative J_i in the even case (Fig. 7a) while they melt into a single island of length $k_1 + k_2 - n$ in the odd case (Fig. 7b).

To obtain $\mathfrak{F}_{n+1}(k)$ is thus sufficient to compute in how many different ways and with what probability a k long island can be generated with multiple melting of islands of lengths at least $n + 1$, but first it is necessary to renormalize $\mathfrak{F}_n(k)$ to take into account the vanishing of the n

long islands :

$$\mathcal{F}_n(k) \to \frac{\alpha_n \gamma_n^{k-1}}{1 - \alpha_n \gamma_n^{n-1}} = \alpha_n \gamma_n^{k-2} \quad \text{with} \quad k \geq n+1 .$$

Let us suppose that ν islands of lengths $\{c_i \geq n+1\}_{i=1,\ldots,\nu}$ unite themselves to build up a k long island. In this case, considering the loss of sites, they have to verify the condition :

$$\sum_{i=1}^{\nu} c_i - n(\nu - 1) = k \tag{A5}$$

and therefore $1 \leq \nu \leq k - n$. It follows that $\mathcal{F}_{n+1}(k)$ can be written as :

$$\mathcal{F}_{n+1}(k) = \sum_{\nu=1}^{k-n} \sum_{c_1,\ldots,c_\nu}^{*} (\alpha_n \gamma_n^{c_1-2}) p_m (\alpha_n \gamma_n^{c_2-2}) p_m \ldots\ldots (\alpha_n \gamma_n^{c_\nu-2}) (1 - p_m)$$

where the star over the summation means that $\{c_i \geq n+1\}$ have to verify the request (A5). This condition has a fundamental importance, because it makes the argument of the sum independent of $\{c_i\}$. It then remains to compute in how many different ways we may choose $\{c_i\}$, corresponding to the different manners to distribute $k - n - \nu$ objects in ν sets, that is $\binom{k-n-1}{\nu-1}$ (see Append. 2). In conclusion one obtains :

$$\mathcal{F}_{n+1}(k) = \frac{1 - \gamma_n}{(2 - \gamma_n)^{k-n}} .$$

This concludes the proof of the self-consistency of our method, and the iterated expressions for α_{n+1} and γ_{n+1} are :

$$\begin{cases} \alpha_{n+1} = (1 - \gamma_n)(2 - \gamma_n)^{n-1} \\ \gamma_{n+1} = \dfrac{1}{2 - \gamma_n} . \end{cases} \tag{A6}$$

It is easy to show that γ_n increases with n converging to 1, and that :

$$\gamma_n = \frac{n(1 - \gamma_1) + 2\gamma_1 - 1}{n(1 - \gamma_1) + \gamma_1} .$$

To obtain the expression of the magnetization m_n as function of n and γ_1, it is not convenient to iterate (A3). Let us stress that at every step we eliminate only groups of spins with null magnetization, and $m_n N$ is then equal to the total number of spins present in our n-th renormalized system. In other terms,

$$m_n = \mathcal{L}_n \frac{\mathcal{N}_n}{N}$$

where $\mathcal{L}_n$ is the average length of the generic island :

$$\mathcal{L}_n = \sum_{k=n}^{\infty} k \alpha_n \gamma_n^{k-1} = n + \frac{\gamma_n}{1 - \gamma_n} = 2n + \frac{2\gamma_1 - 1}{1 - \gamma_1} .$$

As regards $\mathcal{N}'_n$, iterating (A4) and recalling (A6), we get

$$\mathcal{N}'_n = \gamma_n(\gamma_{n-1}\,\gamma_{n-2}\cdots\gamma_2)^2\,\gamma_1(1-\gamma_1)N = \frac{\gamma_1(1-\gamma_1)}{(n(1-\gamma_1)+\gamma_1)\,(n(1-\gamma_1)+2\,\gamma_1-1)}N$$

Combining these results, the magnetization of the n-phase is

$$m_n = \gamma_1 \frac{2\,n(1-\gamma_1)+2\,\gamma_1-1}{(n(1-\gamma_1)+\gamma_1)\,(n(1-\gamma_1)+2\,\gamma_1-1)}\,. \tag{A7}$$

As expected, in the limit $n \to \infty$, m_n converges to 0, that is the spin glass phase. Finally from (A2) and (A3) the expression of ω_n reads

$$\omega_n = \frac{1}{4}\,n(m_n + m_{n+1}) =$$

$$= \frac{1}{2}\,\gamma_1 \frac{n}{n(1-\gamma_1)+\gamma_1} \frac{2\,n^2(1-\gamma_1)^2 + 3\,n\gamma_1(1-\gamma_1) + 2\,\gamma_1 - 1}{(n(1-\gamma_1)+\gamma_1)\,(n(1-\gamma_1)+2\,\gamma_1-1)}\,. \tag{A8}$$

The limit $n \to \infty$ of ω_n represents the last transition point to the phase with $m_\infty = 0$:

$$\lim_{n\to\infty} \omega_n = \frac{\gamma_1}{1-\gamma_1}\,.$$

Recalling that γ_1 is the probability to have a positive coupling in the original system, ω_∞ exhibits the correct behaviour both for $\gamma_1 = 0$ and for $\gamma_1 = 1$ (limits of vanishing disorder). Moreover, the explicit expressions of m_n and $\gamma_n = \frac{1}{2}$ in (A7) and (A8) are given by formulas (3.1) and (3.2) of section 3.

Appendix 2.

In this appendix we show that the asymptotic form (4.6) of $P(m, n)$ can be obtained *via* a combinatorial approach. In this order of things, instead of considering

$$m = \frac{1}{N}\sum_{i=1}^{N}\sigma_i \qquad n = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\,\sigma_{i+1}$$

it is more useful to consider the variables $N^{(-)}$ and k, representing, respectively, the number of spins down and the number of islands with spins down, for a given configuration of N spins. In the following we indicate by $N^{(+)} = N - N^{(-)}$ the number of spins up. With no loss of generality, we assume $m > 0$ (i.e. $N^{(-)} \leqslant N/2$), and therefore the relations between $N^{(-)}$, k and m, n are :

$$m = 1 - 2\,\frac{N^{(-)}}{N} \qquad n = 1 - 4\,\frac{N^{(-)}}{N} \tag{B1}$$

with :

$$1 \leqslant k \leqslant N^{(-)} \quad \text{if} \quad N^{(-)} \geqslant 0$$
$$k = 0 \quad \text{if} \quad N^{(-)} = 0$$

(periodic boundary conditions are assumed). After this preliminary statement, it results that :

$$P(m, n) = \frac{1}{2^N} C(N^{(-)}, k) \tag{B2}$$

where $C(N^{(-)}, k)$ is the number of spin configurations with a given $N^{(-)}$ and k.

Before computing $C(N^{(-)}, k)$, we have to recall some simple combinatorial formulas : R objects can be distributed among ω sets in

$$\binom{R + w - 1}{w - 1}$$

different ways, while using w non empty sets we have

$$\binom{R - 1}{w - 1}$$

configurations.

Let us suppose to have $N^{(-)} \leq N/2$ spins down collected in k distinct islands. For simplicity, in this context we usually call *islands* only the islands of spins down, while we refer to the islands of spins up at the *spaces*. In order to compute $C(N^{(-)}, k)$ one has to calculate in how many different ways the spins down and the spaces can arrange themselves.

We have to distinguish two opposite cases : in the former there are no islands spanning the boundary sites (i.e. 1 and N). In this case $N^{(-)}$ spins can distribute themselves among the k non empty islands in

$$\binom{N^{(-)} - 1}{k - 1} \tag{B3}$$

distincts ways, while there are $k - 1$ non empty spaces between the islands and two spaces at the edges of the strip that can not be simultaneously empty, for a total of $N^{(+)} = N - N^{(-)}$ sites. In other terms, neglecting $k - 1$ sites to ensure $k - 1$ non empty *internal* spaces, one has to distribute $N^{(+)} - (k - 1)$ sites among $k + 1$ spaces, eliminating the combinations with two empty boundary spaces ($N^{(+)} - (k - 1)$ sites among $k - 1$ internal spaces). Let us stress that the least correction is not present if $k = 1$. In conclusion, the number of possible combinations of the spaces is :

$$\binom{N^{(+)} + 1}{k} - \binom{N^{(+)} - 1}{k - 2} (1 - \delta_{k,1}) . \tag{B4}$$

Combining (B3) and (B4), one obtains the number of configurations in the former case (no islands spanning the boundary sites) :

$$C_1(N^{(-)}, k) = \binom{N^{(-)} - 1}{k - 1} \left[\binom{N^{(+)} + 1}{k} - \binom{N^{(+)} - 1}{k - 2} (1 - \delta_{k,1}) \right] . \tag{B5}$$

The latter case (an island spanning the boundary sites) can be regarded as $k + 1$ non empty islands with two of them on the edges of the strip, and, consequently, with k non empty spaces between them. It is thus easy to obtain the number of combinations for the islands :

$$\binom{N^{(-)} - 1}{k} \tag{B6}$$

and for the spaces :

$$\binom{N^{(+)}-1}{k-1} . \tag{B7}$$

Remarking that for $k = N^{(-)}$ this latter case is absent, the total number of possible configurations is :

$$C_2(N^{(-)}, k) = \binom{N^{(-)}-1}{k} \binom{N^{(+)}-1}{k-1} (1 - \delta_{k, N^{(-)}}) . \tag{B8}$$

At this point we have only to sum (B5) and (B8) to obtain $C(N^{(-)}, k)$. After simple algebraic manipulation, it is possible to show that :

$$C(N^{(-)}, k) = C_1(N^{(-)}, k) + C_2(N^{(-)}, k) = \frac{Nk}{N^{(-)}N^{(+)}} \binom{N^{(-)}}{k} \binom{N^{(+)}}{k} \tag{B9}$$

which is correct even for $N^{(-)} \geqslant N/2$. Returning to m and n, and recalling (B2), we get

$$P(m, n) = \frac{1}{2^N} \frac{1-n}{1-m^2} \binom{\frac{1-m}{2}N}{\frac{1-n}{4}N} \binom{\frac{1+m}{2}N}{\frac{1-n}{4}N} . \tag{B10}$$

The expression (4.7) for $S(n, m)$ follows immediately from the asymptotic expansion of (B10) by using the Stirling formula.

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