Complexity in Dynamical Systems with Noise

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We characterize the complexity in dynamical systems with a random perturbation by considering the rate K of divergence of nearby orbits evolving under two different noise realizations. We discuss the meaning of K in the context of the information theory and its physical relevance for the analysis of experimental data. Our definition of complexity becomes crucial for strongly intermittent systems where K is very different from the Lyapunov exponent. This behavior is illustrated by some numerical computations.

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The problem of the effect of a random perturbation on a deterministic evolution law is an issue of great importance since thermal fluctuations or uncontrollable changes of parameters are always present in physical systems, while the roundoff errors have the same role in numerical simulations [1]. From the experience of previous works a first rough conclusion is that the presence of a random term does not change the qualitative behavior of the dynamics [2–8]. In the case of a regular (stable) system the random perturbation just changes the very long time behavior by introducing the possibility of jumps among different attractors (stable fixed points or stable limit cycles). A familiar example is the Langevin equation describing the motion of an overdamped particle in a double well. Even in the opposite limit of chaotic dissipative systems the presence of noise is expected not to change the qualitative behavior in a dramatic way. The typical situation is the following: (a) the strange attractor is just smoothed at small scale $O(\sigma)$, if σ is the strength of the noise, but maintains the fractal structure at larger scales; and (b) the value of Lyapunov exponents differs from the unperturbed one of a quantity $O(\sigma)$.

However, the combined effects of the noise and of the deterministic part of the evolution law can produce highly nontrivial, and often intriguing, behaviors in some cases. We just mention the stochastic resonance where one has a synchronization of the jumps between two stable points [9–11] and the possibility to have the phenomena both of the so called noise-induced order [7] and of the noise-induced instability [5,6].

We believe that one of the main problems is the lack of a well defined method to characterize the "complexity" of the trajectories. Usually [2–5,7], an indication of chaoticity is obtained treating the random term as a usual time-dependent term, and, therefore, considering the separation of two nearby trajectories with the same realization of the noise. To be explicit, consider a 1D map (the generalization to N-dimensional maps or ordinary differential equations is obvious):

$$x(t+1) = f[x(t),t] + \sigma w(t), \qquad (1)$$

where t is an integer and $w(t)$ is an uncorrelated random process; e.g., w are independent random variables uniformly distributed in $[-1,1]$. A possible, well defined way to quantify the complexity is to compute the maximum Lyapunov exponent associated with the separation rate of two nearby trajectories with the same realization of the stochastic term $w(t)$. In the limit of vanishing initial distance between the two trajectories, one obtains the equation for the tangent vector:

$$z(t+1) = f'[x(t),t]\, z(t), \qquad (2)$$

where $f' = df/dx$. The maximum Lyapunov exponent can thus be defined in the standard way:

$$\lambda_\sigma = \lim_{t\to\infty} t^{-1} \ln|z(t)| \qquad (3)$$

and, when $\sigma = 0$, one gets the Lyapunov exponent λ_0 of the unperturbed dynamical system. From the computation of λ_σ as defined by Eqs. (2) and (3) some authors argue that there exists a phenomenon of noise-induced order [7]; i.e., at increasing the strength of the fluctuation σ, λ_σ passes from positive to negative. Even the opposite phenomenon (noise-induced instability) has been observed: At increasing σ, λ_σ can pass from negative to positive [5,6]. We shall discuss these results in the conclusions.

Although the Lyapunov exponent λ_σ is well defined, we believe that (3) is neither unique nor the most useful characterization of the complexity of a noisy system. In addition, it is easy to realize that it is practically impossible to extract λ_σ from the experimental data.

In order to introduce a more natural indicator of complexity in noisy dynamics it is convenient to follow a quite different approach, where two realizations of the noise, instead of only one as in (1), are used. This is exactly what happens when experimental data are analyzed by the Wolf *et al.* algorithm [12].

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Before discussing our alternative definition of chaos in noisy systems, we must briefly recall the characterization of intermittency in deterministic dynamical systems. An effective Lyapunov exponent [13] has been introduced to measure the fluctuations of chaoticity,

$$\gamma_t(\tau) = \frac{1}{\tau} \ln \frac{|z(t+\tau)|}{|z(t)|}. \tag{4}$$

It gives the local expansion rate in the interval $[t, t+\tau]$. The maximum Lyapunov exponent is thus given by a time average along the trajectory $x(t)$: $\lambda_0 = \langle \gamma_t \rangle$ for $\tau \to \infty$.

Let us define our new indicator of complexity in the framework of a deterministic system [i.e., Eq. (1) with $\sigma = 0$] where it coincides with λ_0. Let $x(t)$ be the trajectory starting at $x(0)$ and $x'(t)$ be the trajectory starting at $x'(0) = x(0) + \delta x(0)$ with $\delta_0 = |\delta x(0)|$, and indicate by τ_1 the maximum time such that $|x'(t) - x(t)| < \Delta$. Then, we put $x'(\tau_1 + 1) = x(\tau_1 + 1) + \delta x(0)$ and define τ_2 as the maximum τ such that $|x'(\tau_1 + \tau) - x(\tau_1 + \tau)| < \Delta$, and so on. In our context, we can define the effective Lyapunov as

$$\gamma_i = \tau_i^{-1} \ln(\Delta/\delta_0). \tag{5}$$

However, we sample the expansion rate in a nonuniform way, at time intervals τ_i. As a consequence, the probability of picking γ_i is $p_i = \tau_i / \sum_i \tau_i$ so that

$$\lambda_0 = \langle \gamma_i \rangle = \frac{\sum_i \tau_i \gamma_i}{\sum \tau_i} = \frac{1}{\overline{\tau}} \ln\left(\frac{\Delta}{\delta_0}\right), \quad \overline{\tau} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \tau_i. \tag{6}$$

This definition without any modification can be extended to noisy systems by introducing the rate

$$K_\sigma = \overline{\tau}^{-1} \ln(\Delta/\delta_0), \tag{7}$$

which coincides with λ_0 for a deterministic system ($\sigma = 0$). When $\sigma = 0$ there is no reason to determine the Lyapunov exponent in this apparently odd way, of course. However, the introduction of K_σ is rather natural in the framework of the information theory [14]. Considering again the noiseless situation, if one wants to transmit the sequence $x(t)$ ($t = 1, 2, \dots, T_{\max}$) accepting only errors smaller than a tolerance threshold Δ, one can use the following strategy: (1) Transmit the rules which specify the dynamical system (1), using a finite number of bits which does not depend on the length $T_{\max}$. (2) Specify the initial condition with precision δ_0 using a number of bits $n = \ln_2(\Delta/\delta_0)$ which permits arrival up to time τ_1 where the error equals Δ. Then specify again the new initial condition $x(\tau_1 + 1)$ with a precision δ_0 and so on. The number of bits necessary to specify the sequence with a tolerance Δ up to $T_{\max} = \sum_{i=1}^{N} \tau_i$ is $\simeq Nn$ and the mean information for time step is $\simeq Nn/T_{\max} = K_\sigma / \ln 2$ bits.

In the presence of noise, the strategy of the transmission is unchanged but, since it is not possible to transmit the realization of the noise $w(t)$, one has to estimate the growth of the error $\delta x(t) = x'(t) - x(t)$, where $x(t)$ and $x'(t)$ evolve in two different noise realizations $w(t)$ and $w'(t)$, and $|\delta x(0)| = \delta_0$.

The resulting equation for the evolution of $\delta x(t)$ is

$$\delta x(t+1) \simeq f'[x(t), t]\, \delta x(t) + \sigma \tilde{w}(t), \qquad \tilde{w}(t) = w'(t) - w(t). \tag{8}$$

For the sake of simplicity we discuss the case $|f'[x(i), i]| = \text{const} = \exp \lambda_0$, where (8) gives the bound on the error:

$$|\delta x(t)| < e^{\lambda_0 t}(\delta_0 + \tilde{\sigma}) \quad \text{with } \tilde{\sigma} = 2\sigma/(e^{\lambda_0} - 1). \tag{9}$$

This formula shows that δ_0 and $\tau = \overline{\tau}$ are not independent variables but they are linked by the relation

$$e^{\lambda_0 \tau}(\delta_0 + \tilde{\sigma}) \simeq \Delta. \tag{10}$$

As a consequence, we have only one free parameter, say τ, to optimize the information entropy K_σ in (7), so that the complexity of the noisy system can be estimated by

$$G_\sigma = \min_\tau K_\sigma = \lambda_0 + O(\sigma/\Delta), \tag{11}$$

where the minimum is reached at an optimal time $\tau = \tau_{\text{opt}}$ from the transmitter point of view.

In the case of a deterministic system K_σ does not depend on the value of τ (i.e., it is equivalent to use a long $\overline{\tau}$ and to transmit many bits few times or a short $\overline{\tau}$ and to transmit few bits many times). On the contrary, in noisy systems there exists an optimal time τ_{opt} which minimizes K_σ: Using relation (9) one sees that $\Delta = \exp(\lambda_0 \overline{\tau})(\delta_0 + \tilde{\sigma})$ and K_σ has a minimum for $\tau_{\text{opt}} \simeq 1/\lambda_\sigma$. This result might appear trivial but has a relevant consequence from a theoretical point of view in presence of noise; even if the value of the entropy G_σ changes only $O(\sigma/\Delta)$, there exists an optimal time for the transmission.

The interesting situation happens for strong intermittency when there is an alternation of positive and negative γ during long time intervals. In this case the existence of an optimal time for the transmission induces a dramatic change for the value of G_σ. This is particularly clear when considering the limit case of positive γ_1 in an interval $T_1 \gg 1/\gamma_1$, followed by a negative γ_2 in an interval $T_2 \gg 1/|\gamma_2|$, and again a positive effective Lyapunov exponent and so on. In the expanding intervals, one can transmit the sequence using $\simeq T_1/\gamma_1 \ln 2$ bits, while during the contracting interval one can use only few bits. Since, in the expanding intervals, the transmission has to be repeated rather often and moreover $|\delta x|$ cannot be lower than the noise amplitude σ, at difference with the noiseless case, it is impossible to use the contracting intervals to compensate the expanding ones. This implies that in the limit of very large T_i only the expanding intervals contribute to the evolution of the error $\delta x(t)$, and the information entropy is given by an average of the positive effective Lyapunov exponents:

$$G_\sigma \simeq \langle \gamma \theta(\gamma) \rangle. \tag{12}$$

For the approximation considered above, $G_\sigma \geq \lambda_\sigma = \langle \gamma \rangle$. Note that by definition $G_\sigma \geq 0$ while λ_σ can be negative.

The estimate (12) stems from the fact that δ_0 cannot be smaller than σ so the typical value of τ_i is $O(1/\gamma_i)$ if γ_i is positive. We stress again that (12) holds only for strong intermittency, while for uniformly expanding systems or rapid alternations of contracting and expanding behaviors $G_\sigma \simeq \lambda_\sigma$.

It is not difficult to estimate the range of validity of the two limit cases $G_\sigma \simeq \lambda_\sigma$ and $G_\sigma \simeq \langle \gamma\theta(\gamma)\rangle$. Denoting by $\gamma_+ > 0$ and $\gamma_- < 0$ the typical values of the effective Lyapunov exponent in the expanding and contracting time intervals of lengths T_+ and T_-, respectively, (12) holds if during the expanding intervals there are at least two repetitions of the transmission and the duration of the contracting interval is long enough to allow the noise to be dominant with respect to the contracting deterministic effects. In practice one should require

$$\exp(\gamma_+ T_+) \gg \Delta/\sigma, \qquad \exp(-|\gamma_-|T_-) \gg \Delta/\sigma. \quad (13)$$

In a similar way, $K \simeq \lambda_0$ holds if

$$\exp(-|\gamma_-|T_-) \ll \Delta/\sigma. \quad (14)$$

These results have been checked by some simple numerical computations in three different systems which are shown in Figs. 1, 2, and 3, respectively. Let us stress that we have directly computed K_σ, and, since $\tau_i = O(1/\gamma_i)$, we automatically are very close to the optimal strategy so that $K_\sigma \simeq G_\sigma$, without performing a minimization. The random perturbation $w(t)$ is an independent variable uniformly distributed in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

The first system is given by periodic alternation of two piecewise linear maps of the interval [0,1] into itself:

$$f[x,t] = \begin{cases} ax \bmod 1, & \text{if } (2n-1)T \le t < 2nT, \\ bx, & \text{if } 2nT \le t < (2n+1)T, \end{cases} \quad (15)$$

where $a > 1$ and $b < 1$. Note that in the limit of small T, $G_\sigma \to \max[\lambda_\sigma, 0]$ since it is a non-negative quantity as shown in Fig. 1 where, for $b = \frac{1}{4}$, λ_σ is negative.

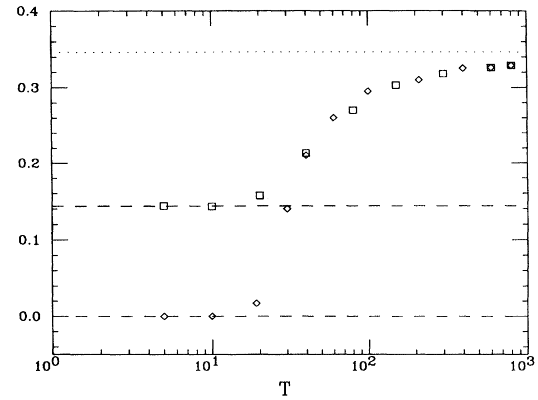

FIG. 1. K_σ versus T with $\sigma = 10^{-7}$ for the map (15). The parameters of map (15) are $a = 2$ and $b = \frac{2}{3}$ (squares) or $b = \frac{1}{4}$ (diamonds). The dotted line indicates the Pesin-like relation (12) while the dashed lines are the noiseless limit of K_σ. Note that for $b = \frac{1}{4}$ the Lyapunov exponent λ_σ is negative.

The second system is given by a random alternation of two logistic maps

$$f[x,t] = r(t)x(1-x), \quad (16)$$

where $r(t)$ is given as follows. We extract at time iT an independent random variable $\tilde{r}_i$ which can assume two values $r_1 = 4$ (chaotic logistic map at the so-called Ulam point, with $\lambda = \ln 2$) or $r = r_0$ ($r_0 < 1$, contracting map) with equal probability. The control parameter is $r(t) = \tilde{r}_i$ in the time interval $[iT, (i+1)T - 1]$.

The third system is strongly intermittent without an external forcing. It is the Beluzov-Zhabotinsky map [4,7] related to a famous chemical reaction:

$$f(x) = \begin{cases} [(1/8 - x)^{1/3} + a]e^{-x} + b, & \text{if } 0 \le x < 1/8, \\ [(x - 1/8)^{1/3} + a]e^{-x} + b, & \text{if } 1/8 \le x < 3/10, \\ c(10xe^{-10x/3})^{19} + b, & \text{if } 3/10 \le x, \end{cases} \quad (17)$$

with $a = 0.506\,073\,57$, $b = 0.023\,288\,527\,9$, and $c = 0.121\,205\,692$. The map exhibits a chaotic alternation of expanding and very contracting time intervals. Although the value of T_- is very small because $|\gamma_-| \gg 1$, the first inequality (13) is unsatisfied because the expanding time intervals are rather short. As a consequence, the asymptotic estimate $G_\sigma \simeq \langle \gamma\theta(\gamma)\rangle$ cannot be observed. In Fig. 3, one sees that, while λ passes from negative to positive values at decreasing σ, G_σ is not sensitive to this transition to "order." Another important remark is that in the usual treatment of the experimental data, if some noise is present, one practically computes G_σ, and the result can be completely different from λ_σ. Let us mention, for example, Ref. [6] where the author studies a one-dimensional nonlinear time-dependent Langevin equation. A simple numerical computation shows that λ_σ is negative while the author claims to find, using the Wolf method, a positive "Lyapunov exponent."

Our results show that the same system can be regarded either as regular (i.e., $\lambda_\sigma < 0$) when the same noise realization is considered for two nearby trajectories or as chaotic (i.e., $G_\sigma > 0$) when two different noise realizations are considered; see, e.g., Fig. 1. The situation is similar to what is observed in fluids with Lagrangian chaos [15]. There, a pair of particles passively advected by a chaotic velocity field might remain closed following together a "complex" trajectory. The Lagrangian Lyapunov exponent is thus zero. However, a data analy-

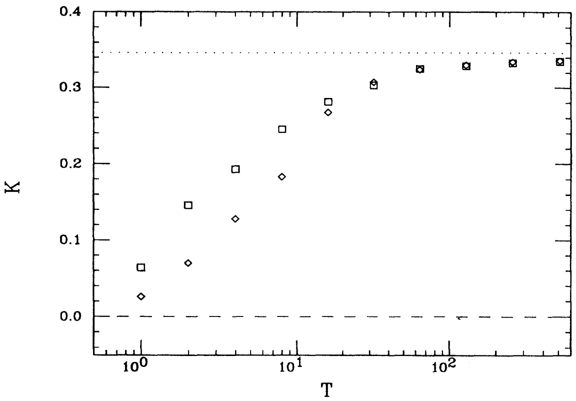

FIG. 2. K_σ versus T with $\sigma = 10^{-7}$ for the map (16). The parameters of the map are $r_1 = 4$ and $r_0 = 0.8$ (squares) or $r_0 = 0.2$ (diamonds). The dotted line indicates the Pesin-like relation (12).

sis gives a positive Lyapunov exponent because of the "Eulerian" chaos. We can say that λ_σ and G_σ correspond to the Lagrangian Lyapunov exponent and to the exponential rate of separation of a particle pair in two slightly different velocity fields, respectively.

In conclusion, we have introduced an indicator of complexity G_σ for chaotic system perturbed by noise. The relation $G_\sigma \simeq \langle \gamma \theta(\gamma) \rangle$ is, in some sense, the time analogous of the Pesin relation $h \simeq \sum_i \lambda_i \theta(\lambda_i)$ between the

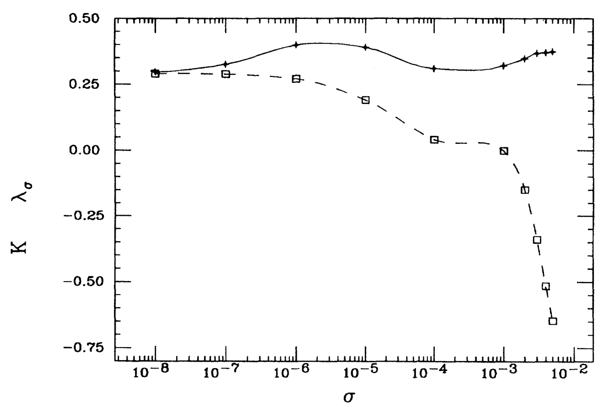

FIG. 3. λ_σ (squares) and K_σ (crosses) versus σ for map (17).

Kolmogorov-Sinai entropy h and the Lyapunov spectrum [16] where the negative Lyapunov exponents do not decrease the value of h since the contraction along the corresponding directions cannot be observed for any finite space partition. In the same way the contracting time intervals, if long enough, do not decrease G_σ. It is important to note that the limit $\sigma \rightarrow 0$ is very delicate. Indeed for small σ, say $\sigma < \sigma_c$, the inequality (14) will be fulfilled and $G_\sigma \simeq \lambda_\sigma \rightarrow \lambda_0$ for $\sigma \rightarrow 0$. However, in strongly intermittent systems T_- can be very long so that the noiseless limit $G_\sigma \rightarrow \lambda_0$ is practically unreachable, as illustrated by Fig. 3.

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