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CONSTRAINED ANNEALING FOR SYSTEMS WITH QUENCHED DISORDER

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We review a method for the study of systems with quenched disorder, such as Ising models with random field or coupling, by means of annealed averages where appropriate constraints are imposed. It allows one to understand the role of the intensive variables of the disorder (mean magnetic field, impurities concentration, frustration ...) and improves the annealed approximation of the quenched free energy.

The constrained annealing can be obtained by introducing generalized Gibbs-like potential depending on Lagrangian multipliers associated to the intensive variables of the disorder. The minimum of the potential gives a lower bound of the quenched free energy which can be a very accurate estimate for a proper choice of the constraints.

The method is first applied to one-dimensional Ising models with random magnetic fields. In this case the frustration is properly taken into account and the quenched free energy density is estimated with a precision higher than the numerical results. As a first step to higher dimensional model we then introduce an Ising model with two competing interactions: nearest neighbour random couplings and a positive infinite range coupling. At low temperature the model exhibits a new type of non-trivial 'ferrimagnetic' order in a region of low temperatures and intermediate disorder strength. The qualitative features of the model (in particular the phase transition line between ferromagnetic and ferrimagnetic phases) is reproduced by the constrained annealing.

Finally we apply our method to d-dimensional Ising models with random nearest neighbour coupling. In this case, we also introduce an alternative new way to obtain constrained annealed averages without recurring to the Lagrange multipliers. It requires to perform quenched averages on small volumes, in a analytic or numerical way. We thus give a sequence of converging lower bounds for the quenched free energy. In particular, in $2d$ the known analytic estimates are considerably improved.

1. Introduction

1.1. *Systems with annealed and quenched disorder*

The statistical mechanics of disordered systems which was initially confined to solid state physics is now an active field of research bridging different disciplines such as physics, biology and information science. From a mathematical point of view, disordered systems are characterized by two types of variables: the *hot* variables which

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arrange themselves for minimizing the free energy and the *cold* variables which have much longer evolution times. A classical example is a spin glass with Hamiltonian

$$H_N = \sum_{i,j} J_{ij}\,\sigma_i\,\sigma_j \tag{1.1}$$

where the N spin variables $\{\sigma = \pm 1\}$ are of the first type and the random couplings J_{ij} are of the second type. This fact corresponds to two different kinds of averages: quenched and annealed averages. The typical free energy is thus given by the quenched average over the disorder *cold* variables

$$-\frac{1}{\beta N}\,\overline{\ln Z_N} = f_N \tag{1.2}$$

where

$$Z_N = \sum_{\{\sigma_i\}} e^{-\beta H_N} \equiv 2^N \langle e^{-\beta H_N} \rangle \tag{1.3}$$

is given by an annealed average over the *hot* variables $\{\sigma_i\}$ which take the two possible values ± 1 with equal probability. In the thermodynamic limit, all the disorder realizations of $\{J_{ij}\}$ (a part a set of zero probability measure) have the same free energy

$$\lim_{N\to\infty} -\frac{1}{\beta N}\,\ln Z_N(\{J_{ij}\}) = \lim_{N\to\infty} f_N = f\,. \tag{1.4}$$

This property is called self-average[16] since $(\ln Z_N)/N$ becomes a non-random quantity for $N \to \infty$.

The calculation of the quenched free energy is a difficult problem even in simple one-dimensional models. In practice, it is much easier to estimate the quenched average $\overline{\ln Z}$ by the annealed average $\ln \overline{Z}$ which is a lower bound of f. However, annealed averages are often very bad approximations and, more importantly, can fail to describe even the qualitative aspects of a disordered system. This is due to the fact that in an annealed average the main contribution comes from a set of disorder realizations with zero probability measure in the limit $N \to \infty$. In these realizations the disorder variables (for instance couplings and fields in a random Ising model) are arranged in order to minimize the free energy while in a typical realization of a 'quenched' model they are frozen in some given configuration. The latter is the realistic assumption, since, in general, the disorder variables have much longer evolution times than the thermodynamic variables (e.g. spins).

As a direct consequence of the freezing, quenched systems are often characterized by a fundamental feature that is missed in annealed systems, the so-called *frustration*.[28] Indeed a generic disordered system has some disorder realizations where the energy minima are reached by many spin configurations, which, in general, exhibit different physical properties: for instance, they could have different magnetization. At sufficiently low temperature, when the entropic term in the

free energy is negligible, all these spin configurations represent a possible equilibrium state for the quenched system, and thus it has a problematic choice (i.e. it is *frustrated*). For simplicity, we usually refer to those disorder realizations that can induce this feature as the *frustrated* ones. To be more exact, one would have called *frustrating* the disorder realizations, since the spins or, in general, the system is *frustrated.*

The classical example is a triangular plaquette with Hamiltonian

$$H = -J_1\sigma_1\sigma_2 - J_2\sigma_2\sigma_3 - J_3\sigma_1\sigma_3 \tag{1.5}$$

with $J_i = \pm 1$, where the frustration can be defined as $\tilde{J}_p = -\prod J_i$. If $\tilde{J}_p = 1$, a coupling realization is frustrated and there are different energy minima in contrast with the case $\tilde{J}_p = -1$, where the configuration with aligned spins is favourite.

In more complicated quenched systems, which generally are a set of elementary interacting plaquettes, frustration is present not only on the elementary plaquettes, but also on plaquettes of different sizes, built up from the elementary ones, for a proper choice of the disorder realization.

Recalling the concept of frustration, we have remarked that it is a consequence of the freezing of the disorder variables. In fact, an annealed system has no frustrated realizations since the couplings in a plaquette, or in general the disorder variables, can evolve into a non-frustrated configuration which permits the system to get a further diminution of the free energy. The absence of frustration in the annealed system considerably reduces the physical relevance of the approximation.

However, beside frustration, there are other relevant disorder variables which are modified in the annealed approximation. For example, the typical realizations of an Ising ferromagnetic model with impurities (negative coupling constants) has a fixed concentration of negative couplings. The realizations which contribute to the annealed approximation have a lower concentration of negative couplings, and, moreover, at low temperature the concentration vanishes and the annealed free energy becomes that of the pure ferromagnetic model. Another example is a ferromagnetic model with locally-independent random magnetic fields. Assuming that the fields are positive or negative with the same probability, the typical realization of the disorder will have vanishing mean magnetization. On the contrary, in the annealed approximation the fields will arrange to point preferably in the same direction and, at vanishing temperature, they will all be aligned.

1.2. *Constrained annealed averages*

An alternative method to study a system with quenched disorder is to consider an annealed system where the most relevant qualitative features of the original models are saved. The idea is to compute annealed averages where frustration, impurity concentration, mean magnetization and other relevant intensive variables of the disorder are fixed to their 'quenched' values. In other terms, although we treat random couplings and/or fields as *hot* (annealed) variables, we require that they

satisfy appropriate constraints which determine the main physical properties of the model. The standard way to impose a constraint in an average is the method of the Lagrange multipliers.[26] The thermodynamic limit of the annealed average thus gives a generalized thermodynamic potential which is a function of the Lagrange multipliers (related to some disorder-intensive variables), playing the role of the chemical potential in ordinary statistical mechanics. The value of the Lagrangian multipliers which minimizes the thermodynamic potential selects the realizations with a correct value of the related disorder-intensive variables, and, at the same time, minimizes the difference between the mean free energy density and its annealed approximations.

However the practical difficulties of implementation increase with dimensionality and at present we are able to obtain analytic results for spin glasses in two and three dimensions, while the infinite range Sherrington–Kirkpatrick model is still a challenge for the method.

The general formulation of the constrained annealing for disordered systems and most of the applications reviewed in this paper are new although the method and similar approaches have been proposed and applied in special contexts by many authors (see Refs. 17, 27, 14, 29 and 10).

We want finally to stress that it is possible to perform constrained annealed averages, even without Lagrange multipliers, in d-dimensional spin glasses. One needs only to perform a quenched average over a sub-system with a finite small size. This is often possible both in an analytic and numerical way. With that second method, we are able to obtain very good lower bounds of the quenched free energy of two-dimensional spin glasses.

1.3. *The models of the review*

We have applied the method of constrained annealed averages to one-dimensional Ising systems with random magnetic fields and couplings, to nearest neighbour Ising spin glass in two or more dimensions and to a one-dimensional model with two competing interactions: a long range ferromagnetic interaction and a short range random interaction.

In one dimension, we have considered the Ising model with a constant positive nearest neighbour coupling J and a local field h_i which can take at random two possible values $H + h$ or $H - h$. At low temperature, for some values of the three constants J, H and h, the system exhibits a frustrated behaviour. This is due to the fact that a single spin has two energetically equivalent choices, either to align with the first neighbour spins or to align with the local magnetic field. The frustration implies that the zero temperature entropy is positive. However, the annealed approximation destroys the typical behaviour associated to the frustration. On the contrary, the constrained annealed approximation is able to properly take into account the frustration and to give an extremely accurate qualitative and quantitative description of the model.

We have then introduced a competing interaction Ising model which has two kinds of couplings: nearest neighbour random couplings $\pm J$ with equal probability and a positive infinite range coupling Λ. At low temperature T the model exhibits a first-order phase transition between a ferromagnetic state (with magnetization $m_1 = 1$ at $T = 0$) and a 'ferrimagnetic' state (with $m_2 = 2/3$ at $T = 0$), when the disorder strength J/Λ is increased. For $5/12 < J/\Lambda < 1$, a whole spectrum of ferrimagnetic ground states with magnetization $m_n = 2/(n+1)$ $(n = 2, \cdots, \infty)$ is present while for $J/\Lambda > 1$ the ground state is given by a trivial one-dimensional spin glass with $m = 0$. The main qualitative features of the model can be described by an annealed average with the constraint that the number of positive couplings is fixed to its quenched value. This constrained annealed model is exactly solved at all temperatures and the diagram of phase is calculated.

At increasing the dimensionality, the application of the method of constrained annealing becomes more and more difficult, since the relevant variables are connected to the frustration on loops of bonds on the lattice. The problem is equivalent to a lattice gauge theory as pointed out by Toulouse and Vannimenus.[29] Some Monte Carlo simulations have been performed following this proposal (see e.g. Refs. 5, 3 and 2), though, as far as we know, no analytic results have been obtained.

In this paper we consider two- and three-dimensional Ising spin glasses, with nearest neighbour coupling and no external magnetic field. The couplings are independent random variables which are either Gaussian or take two possible values ± 1 with equal probability. We also review the results of Ref. 27 in the case of diluted disorder, i.e. $J_i = +1$ with probability p and $J_i = -1$ with probability $1 - p$.

These models have been widely studied from a numerical point of view and also by finite size exact calculations. It is commonly accepted that in two dimensions, when $J_i = \pm 1$ with equal probability, the model has no finite temperature transition to a glassy phase. Nevertheless, its behaviour at zero temperature is still not completely understood. In particular, it is unclear what the scaling behaviour of the heat capacity at vanishing temperature is.

The annealed approximation is very poor and gives a completely wrong description at low temperature. In order to get an analytic close form for the free energy we impose a simple constraint: the frustration associated to the square plaquettes corresponding to the black squares of a chessboard is fixed to the correct quenched value. In this way, we find out a qualitative behaviour of the vanishing temperature heat capacity in agreement with the data obtained by finite size calculations (see Ref. 25).

1.4. *Organization of the paper*

The paper is organized as follows.

In Sec. 2 we introduce the method of Lagrange multipliers for constrained annealed averages, showing that it is able to give good estimates (lower bounds) of

the free energy and a deeper understanding of the role of the intensive variables of the disorder. The physical meaning of our proposal is explained in terms of large deviation properties of the free energy and of the intensive variables of the disorder. In order to clarify the practical aspects, we first test the method in two exactly solvable examples and then in a less trivial one-dimensional model.

In Sec. 3 we systematically apply the method to a one-dimensional ferromagnetic Ising model with random dichotomic magnetic fields. We compute the free energy with a precision which is higher than that obtained by numerical methods. More importantly, we are able to find out the relevant variables of the disorder. These variables completely determine the physical properties of the model. The mean magnetization is also computed.

In Sec. 4, as a first step toward analytic results in higher dimensions, we introduce a one-dimensional Ising model, with competing interactions: a nearest neighbour random coupling $J_i = \pm J$ and a positive infinite range coupling Λ. We exactly solve the model at temperature $T = 0$ and we show that for an intermediate range of values of J/Λ it exhibits a ferrimagnetic order, which also survives at $T \neq 0$. The ferrimagnetic phases are frustrated with zero temperature entropy which does not vanish and with a non-trivial value of the overlap between the degenerate ground states. We then introduce a constrained annealed version of the model which retains the ferrimagnetic order, although without taking into account the frustration. The advantage is that in this case the phase diagram can be computed in an analytic way for all T, without Monte Carlo simulations.

In Sec. 5 we discuss the Ising spin glass model on the d-dimensional lattice $\mathbb{Z}^d$. We consider three cases:

(a) Gaussian coupling models for which we improve the annealed estimates for the free energy and ground state energy;
(b) two-dimensional Ising model with diluted disorder (we essentially review the results of Ref. 27);
(c) dichotomic couplings in two- and three-dimensions where we are able to fix the frustration on some of the elementary plaquettes of the system.

The third case is the most interesting since, in two dimensions, the local frustration is properly taken into account and the qualitative behavior at low temperature is similar to that of the quenched model.

In Sec. 6, we introduce a new method to obtain constrained annealed averages without recurring to Lagrange multipliers. We are thus able to improve the result of Sec. 5 and get accurate estimates of the free energy of a two-dimensional spin glass. Finally, in Sec. 7 the reader can find a summary and a critical discussion on our results. In the same section we also give an outlook on the expected future applications of our method.

2. Lagrange Multipliers and Gibbs Potentials

2.1. *The method*

Let us denote the finite volume free energy at fixed coupling realization $\{J_{i,j}\}$

$$y_N = -\frac{1}{N\beta} \ln Z_N[\{J_{ij}\}] . \tag{2.1}$$

The variables $J_{i,j}$ are used here to indicate any kind of *cold* variables: couplings, magnetic fields, impurities, ... In general, y_N is a random quantity which depends on the configuration $\{J_{i,j}\}$, nevertheless, in the thermodynamic limit $N \to \infty$, almost all the disorder realizations correspond to the same free energy, so that

$$\lim_{N\to\infty} y_N = f \tag{2.2}$$

with probability one. This property is called self-average since y_N becomes a non-random quantity for $N \to \infty$. It is useful to introduce the finite volume mean free energy $f_N \equiv \overline{y_N}$. In terms of the probability $P_N(y)dy$ that the free energy density y_N falls in the interval $[y, y + dy]$, one thus has

$$f_N = \overline{y_N} = -\frac{1}{\beta N}\overline{\ln Z_N} = \int dy\ P_N(y)\ y \tag{2.3}$$

where the integral is over all possible values of y. For $N \to \infty$, the probability of finding $y_N \neq f$ should vanish, because of the self-averaging. Therefore for large N, $P_N(y)$ is peaked around the most probable value f, and one can make the *ansatz*

$$P_N(y) \sim e^{-R(y)N} \tag{2.4}$$

where the rate function $R(y) > 0$ for $y \neq f$, and $R(f) = 0$. This large deviation property holds in the great majority of physical models; in Appendix 1 we discuss its validity in the relevant case of nearest neighbour Ising spin glass in d dimensions. The large deviations property (2.4) allows one to perform a saddle point estimate of the integral (2.3) which gives the expected result

$$\lim_{N\to\infty} f_N = f . \tag{2.5}$$

The case of the annealed free energy

$$f_a \equiv \lim_{N\to\infty} -\frac{1}{N\beta} \ln(\overline{Z_N}) \tag{2.6}$$

is rather different. Taking into account (2.4), one has

$$\overline{Z_N} = \int dy\ P_N(y)\ e^{-\beta y N} \sim \int dy\ e^{-N(R(y)+\beta y)} . \tag{2.7}$$

In the large N limit this integral can be estimated by the saddle point method so that

$$\beta f_a = \min_y [R(y) + \beta y] = R(y^*) + \beta y^* . \tag{2.8}$$

In general y^* differs from the most probable value f and therefore $f_a \neq f$. This equality shows that the main contribution to f_a comes from a set of realizations of the disorder of probability $\sim e^{-R(y^*)N}$ which exponentially vanishes in the thermodynamic limit. This set of realizations can be very different from the most probable set and, therefore, the physical properties of the annealed model can be very different from the quenched one.

In order to improve the annealed estimate (2.8) of the free energy, let us consider intensive disorder variables defined in terms of the microscopic quenched variables. Variables of this type e.g. are: $\frac{1}{N}\sum_{i,j} J_{ij}$ or $\frac{1}{N}\sum_{i,j} J_{ij}^2$ in Ising models with nearest neighbour random couplings and $\frac{1}{N}\sum_i h_i$ in Ising models with random magnetic fields. As we will see in Sec. 6, one of the most relevant intensive disorder variables is the single plaquette frustration $\frac{1}{N}\sum_p \tilde{J}_p$ where $\tilde{J}_p$ indicates the product of the coupling constants J_{ij} on the four bonds $\{i, j\}$ which form a square plaquette on the d-dimensional hypercubic lattice.

Let us indicate by α_N one of these quantities which are supposed to self-average to $\overline{\alpha}$. In the previous cases, if J_{ij} and h_i are independent identically-distributed (i.i.d.) random variables, in the limit of large N α self-averages to $\overline{J}$ or to $\overline{h}$ because of the large number law. It follows that the joint probability

$$\mathcal{P}_N(y, \alpha) \sim \exp[-\mathcal{R}(y, \alpha)N] \tag{2.9}$$

has a maximum at $y = f$, $\alpha = \overline{\alpha}$ which means that $\mathcal{R}(y, \alpha) > 0$ for $(y, \alpha) \neq (f, \overline{\alpha})$, and $\mathcal{R}(f, \overline{\alpha}) = 0$. Furthermore since $P_N(y) = \int \mathcal{P}_N(y, \alpha)\, d\alpha$ one has from saddle point method

$$R(y) = \min_{\alpha}[\mathcal{R}(y, \alpha)] = \mathcal{R}[y, \alpha(y)]\,. \tag{2.10}$$

This equality obviously implies that $\alpha(f) = \overline{\alpha}$. The introduction of the new variable α has no effects in the integral $f = \lim_{N\to\infty} \int dy\, d\alpha\, \mathcal{P}_N(y, \alpha)\, y$ for the quenched free energy but shows that only the disorder realizations which correspond to $\alpha = \overline{\alpha}$ contribute to f in the thermodynamic limit. The introduction of the new variable also has no effect in the calculation of the integral

$$\int dy\, d\alpha\, \mathcal{P}_N(y, \alpha)\, e^{-N\beta y} \tag{2.11}$$

so that one has

$$\beta f_a = \min_{y,\alpha}[\mathcal{R}(y, \alpha) + \beta y] = \mathcal{R}(y^*, \alpha^*) + \beta y^* \tag{2.12}$$

where $\alpha^* = \alpha(y^*)$, as can be seen by comparison with (2.8) and (2.10). The above expression is the same as (2.8) but explicitly shows that the minimum condition is realized for a free energy $y = y^*$ and for $\alpha = \alpha^*$. These two quantities both differ from the quenched averages f and $\overline{\alpha}$. In other words, the main contribution to f_a comes from disorder realizations of exponentially vanishing probability with a wrong value of α. At this point, it is clear that it would be useful to restrict the

annealed average only on the set of realizations with the correct value of α. This is indeed possible by means of a Lagrange multiplier μ. In the following we take $\overline{\alpha} = 0$, to simplify the notation. In order to average over the disorder configurations with $\alpha = \overline{\alpha} = 0$, we start by computing

$$\overline{Z_N e^{-\mu \alpha_N N}} = \int dy\, d\alpha \mathcal{P}_N(y, \alpha)\, e^{-\beta y N}\, e^{-\mu \alpha N} \sim e^{-g(\beta, \mu) N} \tag{2.13}$$

where we have introduced the new Gibbs-like thermodynamic potential

$$g(\beta, \mu) \equiv \lim_{N \to \infty} (-\frac{1}{N} \ln \overline{Z_N e^{-\mu \alpha_N N}}) = \min_{y, \alpha} [\mathcal{R}(y, \alpha) + \beta y + \mu \alpha]\,. \tag{2.14}$$

At given μ and β, the value of the variable α_N which minimizes (2.14) can be expressed as a function of the Lagrange multiplier μ, that is $\alpha(\mu) = -\frac{dg}{d\mu}$. Therefore, the value $\tilde{\mu}$ of the Lagrange multiplier which fixes $\alpha = \overline{\alpha} = 0$ is given by the condition

$$\left. \frac{\partial g(\beta, \mu)}{\partial \mu} \right|_{\tilde{\mu}} = 0\,. \tag{2.15}$$

The relation (2.15) simply says that the constrained free energy

$$f_{ac} = \frac{1}{\beta} g(\beta, \tilde{\mu}) \tag{2.16}$$

is obtained by computing the maximum of the Gibbs potential (2.14). As the minimum of α in (2.14) for $\mu = \tilde{\mu}$ is reached at $\alpha = 0$, one has the constrained annealed free energy f_{ac}

$$\beta f_{ac} = g(\beta, \tilde{\mu}) = \min_{y} [\mathcal{R}(y, 0) + \beta y] = \mathcal{R}(\tilde{y}, 0) + \beta \tilde{y} \tag{2.17}$$

which should be compared with (2.12) where $\alpha = \alpha^*$ is a free parameter different from its self-average $\overline{\alpha} = 0$. Furthermore, by the convexity of the logarithm function, one can prove the inequality

$$f \geq f_{ac} \geq f_a \equiv \frac{1}{\beta} g(\beta, \mu = 0) \tag{2.18}$$

showing that f_{ac} is a better approximation of f than the annealed free energy f_a. In fact, $f_{ac} - f_a$ has an important physical meaning. When αN is considered as the sum of hot variables, the system minimizes the annealed free energy $f_a = \frac{1}{\beta}\, g(\beta, 0)$ while when α is taken as a cold variable, the system cannot arrange itself and it should minimize $f_{ac} = \frac{1}{\beta} g(\beta, \tilde{\mu})$. In other words, $W \equiv f_{ac} - f_a = \int_0^{\tilde{\mu}} \alpha(\mu)\, d\mu$ is the work needed to freeze the variable α. This consideration indicates that a macroscopic variable can be of two types:

(1) $W = 0$. In the thermodynamic limit the variable is not relevant since no work is needed to freeze it. It follows that $\tilde{\mu} = 0$ and $f_{ac} = f_a$.

(2) $W > 0$. One obtains a better approximation of the quenched free energy f by computing f_{ac} instead of f_a, since $\tilde{\mu} \neq 0$ and $g(\beta, \tilde{\mu})$ is strictly larger than $g(\beta, 0)$. Our arguments can be trivially extended to a set of M relevant variables $\{\alpha_i\}$. Such a set will be complete only if $g(\beta, \tilde{\mu}_1, \cdots \tilde{\mu}_M) = \beta f$ and it might be rather difficult to individuate it.

In conclusion our method works as follows. First compute

$$g(\beta, \mu) = \lim_{N \to \infty} \left(-\frac{1}{N} \ln \overline{Z_N e^{-\mu \alpha_N N}} \right) \tag{2.19}$$

then obtain f_{ac} as

$$f_{ac} = \frac{1}{\beta} \max_{\mu} g(\beta, \mu) . \tag{2.20}$$

2.2. *An alternative picture of the method*

The theory of large deviation allows us to give an alternative way of looking at constrained annealing, which leads to a deeper insight on the method of Lagrange multipliers. Let us introduce a class of random variables y_n^μ defined as

$$y_N^\mu \equiv y_N + \frac{\mu}{\beta} \alpha_N = -\frac{1}{N\beta} \ln(Z_N e^{-\mu \alpha_N N}) \tag{2.21}$$

where y_N is the ordinary finite volume free energy and α_N is one of the intensive disorder variables before introduction. Since α_N self-averages to zero, the variables y_N^μ self-average to the quenched free energy density f, independently of μ. The averaged $\overline{y_N^\mu}$ also converges to f. By the convexity of the logarithm one has the inequality

$$\overline{y_N^\mu} \geq -\frac{1}{N\beta} \ln(\overline{Z_N e^{-\mu \alpha_N N}}) = \frac{1}{\beta} g(\beta, \mu) \tag{2.22}$$

which holds for any μ, so that in the thermodynamic limit

$$f \geq \frac{1}{\beta} \max_{\mu} [g(\beta, \mu)] \equiv f_{ac} \geq \frac{1}{\beta} g(\beta, 0) \equiv f_a . \tag{2.23}$$

We thus have shortly re-obtained the fundamental inequalities (2.18).

The physical meaning of this alternative picture is that a disordered system can be described by means of the class of random variables y_N^μ. These variables equal the ordinary free energy in the thermodynamic limit, but have different finite volume fluctuations. Inside this class, one chooses the less fluctuating variable corresponding to $\tilde{\mu}$ for which the difference between the free energy and its annealed approximation is minimal.

For any variable y_N^μ, it is possible to consider the associated probability density

$$P_N^\mu(y) \approx e^{-R^\mu(y) N} \tag{2.24}$$

and, using this large deviation property, the Gibbs potential $g(\beta, \mu)$ is obtained as

$$g(\beta, \mu) = \min_y [R^\mu(y) + \beta y] . \tag{2.25}$$

A comparison with (2.14) allows one to see that

$$R^\mu(y) = \min_\alpha [\mathcal{R}(y, \alpha) + \mu\alpha] . \tag{2.26}$$

In Appendix 2 we discuss the large deviations property (2.24) for the relevant case of nearest neighbour Ising spin glasses. Our conclusions, by virtue of (2.26), also applie to the large deviation property (2.9).

To illustrate our method, in the following we study two exactly solvable models where the introduction of a single variable α_N allows us to recover the exact free energy.

2.3. *Two exactly solvable examples*

Let us start with a one-dimensional nearest neighbour model without magnetic field, whose Hamiltonian is

$$H = \sum_i J_i \sigma_i \sigma_{i+1} \tag{2.27}$$

with independent random couplings which assume two possible values, say $J_i = 1$ or $J_i = 2$ with equal probability. The finite volume partition function is

$$Z_N = \sum_{\{\sigma_i\}} \exp(-\beta H) = 2^N \prod_i \cosh(\beta J_i) . \tag{2.28}$$

The quenched free energy f of this model is thus exactly computable:

$$f = \lim_{N\to\infty} -\frac{1}{N\beta} \overline{\ln Z_N} = -\frac{1}{2\beta} (\ln(\cosh\beta) + \ln(\cosh 2\beta)) . \tag{2.29}$$

The annealed free energy can also be easily computed:

$$f_a = \lim_{N\to\infty} -\frac{1}{N\beta} \ln \overline{Z_N} = -\frac{1}{\beta} \ln \left(\frac{\cosh\beta + \cosh 2\beta}{2} \right) , \tag{2.30}$$

which is smaller than f. In order to compute the constrained free energy we introduce the intensive disorder variable

$$\alpha_N = \frac{1}{N} \sum_i J_i - \frac{3}{2} \tag{2.31}$$

where 3/2 is subtracted in order to have a vanishing average variable ($\overline{\alpha}_N = 0$). Since the couplings are independent, the large number law ensures that α_N self-averages to zero and gets in the thermodynamic limit ($\alpha_N \to 0$ with probability one). Let us remark that α_N gives the relative concentration of the two types of couplings. In fact, $\alpha_N = n_1 + 2n_2 - 3/2$ where n_1 is the concentration of couplings

of magnitude one and $n_2 = 1 - n_1$ is the concentration of couplings of magnitude two.
Using α_N we can compute

$$g(\beta, \mu) = \lim_{N\to\infty} -\frac{1}{N} \ln \overline{Z_N e^{-\mu\alpha_N N}} = -\ln\left(\frac{e^{\mu/2}\cosh\beta + e^{-\mu/2}\cosh 2\beta}{2}\right) . \quad (2.32)$$

The maximization with respect to μ gives

$$f_{ac} = -\frac{1}{\beta} \max_{\mu} g(\beta, \mu) = -\frac{1}{2\beta}(\ln(\cosh\beta) + \ln(\cosh 2\beta)) = f . \quad (2.33)$$

We thus see that fixing the correct value of α, one obtains the quenched free energy by the constrained annealed average. This means that the only relevant disorder variable is the relative concentration of bonds. The ordering of the couplings is thus irrelevant for what concerns this model.

Let us now consider, as second example, the infinite range Random Field Ising Model with Hamiltonian

$$H = \sum_{i>j} J\sigma_i\sigma_j + \sum_i h_i\sigma_i \quad (2.34)$$

where J is a constant coupling and the magnetic fields h_i are i.i.d. random variables with zero mean value. Also in this model $\ln\overline{Z} \neq \overline{\ln Z}$, indicating the presence of at least one relevant variable α_N. Let us limit ourselves to consider the disorder realizations of $\{h_i\}$ which satisfy the law of the large numbers, that is

$$\alpha_N = \frac{1}{N}\sum_i^N h_i = 0 . \quad (2.35)$$

We can thus compute the annealed average over the h-distribution

$$\overline{Ze^{-\mu\alpha_N N}} = \overline{\sum_{\{\sigma_i=\pm 1\}} \exp\left[\frac{1}{N}\sum_{i>j} J\sigma_i\sigma_j + \sum_i h_i(\sigma_i + \mu)\right]} \quad (2.36)$$

where $\beta = 1$ is chosen for simplicity. Using the independence of the random variables h_i, the average over the disorder can be performed:

$$\sum_{\{\sigma_i=\pm 1\}} \exp\left[\frac{1}{N}\sum_{i>j} J\sigma_i\sigma_j\right] \overline{\exp[\sum_i h_i(\sigma_i + \mu)]} . \quad (2.37)$$

To be explicit, we consider the binomial distribution $h_i = \pm h$ with respective weights $1/2$, so that (2.37) takes the form

$$\sum_{\{\sigma_i=\pm 1\}} \exp\left[\frac{1}{N}\sum_{i>j} J\sigma_i\sigma_j\right] \prod_i^N \cosh[h(\sigma_i + \mu)] . \quad (2.38)$$

One should note the identity $\cosh[h(\sigma_i+\mu)] = \exp(A(\mu)+B(\mu)\sigma_i)$, where A and B are solutions of the equations $\cosh[h(1+\mu)] = \exp(A(\mu)+B(\mu))$ and $\cosh[h(-1+\mu)] = \exp(A(\mu)-B(\mu))$, obtained by taking $\sigma_i = \pm 1$. After some trivial algebraic manipulations, one thus gets

$$\overline{Ze^{-\mu\alpha N}} = \sum_{\{\sigma_i=\pm 1\}} \exp\left[N\left(A(\mu)+\frac{J}{2}m^2+B(\mu)\,m\right)\right] \tag{2.39}$$

where $m \equiv \sum_i \sigma_i/N$ is the magnetization density. The sum over the thermal configurations can be transformed into an integral over the magnetization, that is,

$$\int P(m)dm \exp\left[N(A(\mu)+\frac{J}{2}m^2+B(\mu)m)\right] \tag{2.40}$$

where $P(m)$ is given by the binomial factor and can be approximated for large N by the Stirling formula as $P(m) \sim \exp[-s(m)N]$ with

$$s(m) = \frac{(1-m)}{2}\ln\frac{(1-m)}{2} + \frac{(1+m)}{2}\ln\frac{(1+m)}{2}. \tag{2.41}$$

The usual saddle point estimation of (2.40) gives

$$g(\beta=1,\mu) = A(\mu)+\frac{J}{2}(m^*)^2 + B(\mu)\,m^* - s(m^*) \tag{2.42}$$

where m^* is the value of m maximizing the exponent in the integral. The exact solution for f (see Ref. 1) coincides with $g(\beta, \tilde{\mu})$ where $dg/d\mu|_{\tilde{\mu}} = 0$. Our result shows that $\sum_i h_i$ is the only new relevant thermodynamic variable in this mean field model.

2.4. *One more example*

Let us now describe a less trivial model where one constraint is not sufficient to get the quenched free energy by the Gibbs potential. It is the one-dimensional Ising model with Hamiltonian

$$H = \sum_i J\sigma_i\sigma_{i+1} + \sum_i h_i\sigma_i \tag{2.43}$$

where J is a positive coupling and $h_i = a + b\,x_i$ is a random field with a, b arbitrary constants and x_i i.i.d. random variables identically distributed according a standard Gaussian. One can repeat the previous calculation for the Gibbs potential introducing the variable $\alpha = \sum x_i/N$ up to obtain

$$e^{-g(\beta=1,\mu)\,N} = \sum_{\{\sigma_i=\pm 1\}} \prod_i^N \exp[J\sigma_i\,\sigma_{i+1} + a\sigma_i]\;\overline{\exp[x_i(b\sigma_i+\mu)]} \tag{2.44}$$

$$= e^{(b^2+\mu^2)N/2} \sum_{\{\sigma_i=\pm 1\}} \prod_i^N \exp[J\sigma_i\,\sigma_{i+1} + (a+b\mu)\sigma_i] \tag{2.45}$$

where the disorder average is performed by an integral over the standard Gaussian $P(x) = (2\pi)^{-1/2} \exp(-x^2/2)$. The resulting thermodynamic potential is given by the solution of a one-dimensional Ising model under constant magnetic field $a + b\mu$. The free energy ($\beta = 1$) of such a model is well-known to be $-\ln\gamma$ where

$$\gamma(\mu) = e^J \cosh(a + b\mu) + [e^{2J} \cosh^2(a + b\mu) - 2\sinh(2J)]^{1/2}, \tag{2.46}$$

so that the Gibbs thermodynamic potential is

$$g(\beta = 1, \mu) = -\frac{b^2 + \mu^2}{2} - \ln\gamma(\mu). \tag{2.47}$$

Although $g(\beta, \tilde{\mu})$ is a much better approximation of the quenched free energy than the annealed free energy $g(\beta, \mu = 0)$, it is still different from f, as shown in Fig. 1. In fact, we can impose a further natural constraint on the system by considering only the disorder realizations that have the correct mean value and variance for the total magnetic field, i.e. the realizations where, besides $\alpha = 0$, one also has $\alpha_2 = \sum_i^N (x_i^2 - 1)/N = 0$. To do it, we introduce a second Lagrange multiplier μ_2 related to the new variable α_2, as well as the generalized thermodynamic potential $g_2(\beta, \mu, \mu_2)$ defined by the relation

$$e^{-g_2(\beta=1,\mu,\mu_2)N} = \sum_{\{\sigma_i=\pm1\}} \prod_i^N \exp[J\sigma_i\sigma_{i+1} + a\sigma_i]\overline{\exp[x_i(b\sigma_i + \mu) + \mu_2(x_i^2 - 1)]}. \tag{2.48}$$

After some trivial Gaussian integrations, the problem is again reduced to the solution of an appropriate one-dimensional Ising model without disorder. One thus finds

$$g_2(\beta = 1, \mu, \mu_2) = \mu_2 - \frac{b^2 + \mu^2}{2(1 - 2\mu_2)} + \frac{1}{2}\ln(1 - 2\mu_2) - \ln\gamma(\mu, \mu_2) \tag{2.49}$$

where $\gamma(\mu, \mu_2)$ is obtained from (2.46) substituting the magnetic field $a + b\mu$ with the new magnetic field $H = a + b\,\mu/(1 - 2\mu_2)$. The potential g_2 becomes equal to g for $\mu_2 = 0$. The maxima of g_2 are reached at $\mu = \tilde{\mu}_1$, $\mu_2 = \tilde{\mu}_2$ given by the relations

$$\left.\frac{\partial g_2}{\partial \mu}\right|_{\tilde{\mu}_1,\tilde{\mu}_2} = \left.\frac{\partial g_2}{\partial \mu_2}\right|_{\tilde{\mu}_1,\tilde{\mu}_2} = 0. \tag{2.50}$$

Figure 1 shows the potentials $-g(\beta = 1, \mu)$ and $-G(\Lambda) = \min_\mu -\tilde{g}_2(\beta = 1, \mu, \Lambda)$ with $\tilde{g}_2(\beta = 1, \mu, \Lambda(\mu, \mu_2)) = g_2(\beta = 1, \mu, \mu_2)$ and $\Lambda = \mu/(1 - 2\mu_2)$. Their minima give increasingly better estimates of the quenched free energy. The exact solution of the model is not known. However, two constraints are not sufficient to obtain f by the corresponding generalized potential. In the next section we will consider again the one-dimensional Ising model (with dichotomic disorder) and we will find out what the relevant intensive variables needed to properly describe the model are.

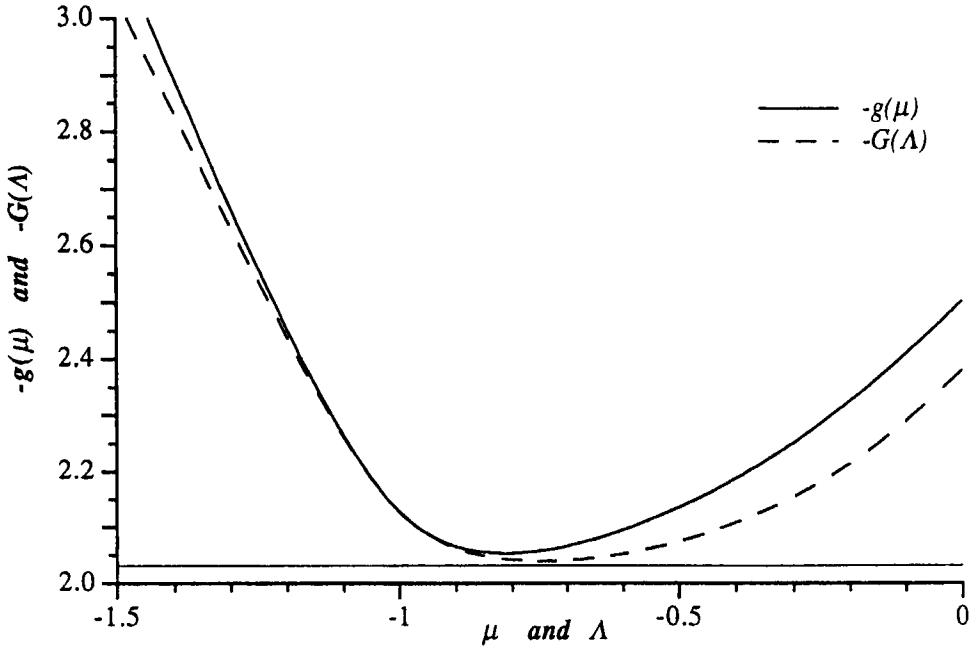

Fig. 1. 1*d* Ising model (2.43) with $J = 1$ and random magnetic field $h_i = a + bx_i$ ($a = 1$, $b = 1$ and x_i normal Gaussian variables). The full line is the Gibbs potential $-g(\beta = 1, \mu)$ and the dashed line is $-G(\Lambda) = \min_\mu -\tilde{g}_2(\beta = 1, \mu, \Lambda)$ with $\tilde{g}_2(\beta = 1, \mu, \Lambda(\mu, \mu_2)) = g_2(\beta = 1, \mu, \mu_2)$ and $\Lambda = \mu/(1 - 2\mu_2)$. The straight line indicates the numerical result for the quenched free energy $f = 2.031 \pm 0.001$. Our approximations are obtained by taking the minima of the thermodynamic potentials, that is $-g(\beta = 1, \tilde{\mu} = -0.81) = 2.0530$ and $-G(\tilde{\Lambda} = -0.75) = 2.0389$ (corresponding to the values $\tilde{\mu}_1 = -0.489$, $\tilde{\mu}_2 = 0.198$). The annealed free energy is obtained by $-g(\beta = 1, \mu = 0) = 2.5029$.

3. One-Dimensional Ising Spin Glass

3.1. *One constraint*

Here, we study the one-dimensional Ising model with a random field and Hamiltonian

$$H = -\sum_i J\sigma_i\sigma_{i+1} - \sum_i h_i\sigma_i \tag{3.1}$$

where $J = 1$ is the positive coupling and $h_i = H + h\eta_i$ is a random field ($H > 0$, $h > 0$) and η_i are independent random variables which assume the values $\eta_i = \pm 1$ with equal probability.

It is well-known that the quenched free energy can be written as a product of transfer random matrices[7]

$$f = -\lim_{N\to\infty} \frac{1}{\beta N} \overline{\ln \operatorname{Tr} \prod_{i=1}^{N} \mathbf{A}_{\eta_i}} \tag{3.2}$$

where the average is over the η_i distribution and

$$\mathbf{A}_{\eta_i} = \begin{pmatrix} e^{\beta(J+H+h\eta_i)} & e^{\beta(-J+H+h\eta_i)} \\ e^{\beta(-J-H-h\eta_i)} & e^{\beta(J-H-h\eta_i)} \end{pmatrix} . \tag{3.3}$$

The quenched free energy is thus given by $f = -\lambda/\beta$ where λ is the maximum Lyapunov exponent of the matrix product. The multiplicative ergodic theorem[18] assures that the thermodynamic limit (3.2) exists and is unique for almost all η_i realizations (a part a set of zero probability measure).

The method of constrained annealed averages allows one to obtain the Gibbs potential

$$g(\beta, \mu) = -\frac{1}{N} \ln \overline{\left[e^{-\mu \alpha_N N} \mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_{\eta_i} \right]} \tag{3.4}$$

where in this section $\mu = \{\mu_1, \cdots \mu_M\}$ indicates M Lagrange multipliers which fix M self-averaging quantities $\alpha = \{\alpha_1, \cdots \alpha_M\}$ to their mean values $\overline{\alpha} = 0$. The maximum with respect to all components of μ gives the constrained annealed estimate of the free energy.

In our case, the simplest quantity to be considered is

$$\alpha_1 = \frac{1}{N} \sum_{i=1}^{N} \eta_i \tag{3.5}$$

with $\overline{\alpha_1} = 0$. This constraint is derived by the large number law. In this case the variable μ has a single component and the maximization of $g(\beta, \mu)$ leads to the estimate

$$\beta f_{ac}^{(1)} = \max_{\mu_1} [g(\beta, \mu_1)] \tag{3.6}$$

with

$$g(\beta, \mu_1) = -\ln \left[\text{max eigenvalue } \frac{1}{2} (\mathbf{A}_1 e^{-\mu_1} + \mathbf{A}_{-1} e^{\mu_1}) \right] . \tag{3.7}$$

The previous expression is obtained by a calculation which is identical to an annealed average without constraints, so that, after averaging over the disorder, one obtains a transfer matrix corresponding to a particular one-dimensional non-random system. Its eigenvalue is given by the secular equation which requires the solution of a second order polynomial, while the minimization over μ_1 corresponds to the solution of a 4th order polynomial. In the case of average magnetic field $H = 0$, it is possible to show that the value of the multiplier that realizes the minimum is ${\mu_1}^* = 0$. This means that imposing $\alpha_1 = 0$ does not require any thermodynamic work, and there is no difference between annealed averages with or without constraint, as a consequence of the invariance, at $H = 0$, under a simultaneous inversion of all the fields and spins.

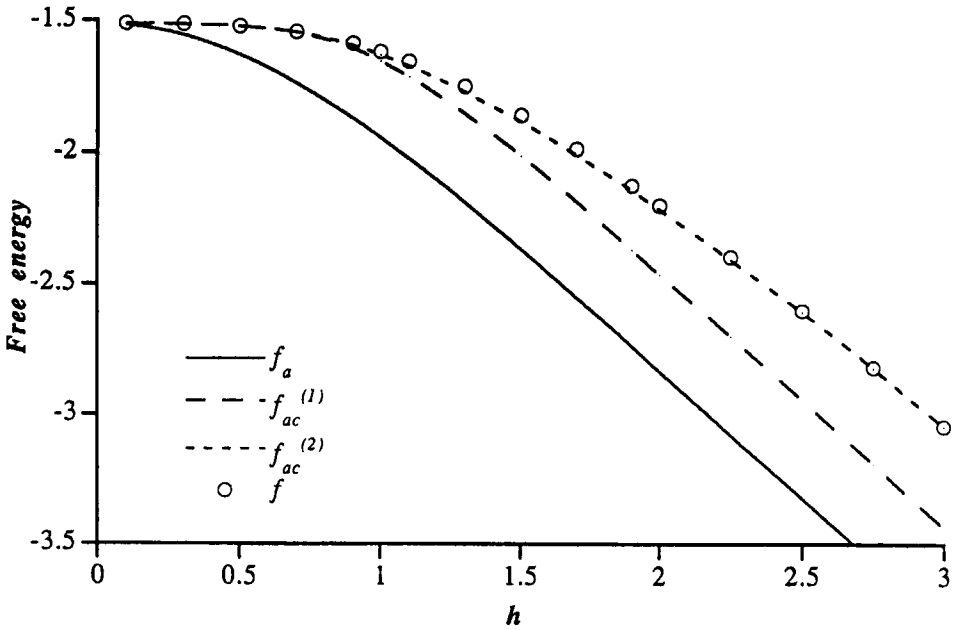

Fig. 2. 1*d* Ising model (3.1): annealed averages without constraints f_a (full line), and with constraints $f_{ac}^{(1)}$ (dashed line), $f_{ac}^{(2)}$ (dotted line) compared with the numerical results for the quenched free energy f (circles) for $\beta = 1$ and $H = 0.5$ as a function of h.

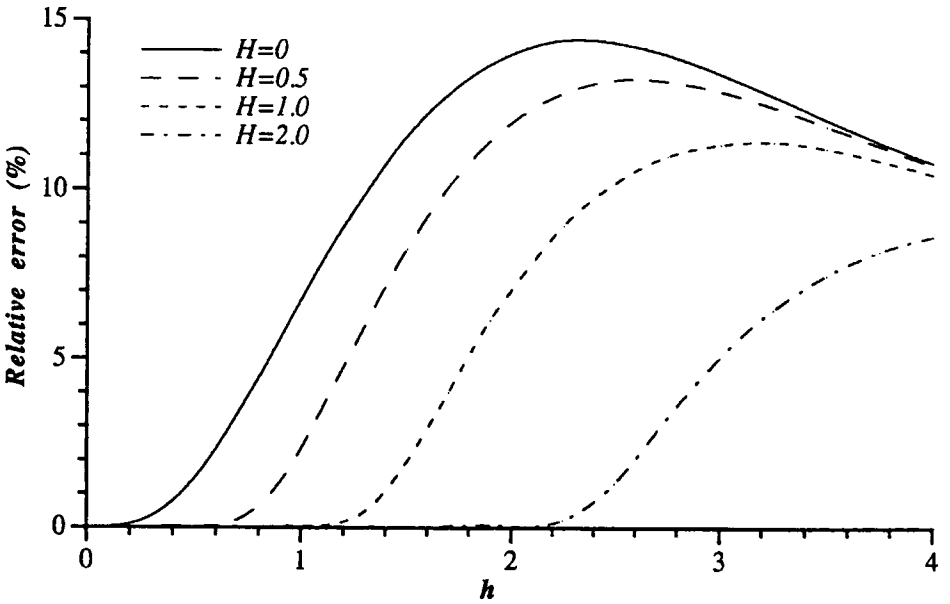

Fig. 3. 1*d* Ising model (3.1) with $\beta = 1$: relative difference between $f_{ac}^{(1)}$ and the numerical value of the quenched free energy f, as function of H and h.

The annealed average without constraints $f_a \equiv g(\beta, \mu_1 = 0)/\beta$ can be very different from f while $f_{ac}^{(1)}$ is in general a reasonable estimate, as shown in Fig. 2. In Fig. 3, we show the relative error of $f_{ac}^{(1)}$ with respect to the numerical value of f obtained by a numerical simulation, at varying H and h with inverse temperature

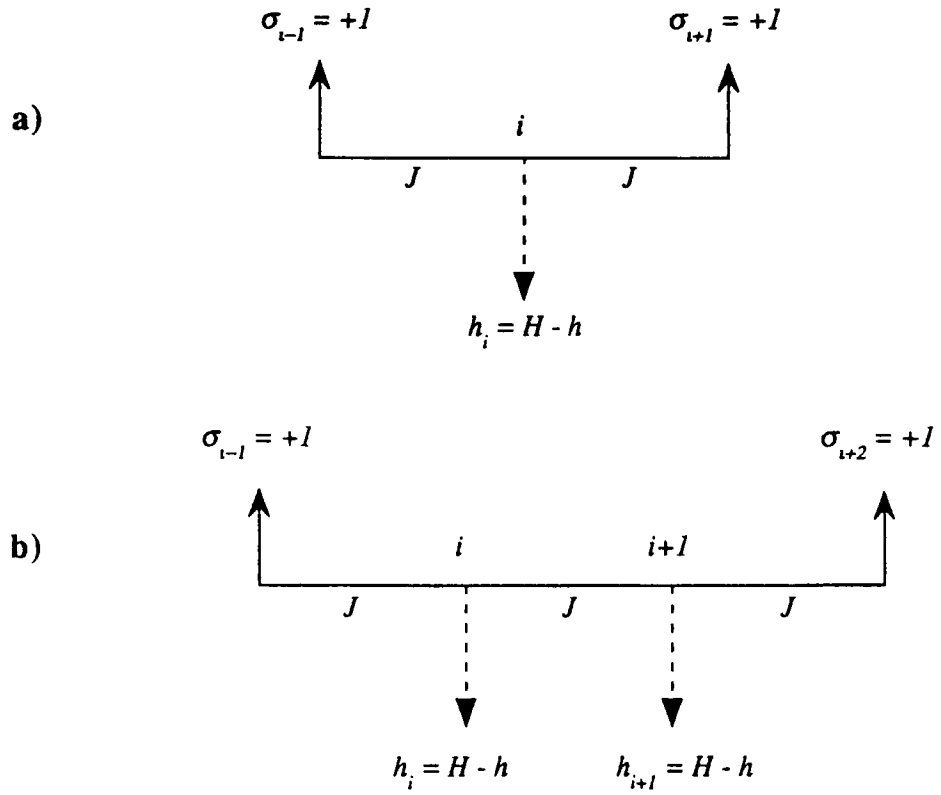

Fig. 4. 1d Ising model (3.1): illustration of a frustrated spin configuration in : (a) an elementary plaquette of one site; (b) a plaquette of two sites. The spins are the arrows; the dashed vertical line represents the random field $H - h$; the solid line represents the positive J coupling.

$\beta = 1$. The error is maximum when the parameters satisfy the relation $h = H + 2J$. This condition is that which makes frustrated a spin in a site which roughly speaking represents the elementary plaquette of the model. It also leads to a non-zero entropy at $T = 0$.[8] Indeed, consider the case of a site i with random field $h_i = H - h$ (that is $\eta_i = -1$) and with two spins up in the 2 neighbour sites, see Fig. 4(a). The energy of the configuration $\sigma_i = +1$ is $E_+ = -2J - H + h$ while if $\sigma_i = -1$ the energy is $E_- = -E_+$. The two configurations are degenerate when $E_+ = E_- = 0$. However, the constrained annealed system can still move its random fields to minimize free energy though the number of sites where $\eta_i = 1$ is fixed to be $N/2$. It follows that it does not exhibit the local frustration of the quenched system since it can separate itself into two parts (e.g the first $N/2$ sites with $\eta_i = +1$ followed by a sequence of $N/2$ variables $\eta_i = -1$).

3.2. *More constraints*

In this situation, if the temperature is not too large, the constraint $\alpha_1 \equiv \frac{1}{N}\sum \eta_i = 0$ is not sufficient to obtain a good qualitative description of the quenched system. In fact, we must impose a further constraint on the system that does not permit the formation of these islands of random variables of the same sign. Such islands lead to a predominance of $\eta_i\eta_{i+1} = 1$ with respect to $\eta_i\eta_{i+1} = -1$ while, in a

typical realization of the quenched system, the law of large numbers implies that $\alpha_2 \equiv \frac{1}{N}\sum_i \eta_i\eta_{i+1} = 0$. In terms of the product of matrices, imposing the constraints $\alpha_1 = 0$ and $\alpha_2 = 0$ is equivalent to considering only sequences where the number of every possible couple of consecutive matrices appears with a frequency which, in the thermodynamic limit, is equal to the probability. In our case each type of couple has probability 1/4. The estimate of the free energy with two constraints is

$$\beta f_{ac}^{(2)} = \max_{\mu_1,\mu_2}[g(\beta, \mu_1, \mu_2)] \tag{3.8}$$

where

$$g(\beta, \mu_1, \mu_2) = -\frac{1}{N}\ln\left[\overline{e^{-(\mu_1\alpha_1+\mu_2\alpha_2)N}\mathrm{Tr}\prod_{i=1}^{N}\mathbf{A}_{\eta_i}}\right] \leq L_1\,. \tag{3.9}$$

As in the case of one constraint (see Appendix 3) g is given by minus the logarithm of the maximum eigenvalue of the 4×4 transfer matrix

$$\mathbf{G}_2(\mu_1,\mu_2) = \frac{1}{2}\begin{pmatrix} \mathbf{A}_1 e^{-\mu_1-\mu_2} & \mathbf{A}_1 e^{-\mu_1+\mu_2} \\ \mathbf{A}_{-1} e^{\mu_1+\mu_2} & \mathbf{A}_{-1} e^{\mu_1-\mu_2} \end{pmatrix}. \tag{3.10}$$

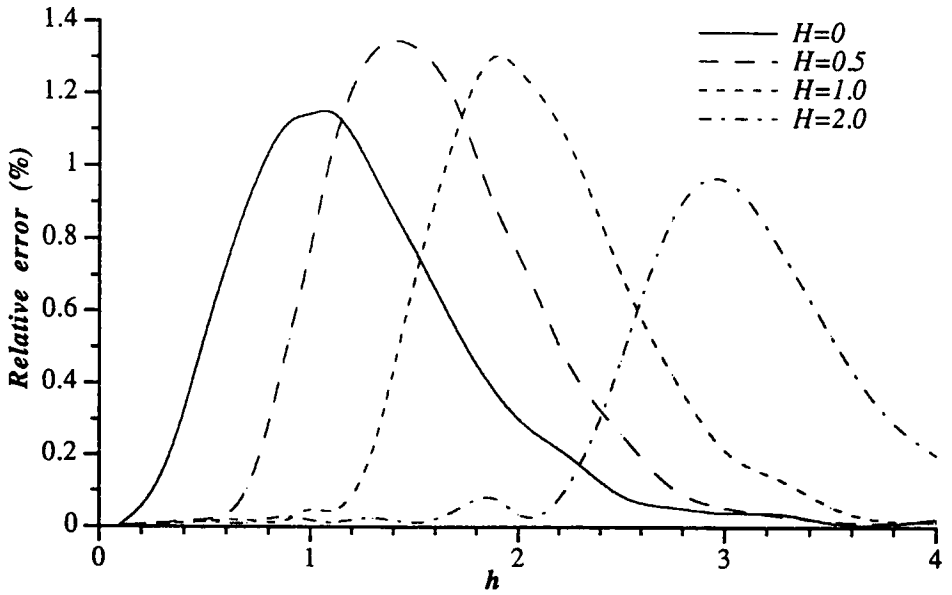

Fig. 5. 1*d* Ising model (3.1) with $\beta = 1$: relative difference between $f_{ac}^{(2)}$ and the numerical value of the quenched free energy f, as a function of H and h.

In Fig. 2 one sees the new constraint $\alpha_2 = 0$ permits us to obtain a very accurate estimate of the free energy f. Figure 5 gives the relative error of $f_{ac}^{(2)}$ with respect to free energy f as a function of H and h: it is always below 1.5 per cent. With two constraints, the error is maximum when the parameters verify the relation

$h = H + J$. As before, this condition is related to a frustrated situation, involving now the spins on a non-elementary plaquette of two sites [see Fig. 4(b)]. Indeed, consider two random fields $h_i = h_{i+1} = H - h$ and two spins up on the neighbour sites. If $h = H + J$, there exist two energetically equivalent configurations where the two spins of the plaquette $(i, i+1)$ have the same direction (up or down). The annealed system with constraints $\alpha_1 = 0$ and $\alpha_2 = 0$ can avoid this frustrated situation, selecting configurations of islands of size $N/4$: two of them with fields, respectively equal to $H + h$ and $H - h$, and the other two islands with variables η_i which have alternated sign.

Our constraints can be easily generalized to avoid corresponding annealed systems from choosing particular disorder realizations of equilibrium which ignore frustration over plaquettes of larger and larger sizes. For plaquettes of size $n - 1$, this goal can be achieved by imposing that in the product of random matrices (3.4), each type of n-ple of consecutive matrices appears with the 'right' frequency (in the thermodynamic sense), e.g. among the triples $\mathbf{A}_1\,\mathbf{A}_1\,\mathbf{A}_1$ has to appear $N/8$ times when $N \to \infty$.

In terms of the η's, we have to perform annealed averages with constraints on all the 2^{n-1} quantities of the type

$$\frac{1}{N}\sum_{i=1}^{N}\eta_i\eta_k\cdots\eta_s \tag{3.11}$$

with $i < k < \cdots < s < i + n$. For example for $n = 3$, we have to impose the correct frequencies for $\mathbf{A}_1$ (and $\mathbf{A}_{-1}$) by the multiplier μ_1, for the couples of matrices by μ_2 and for the term of matrices by two new Lagrange multipliers μ_3 and μ_4 related to

$$\frac{1}{N}\sum_{i=1}^{N}\eta_i\,\eta_{i+1}\,\eta_{i+2} \qquad \text{and} \qquad \frac{1}{N}\sum_{i=1}^{N}\eta_i\,\eta_{i+2}\,. \tag{3.12}$$

In practice, the problem can be reduced to the determination of the largest eigenvalue of a matrix $\mathbf{G}_n$ of size $2^n \times 2^n$, as proved in Appendix 3. The upper bound $f_{ac}^{(n)}$ is then given by a maximization on μ and obviously

$$f \geq f_{ac}^{(k)} \geq f_{ac}^{(k')} \qquad \text{for } k > k'\,. \tag{3.13}$$

The computation of $f_{ac}^{(3)}$ for $H = 0.5$ is shown in Fig. 6, where one sees that there is a considerable improvement over the estimate $f_{ac}^{(2)}$. However the frustration on plaquettes larger than $n = 2$ still produces a difference between $f_{ac}^{(3)}$ and λ which attains its maximum (0.2 per cent) when $h \sim 1$.

For a vanishing magnetic field $H = 0$, as a generalization of what was discussed for $n = 1$, it is possible to prove by symmetry arguments that the constraints on products of an odd number of η's are irrelevant so that the corresponding Lagrange multipliers are zero.

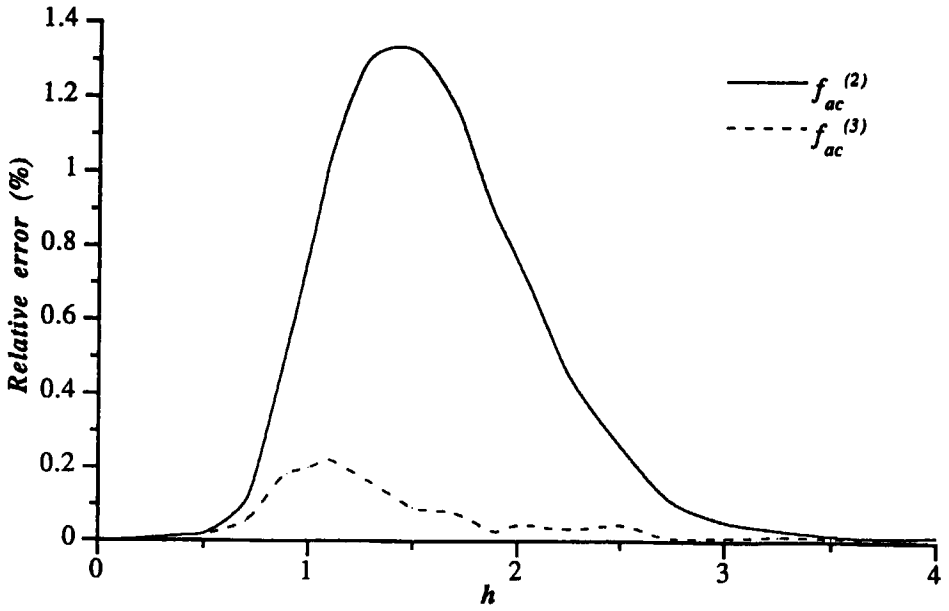

Fig. 6. 1*d* Ising model (3.1) with $H = 0.5$ and $\beta = 1$: relative difference between $f_{ac}^{(3)}$ and the numerical value of the quenched free energy f (dashed line). For comparison, the full line indicates the relative difference between $f_{ac}^{(2)}$ and f. The relative error is maximum (0.2 per cent) at $h \sim 1$.

More importantly, the constraints on the n-ples of matrices that we have considered are very general in a wide class of functions. To be explicit, let us consider the case of the couples. In this case, a pair of constraints on two independent sufficiently regular functions of two variables r and s,

$$\frac{1}{N}\sum_{i=1}^{N} r(\eta_i, \eta_{i+1}) , \qquad \frac{1}{N}\sum_{i=1}^{N} s(\eta_i, \eta_{i+1}) \tag{3.14}$$

are equivalent to impose $\alpha_1 = 0$ and $\alpha_2 = 0$, i.e. one exactly has the same disorder realizations in the two ensembles of full η probability measure selected by the different constraints. The generalization to n-ples with $n > 2$ is immediate and this equivalence provides a strong indication that our results are quite generic. We thus expect that the sequence of estimates f_a, $f_{ac}^{(1)} \cdots f_{ac}^{(n)}$ converges to f. This conjecture is based on the physical argument that when $n \to \infty$ the quenched system has the same degree of frustration of the constrained annealed system since the former as well as the latter can be regarded as the superposition of frustrated plaquettes of different sizes $k \leq n$. The convergence of $f_{ac}^{(n)}$ toward the free energy seems to be extremely fast, probably exponential in n, at least for non-zero temperature. For instance, in the random field Ising model at inverse temperature $\beta = 1$, the maximum relative error of f_a is 30 per cent, of $f_{ac}^{(1)}$ is 10 per cent, of $f_{ac}^{(2)}$ is 1.5 per cent and of $f_{ac}^{(3)}$ is 0.2 per cent.

3.3. *Other thermodynamical quantities*

The method of constrained annealed averages is also able to give good estimates of other thermodynamical quantities, such as the quenched magnetization, defined as

$$m = \lim_{N\to\infty} \overline{\left\langle\!\left\langle \frac{1}{N}\sum_{i=1}^{N}\sigma_i \right\rangle\!\right\rangle} \tag{3.15}$$

where $\langle\!\langle \cdots \rangle\!\rangle$ represents the thermodynamic average with respect to the Gibbs measure. The magnetization m is related to the derivative of the free energy,

$$m = -\frac{\partial f}{\partial H} \tag{3.16}$$

so that the difficulty of calculating the magnetization is almost the same as f. It is thus natural to introduce an estimate m_n of the quenched magnetization substituting f with an upper bound $f_{ac}^{(n)}$

$$m_n = -\frac{\partial f_{ac}^{(n)}}{\partial H}\,. \tag{3.17}$$

However, m_n is not an upper bound of m at variance with $f_{ac}^{(n)}$ and f, although we expect that m_n is quickly convergent to m at increasing n.

To be explicit, let us consider the case $n = 2$

$$m_2 = \frac{1}{\beta}\frac{\partial}{\partial H}\left\{ \lim_{N\to\infty}\frac{1}{N}\ln \overline{\mathrm{Tr}\left(\prod_{i=1}^{N}\mathbf{A}_{\eta_i}\right) e^{-N(\mu_1^*\alpha_1+\mu_2^*\alpha_2}} \right\} \tag{3.18}$$

where (μ_1^*, μ_2^*) are the values of the Lagrange multipliers that realize the minimum (3.9). After simple algebraic manipulations, one obtains

$$m_2 = \lim_{N\to\infty} \frac{\overline{\mathrm{Tr}\left(\boldsymbol{\sigma}_z \prod_{i=1}^{N}\mathbf{A}_{\eta_i}\right) e^{-N(\mu_1^*\alpha_1+\mu_2^*\alpha_2}}}{\overline{\mathrm{Tr}\left(\prod_{i=1}^{N}\mathbf{A}_{\eta_i}\right) e^{-N(\mu_1^*\alpha_1+\mu_2^*\alpha_2}}} \tag{3.19}$$

where

$$\boldsymbol{\sigma}_z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.20}$$

is a Pauli matrix. As previously discussed, the denominator in (3.19) can be reduced to the trace of the Nth power of the transfer matrix $\mathbf{G}(\mu_1^*, \mu_2^*)$. It is easy to show that even the numerator can be expressed in a similar way, so that

$$m_2 = \lim_{N\to\infty} \frac{\mathrm{Tr}\,\mathbf{D}\,\mathbf{G}^N(\mu_1^*, \mu_2^*)}{\mathrm{Tr}\,\mathbf{G}^N(\mu_1^*, \mu_2^*)} = \max \text{ eigenvalue } \mathbf{S}^{-1}(\mu_1^*, \mu_2^*)\,\mathbf{D}\,\mathbf{S}(\mu_1^*, \mu_2^*) \tag{3.21}$$

where $\mathbf{D}$ is a 4×4 matrix:

$$\mathbf{D} = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \tag{3.22}$$

and $\mathbf{S}(\mu_1^*, \mu_2^*)$ is the matrix of the similarity transformation which diagonalizes $\mathbf{G}(\mu_1^*, \mu_2^*)$. From the computational point of view, m_2 can be easily evaluated, once $f_{ac}^{(2)}$ is known. In the general case for m_n, $\mathbf{S}(\mu)$ is the $2^n \times 2^n$ matrix of the similarity transformation which diagonalizes the related transfer matrix $\mathbf{G}(\mu)$, and $\mathbf{D}$ is a diagonal matrix with entries equal to ± 1, alternated along the diagonal. In Fig. 7, we compare $m_0, \cdots, m_3$ with the values of the quenched magnetization obtained by deriving with respect to H a polynomial fit of numerical calculation of the free energy, for $h = 2$ and $\beta = 1$, as a function of H.

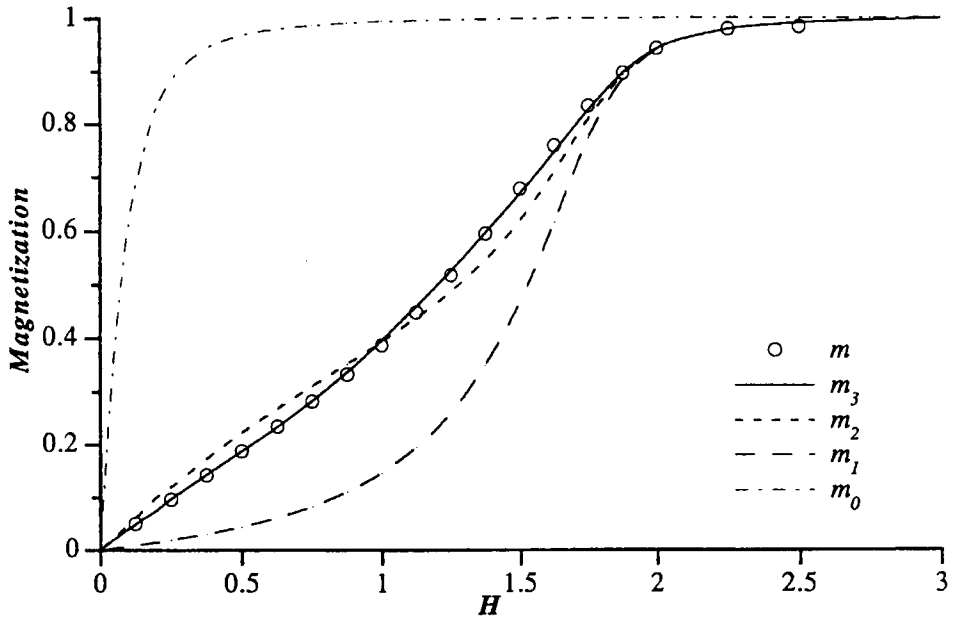

Fig. 7. 1*d* Ising model (3.1) with $\beta = 1$: quenched magnetization m as a function of H, compared with the annealed estimates m_0, m_1, m_2, m_3, at $h = 2$ and $\beta = 1$.

In conclusion, we have estimated the quenched free energy and the magnetization by annealed averages with constraints which prevent the disappearance of frustration. This method is extremely efficient in a one-dimensional random system with short range interactions, where all the interesting physical properties of a spin glass can be reproduced, without knowing the exact solution of the quenched system.

4. The Competing Interaction Model

4.1. *The quenched system*

In the previous section we have seen that the method of constrained annealed averages is able to accurately describe the mean features of a one-dimensional Ising

model with short-range interactions. Indeed our goal is to apply the method to higher dimensional problems where new and richer behaviours appear. As an intermediate step, in this section we consider a one-dimensional Ising model with nearest neighbour (n.n.) random couplings $\pm J$ which has ferromagnetic infinite range interactions. The two limit cases are both trivially solvable: the one-dimensional disordered Ising model by a simple computation and the mean field model by the Curie–Weiss approach. However, the competition of the two types of interactions leads to a rather reach behavior. Beside the paramagnetic and ferromagnetic phases (which are present in the mean field system), and spin glass phase (which is present at zero temperature in the one-dimensional system), this model exhibits a new type of ferrimagnetic order in a region of low temperatures and intermediate disorder strength. It is a non-trivial consequence of the disorder, and could appear in more complicated systems which share the same ingredients of this model.

The Hamiltonian of the model is

$$H = -\sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1} - \frac{\Lambda}{N} \sum_{i>j} \sigma_i \sigma_j \tag{4.1}$$

where Λ is a positive infinite range coupling and J_i are i.i.d. random variables which assume the two values $J_i = \pm J$ with equal probability. The partition function is a random variable depending on the $\{J_i\}$ realization of disorder variables J_i. It can be written as

$$Z_N = \sum_{\sigma_i} \exp\left[\beta \sum_i J_i \sigma_i \sigma_{i+1} + \frac{\beta\Lambda}{2N}\left(\sum_i \sigma_i\right)^2 - \frac{\beta\Lambda}{2}\right]. \tag{4.2}$$

The free energy can be numerically computed through the product of random transfer matrices,[7] since (4.2) can be transformed in an integral over an auxiliary variable Φ. Neglecting the factors which vanish in the thermodynamic limit, it reads

$$Z_N = \int_{-\infty}^{+\infty} d\Phi \, Z_\Phi \, e^{-\frac{1}{2}\Phi^2 N} \tag{4.3}$$

with

$$Z_\Phi = \sum_{\sigma_i} \exp\left(\sum_i \beta J_i \sigma_i \sigma_{i+1} + (\beta\Lambda)^{1/2} \Phi \, \sigma_i\right). \tag{4.4}$$

The quenched free energy is given by the saddle point estimate of the integral (4.3),

$$-\beta f = \max_{\Phi}\left[\lambda(\Phi) - \frac{\Phi^2}{2}\right] \tag{4.5}$$

where

$$\lambda(\Phi) = \lim_{N\to\infty} \frac{1}{N} \ln Z_\Phi = \lim_{N\to\infty} \frac{1}{N} \ln \left\| \prod_{i=1}^{N} \mathbf{T}_i \right\| \tag{4.6}$$

is the maximum Lyapunov exponent of the product of random transfer matrices

$$\mathbf{T}_i = \begin{pmatrix} \exp(\beta J_i + (\beta\Lambda)^{1/2}\Phi) & \exp(-\beta J_i + (\beta\Lambda)^{1/2}\Phi) \\ \exp(-\beta J_i - (\beta\Lambda)^{1/2}\Phi) & \exp(\beta J_i - (\beta\Lambda)^{1/2}\Phi) \end{pmatrix} \tag{4.7}$$

corresponding to the random Ising model (4.4). However, we can describe the main features of the phase diagram without a numerical calculation by some qualitative arguments at zero temperature, and by the analysis of an exactly solvable annealed model, as discussed in this and in the following sections.

4.2. *The ferrimagnetic phases*

Our model obviously has some trivial phases: at high temperature T, the paramagnetic phase of magnetization $m = 0$ and, at low temperature and weak disorder (J small enough), the ferromagnetic phase. On the other hand, when $\Lambda = 0$ and $T = 0$ the system is in a spin glass phase with $m = 0$ where the up or down position of a spin σ_i is determined by the nature (ferro- or antiferromagnetic) of the coupling J_i. This phase is of antiferromagnetic type, since one can define an analogous of the staggered magnetization which is equal to unity at $T = 0$, even if the true magnetization m remains zero. In particular, the overlap

$$q^{\alpha,\xi} \equiv \frac{1}{N}\sum_i \sigma_i^\alpha \sigma_i^\xi \tag{4.8}$$

between two equilibrium states α and ξ is $q = \pm 1$ in both ferromagnetic and glassy phases.

It is natural to expect that a third kind of phase might appear at intermediate values of $\omega \equiv J/\Lambda$ and low temperature with a non-trivial magnetic order, neither ferromagnetic nor glassy. In other terms, as a consequence of the disorder, the ground states can have a magnetization $m \neq 0, \neq 1$. Using a language borrowed from solid state physics, such a phase of a disordered system can be called ferrimagnetic. Notice that in the following, we limit ourselves to consider states with magnetization $m \geq 0$, since the system is invariant for spin inversion, $\sigma_i \to -\sigma_i$ for all i's.

In this section, we show that at $T = 0$, the model exhibits three types of behaviour at varying disorder strengths:

(1) ferromagnetic phase for $0 \leq \omega < 5/12$
(2) ferrimagnetic phases for $5/12 < \omega < 1$
(3) spin glass phase for $\omega > 1$

In particular, there is a first-order phase transition at $\omega_1 = 5/12$ between ferromagnetism ($m_1 = 1$) and ferrimagnetism ($m_2 = 2/3$). Moreover, there exists a

whole spectrum of ferrimagnetic phases with magnetization

$$m_n = \frac{2}{n+1} \qquad n = 2, 3, \cdots, \infty \tag{4.9}$$

for disorder strength:

$$\omega \in]\omega_n, \omega_{n+1}[\qquad \text{with } \omega_n = \frac{n(n+3/2)}{(n+1)(n+2)} \tag{4.10}$$

as shown in Fig. 8. At zero temperature, the calculation is possible because one has to consider only the energy of the different configurations to determine the equilibrium states. The key idea is that the system has a ferrimagnetic state with magnetization $m \neq 1$ which is energetically convenient if the disorder is not too small, as a result of the competition between the antiferromagnetic nearest neighbour coupling $J_i = -J$ and the mean field couplings $\Lambda > 0$.

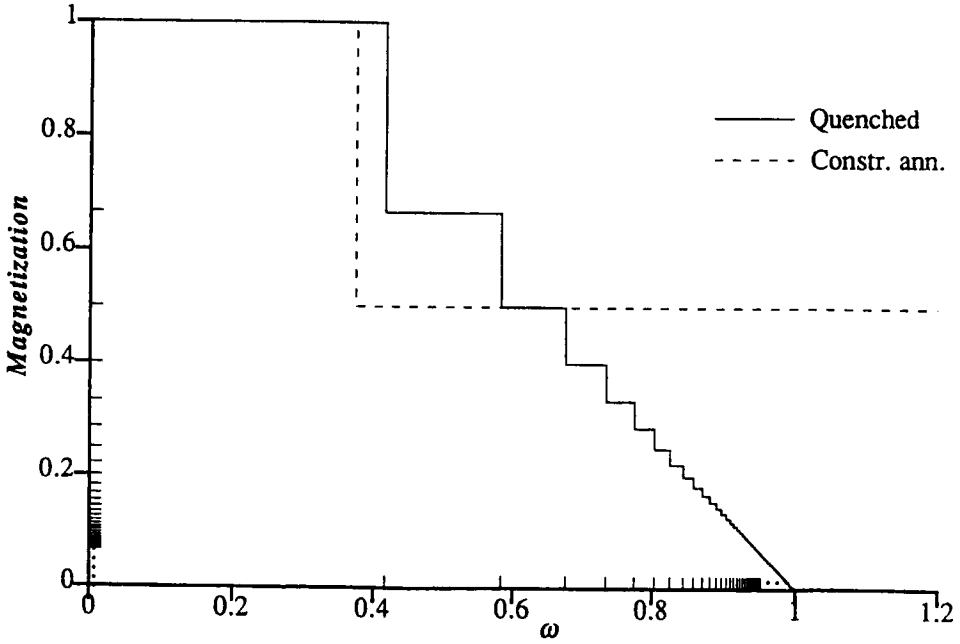

Fig. 8. 1*d* Ising model (3.1): magnetization $m(\omega)$ at zero temperature. The jumps at ω_n given by (3.2) correspond to the first-order phase transitions between the n-phase and the $n+1$-phase. The transition at $\omega_\infty = 1$ toward the spin glass phase is the only continuous one. The dashed line is the constrained annealed approximation.

Here we limit the discussion to the rise of the first ferrimagnetic state m_2 via a phase transition at ω_1. In Ref. 20 the interested reader can find the full proof that an infinity of ferrimagnetic phases exists as anticipated by (4.9) and (4.10).

For the sake of simplicity, suppose that initially $\omega \equiv J/\Lambda$ is close to zero. In this case at $T = 0$ the infinite range interaction prevails and the equilibrium state is given by all spins up. Inserting $\sigma_i = 1$ for all i's in (4.1), we see that the energy

density of the ferromagnetic state is

$$E_F = \lim_{N\to\infty} \frac{1}{N}\left\{-\sum_i J_i - \frac{\Lambda}{2}N\right\} = -\frac{\Lambda}{2} \tag{4.11}$$

since in the thermodynamic limit

$$\lim_{N\to\infty} \frac{1}{N}\sum J_i = 0\,. \tag{4.12}$$

At increasing ω, some of the spins will flip in order to lower the energy due to the n.n. interaction. To illustrate the mechanism, Fig. 9(a) shows a ferromagnetic state, where the first spins to flip are σ_3 and σ_{10} since they have two lateral negative J_i, so that their jumps produce the maximal energy loss. This consideration is general, as the first spins to flip are always the spins with two lateral negative n.n. couplings. However, the situation is more complicated for the island of negative J_i's of size larger than two. For instance, as illustrated in Fig. 9(b), in an island of 3 negative J_i's, only one of the two spins with two negative lateral n.n. couplings flips; in an island of size 4, only two non-consecutive spins flip to lower the energy.

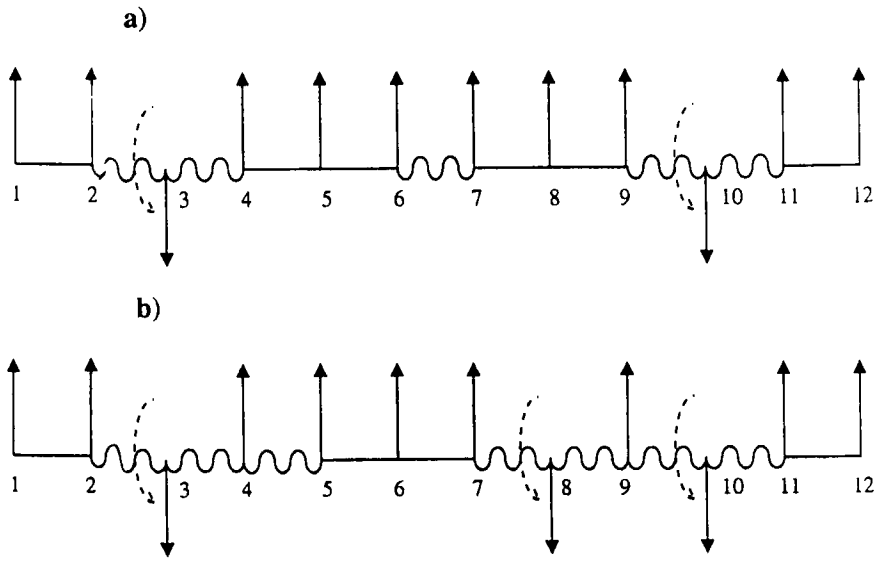

Fig. 9. The competing interactions model (4.1): description of the transition from the ferromagnetic to the first ferrimagnetic state. The full line are the positive n.n. couplings $J_i = +J$ and the wrinkled lines are the negative ones $J_i = -J$. The arrows represent the spins. (a): disorder realization with islands of $k \leq 2$ negative J_i's. At increasing disorder strength ω and for $T = 0$, the first spins to flip are located in the middle of the $k = 2$ islands. (b): disorder realization with some long islands of negative J_i's. The flipping spins are distributed in an alternated way among the islands of negative couplings.

In general, immediately above a critical value ω_1, the antiferromagnetic order will predominate in each island of k negative couplings $J_i = -J$, so that a number $S_k = k/2$ of spins are down if k is even, and a number $S_k = (k-1)/2$ if k is odd. Let us stress that the 'even' island has only one spin configuration of minimal

energy, unlike the 'odd' island which has $(k+1)/2$ equilibrium configurations of minimal energy since one negative n.n. coupling is unsatisfied in the rth site, with odd $1 \leq r \leq k$.

Calling $\mathcal{N}$ the total number of spins which flip in all the islands, the resulting configuration has magnetization

$$m = 1 - 2\frac{\mathcal{N}}{N}. \tag{4.13}$$

It is easy to be convinced that the number of islands of negative n.n. couplings is $N/4 + O(N^{1/2})$. Since the probability that an island is made of k negative bonds is $(1/2)^k$, the number of islands of k n.n. couplings is

$$N_k = (1/2)^k \frac{N}{4}, \tag{4.14}$$

and, if the antiferromagnetic order is present in all the islands, the total fraction of spins down can easily be estimated by

$$\lim_{N\to\infty} \frac{\mathcal{N}}{N} \leq \sum_{k=1}^{\infty} \frac{1}{4} (1/2)^k S_k = 1/6. \tag{4.15}$$

Inserting (4.15) into (4.13), one has $\frac{2}{3} \leq m \leq 1$. At $T = 0$, in order to determine whether a ferrimagnetic state m_2 or the ferromagnetic one $m_1 = 1$ is of equilibrium, we should compare their respective energy densities which are given by

$$E(m) = -4J\frac{(1-m)}{2} - \frac{\Lambda}{2}m^2 \qquad \text{for } 2/3 \leq m \leq 1 \tag{4.16}$$

as shown in Fig. 10. $E(m)$ is a convex function of m (a parabola) so that its minimum is given by one of its two extrema (either at $m = 1$ or at $m = 2/3$) separated by an energy gap $\Delta E \equiv E_{\max} - \max[E(2/3), E(1)]$ where $E_{\max}$ is the maximum of $E(m)$ for $m \in [2/3, 1]$, reached at $m = 2\omega$.

The two states have the same energy at $\omega_1 = 5/12$ where there is a first-order phase transition, since the magnetization has a discontinuous jump from $m_1 = 1$ to $m_2 = 2/3$.

However the first ferrimagnetic phase ($n = 2$) cannot exist when the disorder is too strong. Indeed, its energy is obtained by inserting $m = 2/3$ into (3.8), so that

$$E_{\text{ferri}} = -\frac{2}{3}J - \frac{2}{9}\Lambda. \tag{4.17}$$

On the other hand the spin glass phase with $m = 0$ has energy $E_g = -J$, because all the spins are aligned with the n.n. random couplings. It follows that for $\omega > 2/3$, the first ferrimagnetic phase has higher energy than the spin glass phase. Actually, the first ferrimagnetic phase does not minimize the energy already for $\omega > \omega_2 = 7/12$ (where a second ferrimagnetic phase with $m_3 = 1/2$ appears), while the spin glass phase prevails only for $\omega > 1$. In fact, a numerable infinity of different ferrimagnetic

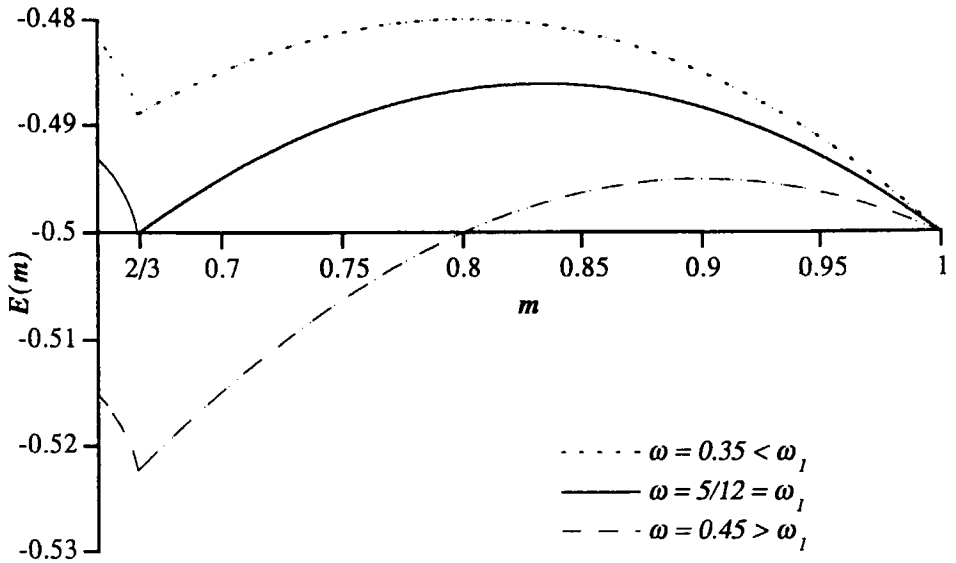

Fig. 10. The competing interactions model (4.1): energy density $E(m)$ as a function of the magnetization m for the quenched model at $\Lambda = 1$ for different values of ω. In the first case ($\omega = 0.4 < \omega_1 = 5/12$)the energy minimum is reached for $m = 1$, in the second case ($\omega = \omega_1$) the minimum is obtained both for $m = 2/3$ and for $m = 1$, and in third case ($\omega = 0.45 > \omega_1$) for $m = 2/3$.

phases are exhibited by the quenched model for $5/12 < \omega < 1$ with magnetization given by (4.9).

This result is obtained by a renormalization procedure which generalizes the argument used for the first ferrimagnetic phase. The transition from an n-phase to the $(n+1)$-phase is of a first-order with an energy gap decreasing with n. The last transition toward the spin glass phase thus is continuous and one has a critical point at $\omega = 1$, $T = 0$.

We can also consider the entropy and the overlap of a ferrimagnetic state at $T = 0$. These two quantities are indicative for the presence of many different minima of the energy as a consequence of the frustration. In the ferromagnetic phase the system is completely ordered and there are only two minima so that the entropy $S(T = 0) = 0$ and the overlap $q(T = 0) = \pm 1$. This is also true for the spin glass phase. On the contrary, in the first ferrimagnetic state one can show that the entropy is

$$S(T = 0) = 0.034\ldots \tag{4.18}$$

and the thermal average of the overlap is

$$\langle q(T = 0) \rangle = \frac{19}{27} - \frac{2}{3} \ln \frac{3}{4} = 0.895\ldots . \tag{4.19}$$

The non-zero value of the entropy and the non-trivial value of the overlap indicate that a ferrimagnetic phase share an important aspect of high dimensional disordered systems, namely the existence of many degenerate ground states. Moreover ferrimagnetism and first-order phase transitions survive at non-zero temperature,

in contrast with the spin glass phase which disappears at $T > 0$, because the system for $\omega > 1$ is basically a one-dimensional system. We thus expect that in the diagram of phase $\tau \equiv T\Lambda$ versus $\omega \equiv J/\Lambda$, there are a set of lines of coexistence of two different ferrimagnetic phases, which end in critical points where the phase transition is of second order. In order to check this conjecture, we study an annealed version of the model with the method of constrained annealed averages.

4.3. *The constrained annealed ferrimagnetic system*

The annealed version of the competing interactions model is a quantitatively poor approximation, and, more importantly, fails to describe even the qualitative aspects of the quenched system. In fact, the annealed version has no ferrimagnetic phase. This is due to the fact that the J_i's are treated as *hot* variables. At low temperature the J_i's are allowed to arrange themselves in a prevalently ferromagnetic configuration.

In order to reproduce the ferrimagnetism in a solvable model, we should consider random couplings J_i which are only partially frozen, in the sense that they must satisfy appropriate constraints although minimizing the (annealed) free energy.

In our case, the minimal requirement is that the system has the right number of negative n.n. couplings in the thermodynamic limit, in order to reproduce the fundamental aspects of the quenched system. Let us stress again that we do not pretend to obtain a quantitative agreement between the quenched and the constrained annealed model. The purpose is to understand what the important effects from a qualitative point of view are. In a certain sense, a realistic annealed system is a more convenient model of the physics of a disordered system than a quenched system, since it contains the essential ingredients with the advantage of being exactly solvable.

As discussed, in our case we have to impose the correct number of negative n.n. couplings, using the intensive variable of the disorder

$$\alpha \equiv \frac{1}{N} \sum_{i=1}^{N} J_i \tag{4.20}$$

which is a self-averaging quantity to the value $\overline{\alpha} = 0$, as a consequence of the large number law.

In order to compute the constrained free energy f_{ac}, it is thus convenient to introduce a Grand-partition function which depends on the Lagrange multipliers μ,

$$\Omega_N(\mu) = \overline{e^{-\mu\beta\alpha_N N} Z_N} \tag{4.21}$$

where Z_N is the canonical partition function defined in (4.2). Notice that we have redefined the Lagrangian multiplier in order that $\mu \to \beta\mu$. This is only to have more treatable thermodynamical variables in the vanishing temperature limit.

Since the random variables are independent, we can easily perform the disorder average which reads

$$\Omega_N(\mu) = \sum_{\{\sigma_i=\pm 1\}} \prod_i^N \cosh(\beta J(\sigma_i\sigma_{i+1} - \mu)) \exp\left\{\frac{\beta\Lambda}{2N}\left(\sum_i \sigma_i\right)^2\right\} \tag{4.22}$$

where we have not written the constant factor $\exp(-\beta\Lambda/2)$ in the partition function (4.2) which gives a negligible contribution to the thermodynamic potentials when $N \to \infty$. This formula indicates that the system has two relevant thermodynamic parameters, the magnetization

$$m = \frac{1}{N}\sum_i \sigma_i \tag{4.23}$$

and the one-step correlation

$$n = \frac{1}{N}\sum_i \sigma_i\sigma_{i+1}\,. \tag{4.24}$$

The Grand-partition function (4.21) can be written in terms of n, m as

$$\sum_{nN=-N}^{N} \sum_{mN=-N}^{N} P(n, m)\,[\cosh\beta J(1-\mu)]^{N\frac{(1+n)}{2}}\,[\cosh\beta J(1+\mu)]^{N\frac{(1-n)}{2}}\, e^{\beta\frac{\Lambda}{2}m^2 N} \tag{4.25}$$

where $(1 \pm n)/2$ is the number of times that the variable $\sigma_i\sigma_{i+1} = \pm 1$ and $P(n, m)$ is the joint probability that a spin configuration has a given value of n and m, and is non-zero for a given m only when $n \in [2m-1,\, 1]$. Let us notice that the two sums in (4.25) are limited to those values of n and m compatible with their definitions. This probability is obviously symmetric with respect to m, so that we can limit the discussion to the case $m \geq 0$. The calculation of $P(n, m)$ (see Ref. 20) gets for large N

$$P(n, m) \sim e^{-S(n, m)N} \tag{4.26}$$

with

$$\begin{aligned} -S(n, m) = {} & \frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\ln\left(\frac{1-m}{2}\right) + \\ & -\frac{1-2m+n}{4}\ln\left(\frac{1-2m+n}{4}\right) - \frac{1+2m+n}{4}\ln\left(\frac{1+2m+n}{4}\right) + \\ & -\ln(2) - \frac{1-n}{2}\ln\left(\frac{1-n}{4}\right)\,. \end{aligned} \tag{4.27}$$

Inserting (4.26) into (4.25) the Grand-partition function becomes

$$\Omega_N(\mu) = \sum_{nN=-N}^{N} \sum_{mN=-N}^{N} e^{-g(m,n,\mu)N} \tag{4.28}$$

where we have introduced the Gibbs thermodynamic potential

$$g(m, n, \mu) = +S(n, m) - \frac{\beta\Lambda m^2}{2} - \frac{1+n}{2} \ln\cosh\beta J(1-\mu) - \frac{1-n}{2} \ln\cosh\beta J(1+\mu) . \tag{4.29}$$

In the limit $N \to \infty$, the sum (4.27) can be estimated by an integral so that the saddle point method gives

$$\Omega_N(\mu) \sim \exp\left[-N \min_{m\in[0,1]} \min_{n\in[2m-1,1]} g(\mu, n, m)\right] . \tag{4.30}$$

This expression gives the formal solution of the constrained annealed system:

$$f_{ac}(\beta) = \min_{m\in[0,1]} \min_{n\in[2m-1,1]} \max_{\mu\in R} \frac{1}{\beta} g(\mu, n, m) . \tag{4.31}$$

The discussion of the phase diagram of (4.29) becomes particularly simple at zero temperature, since the entropic factor $S(n, m)$ disappears in $-g(\mu, n, m)/\beta$ when $\beta \to \infty$. As a consequence, one has

$$\lim_{\beta\to\infty} \frac{g(\mu, n, m)}{\beta} = -\frac{(1+n)}{2} J|1-\mu| - \frac{(1-n)}{2} J|1+\mu| - \frac{\Lambda}{2} m^2 . \tag{4.32}$$

The maximization with respect to μ and the minimization with respect to n and m gives for $\omega < 3/8$ the ferromagnetic state

$$m = 1 \qquad n = 1 \tag{4.33}$$

while for $\omega > 3/8$ one obtains a ferrimagnetic state with

$$m = 1/2 \qquad n = 0 . \tag{4.34}$$

Let us stress that in the constrained annealed model the spin glass phase is not present and only one ferrimagnetic phase exists for $3/8 < \omega$. This is the price we must pay for obtaining an exactly solvable model at any temperature. However, the ferrimagnetic order survives and that is our main goal.

As in the previous section, we also use combinatorial arguments to determine the phase diagram at zero temperature, instead of using (4.29), in order to make transparent the nature of our approximations. The constrained annealed system can organize the random couplings under the constraint that one half of them are negative. The ferromagnetic part of the energy is independent of the spatial organization of the couplings and also of the organization of the spins, depending only on m. Therefore, for any given value of m, the J_i's will organize themselves in order to minimize the contribution to the energy due to the n.n. interaction. It is easy to convince oneself that the energetically favourite configuration is that one which has only islands of an even number of negative n.n. couplings. In this

case, the spins can organize in order for all to be positive except in *some* of the islands of antiferromagnetic J_i's where they alternate up and down. This is due to the fact that islands of an even number of negative couplings are not frustrated as discussed in Sec. 5 and a ground state without frustration has a lower energy than a frustrated one. The annealed system with the constraint (4.20) is thus able to escape frustration at difference with the quenched model. As a consequence at $T = 0$ the entropy vanishes and the overlap is $q = \pm 1$ even in the ferrimagnetic state. This is the main qualitative failure of our annealed approximation. It is worth noting that it would be possible to introduce further constraints to take into account the frustration effects.

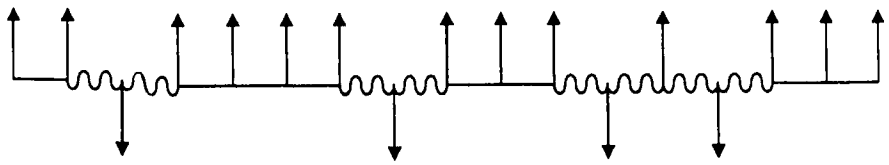

Fig. 11. The competing interactions model (4.1): disorder realization and spin configuration for the constrained annealed model at $T = 0$ and $\omega > 3/8$. The full line are the positive n.n. couplings $J_i = +J$ and the wrinkled lines are the negative ones $J_i = -J$. The arrows represent the spins. The alternated flipped spins are distributed only in the (even) islands of negative couplings.

In order to determine the ground states, we can limit ourselves to the configurations with magnetization $m \geq 1/2$. Indeed, at $m = 1/2$ one fourth of the spin must be down and in *all* the (even) negative islands the spin alternates up and down (see Fig. 11). The contribution to the energy due to the n.n. coupling is now $-JN$ and cannot be reduced. On the other hand, the reduction of m to values lower than $1/2$ would raise the energy because of the infinite range positive interaction. As a consequence, $m = 1/2$ is the lowest possible value for the magnetization at $T = 0$ for all finite ω's. The energy as a function of m has the same qualitative shape as in (4.16) with the difference that the interval for m is [1/2, 1] instead of [2/3, 1]. $E(m)$ is a convex function and its minima are given by the extrema. We thus have to compare the two energy densities at the interval limits, which are

$$E(m = 1) = -\Lambda/2 \tag{4.35}$$

and

$$E(m = 1/2) = -\Lambda/8 - J\,. \tag{4.36}$$

They coincide at $\omega = 3/8$ where a first-order phase transition occurs.

With the quenched model also, one can argue that this phase transition survives at non-zero temperature until a critical value T_c. The advantage is now that this hypothesis can be checked without a Monte Carlo simulation. In fact, the free energy (4.29) is an extremum of the Gibbs potential g, which can be computed in

two steps. First, we determine

$$f(n, m) = \max_{\mu \in R} \beta^{-1} g(\mu, n, m) . \tag{4.37}$$

This can easily be done by solving the equation

$$\left. \frac{\partial g}{\partial \mu} \right|_{\mu^*} = 0 \tag{4.38}$$

which gives the maximization value of the Lagrange multiplier as a function of n,

$$\mu^* = \frac{\tau}{2\omega} \operatorname{arcsinh} \left(n \sinh \frac{2\omega}{\tau} \right) \tag{4.39}$$

where we have introduced the reduced temperature $\tau \equiv T/\Lambda$. The constrained annealed free energy is now the maximum with respect to n and m of

$$f(n, m) = \frac{1}{\beta} g(\mu^*(n), n, m) . \tag{4.40}$$

To find it, we should solve the two equations

$$\frac{\partial g(\mu^*(n), n, m)}{\partial n} = 0 \quad \text{and} \quad \frac{\partial g(\mu^*(n), n, m)}{\partial m} = 0 . \tag{4.41}$$

By the first one, m^* is obtained as a function of n, that is

$$m^* = n^{1/2} \cosh \beta J (1 - \mu^*) . \tag{4.42}$$

In conclusion, only one implicit equation in the variable n is left:

$$\tanh \frac{m^*}{\tau} = m^* \left(1 + \tanh \frac{2\omega\mu^*}{\tau} \tanh \frac{\omega(1 - \mu^*)}{\tau} \right) \tag{4.43}$$

which can be easily solved numerically. When the solutions are more than one we have to choose the solution which minimizes $f(n, m(n))$. This procedure permits us to obtain the magnetization as a function of the temperature at fixed ω. In Fig. 12 one sees that the magnetization as a function of the temperature has a jump for ω in the interval $[3/8, \omega_c]$, while at $\omega_c = 0.4$, $m(T)$ is a continuous function with a discontinuous derivative at $\tau_c = 0.18$. On the other hand, for $\omega > \omega_c$ and $\omega < \omega_1$, the magnetization as a function of the temperature is an analytic function for $\tau < 1$.

Figure 13 shows the phase diagram in the ω–τ plane where there is a line of first-order phase transitions which end at the critical point ω_c, τ_c where the transition is continuous. Moreover, for $\tau = 1$, independently of the value of ω we have a second-order phase transition to the paramagnetic phase, the same as in the mean field model given by $\omega = 0$. This can easily be proved by expanding $f(n, m)$ in powers of n and m and finding their minimum. Finally from (4.39) and (4.41), one can show that in the limit $\omega \to \infty$ for non-zero Λ the magnetization is a monotonous decreasing function of τ from $m(\tau = 0) = 1/2$ to $m(\tau = 1) = 0$.

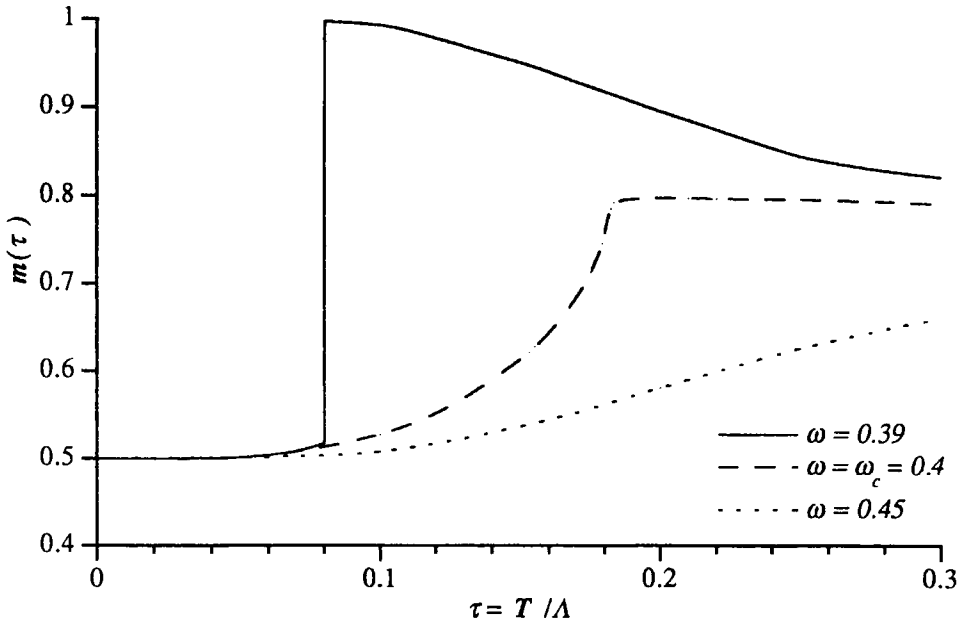

Fig. 12. The competing interactions model (4.1): magnetization $m(\tau)$ for the constrained annealed model as a function of the reduced temperature $\tau = T/\Lambda$, for different disorder strength $\omega = J/\Lambda$: at $\omega = 0.39$ (discontinuous jump at $\tau = 0.08$); at $\omega = \omega_c = 0.4$ (discontinuous m-derivative at $\tau = \tau_c = 0.18$); at $\omega = 0.45$ (m is an analytic function for $\tau < 1$).

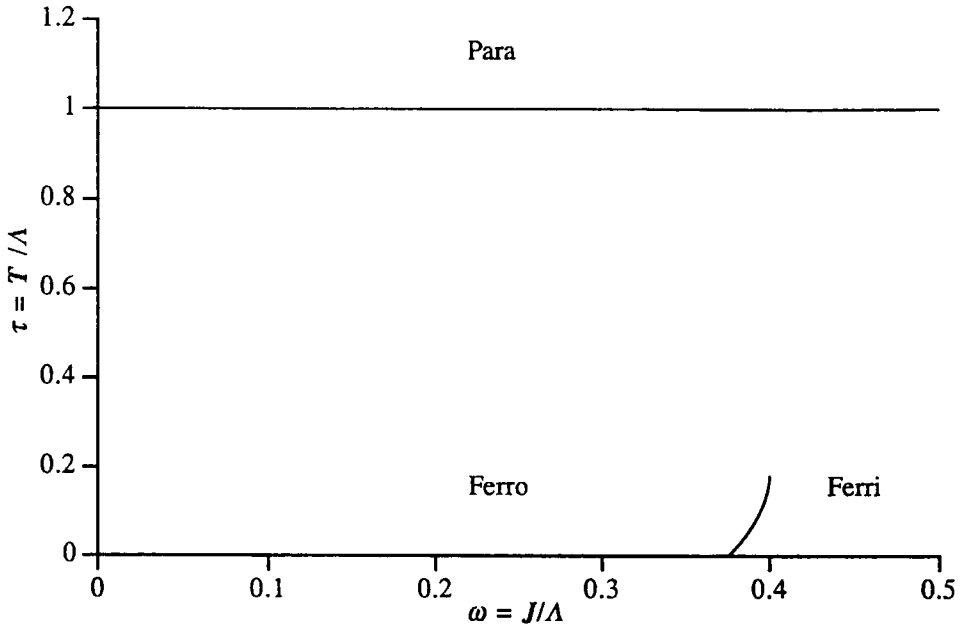

Fig. 13. The competing interactions model (4.1): phase diagram $\tau = T/\Lambda$ versus $\omega = J/\Lambda$ for the constrained annealed model. The dashed line is the line of the second-order phase transition between ferromagnetic and paramagnetic phases. The full line is the line of first-order phase transition between ferromagnetic and ferrimagnetic phases ending with a critical point.

4.4. *Final considerations on the competing interaction model*

We have studied a one-dimensional Ising model with nearest neighbour random couplings where some aspects of high dimensional systems are introduced by considering an infinite range positive coupling. As a consequence of the competition between the two interactions, our model at low temperature and in a range of intermediate disorder exhibits an infinity of disordered ferrimagnetic phases with magnetization $m_n = 2/(1+n)$, $(n = 2, 3, \cdots, \infty)$ which are separated by lines of first-order phase transitions. These phases are a real high dimensional effect since they are frustrated and have many degenerate ground states (the zero temperature entropy does not vanish). They could appear in more complicated systems which share the same ingredients of our toy model. For instance, long period commensurate and incommensurate phases are observed in $X-Y$ models with competing interactions and anisotropy without disorder.[12] It would be interesting to understand whether the same type of behaviours might arise in spin glass models.

Although our model can be studied in a lengthy numerical way at non-zero temperature via products of random matrices we have decided to test the method of the constrained annealed averages to reproduce its behaviour. The constraint is the simplest one (the number of positive couplings is fixed by the law of large numbers) and thus is not able to retain the frustration of the quenched disorder system. However, the ferrimagnetic order survives and the main qualitative feature of the model is described by this first rough approximation. The introduction of a further constraint such as $\frac{1}{N}\sum J_i J_{i+1} = 0$ can also take into account the gross effects of frustrations, although the analytic calculations become rather tedious, and are far beyond the purpose of this paper. More importantly in our opinion is the fact that our results pave the way for the analysis of new disordered phases of ferrimagnetic type in annealed models of spin glasses, without recurring to numerical simulations.

5. Spin Glasses in Two and More Dimensions

The ferrimagnetic model of the previous section is a first step towards the application of the method of constrained annealed averages in systems which are not one-dimensional. However, the challenge is two- and three-dimensional Ising models with independent random couplings, where the relevant variables α_N to be constrained should be related to frustration, as first suggested by Toulouse and Vannimenus in 1976.[29] As far as we know, there are only Monte Carlo simulations[5,2,3] which used that idea while no analytic calculations have been performed on that ground. Here we want to show that is possible to get analytic results in dimension $d \geq 2$ with constraints connected to frustration, after a brief review of the simpler findings obtained for Gaussian couplings[22] and for the $2d$ Ising model with diluted disorder[27] using the standard constraints of one-dimensional type.

5.1. *Constraints for disordered Ising models*

Throughout Sec. 5 we consider d-dimensional Ising models with nearest neighbour couplings J_{ij} which are independent identically distributed (i.i.d.) random variables, in the absence of external magnetic field. In particular, the probability distribution of the coupling is either Gaussian or binomial (that is $J_{ij} = J$ with probability p and $J_{ij} = -J$ with probability $1-p$).

Let us briefly sketch what the dependence of the free energy on the disorder variables is in order to determine the relevant variables to be frozen in the annealed averages. The partition function of the random Ising models is

$$Z_N = \sum_{\{\sigma\}} \prod_{i,j} \exp\{\beta J_{ij}\sigma_i\sigma_j\} \tag{5.1}$$

where the product runs over i, j which are nearest neighbours. Using very basic relations of the hyperbolic functions, (5.1) becomes

$$Z_N = \sum_{\{\sigma\}} \prod_{i,j} \cosh(\beta J_{ij})[1 + \sigma_i\sigma_j \tanh(\beta J_{ij})]\,. \tag{5.2}$$

This form is useful since it shows that the non-trivial part of the partition function is given by $\prod_{i,j}\,[1 + \sigma_i\sigma_j \tanh(\beta J_{ij})]$. A typical term in that product is $\sigma_i\sigma_j\sigma_j\sigma_l \cdots \tanh(\beta J_{ij})\, \tanh(\beta J_{jl}) \cdots$, where the bonds $\{i, j\}$, $\{j, l\} \cdots$ can form either a closed or an open line on the lattice. If the bonds form an open line, at least two distinct spin variables survive as a factor of the product of the various $\tanh(\beta J_{ij})$, and the related term gives a zero contribution to the partition function, after performing the sum over the σ configurations. As a consequence, one has to consider only the terms corresponding to bonds which form a closed line (where the product of the σ's is equal to unity since they appear twice) so that (5.2) can be written as a sum over all the possible loops $L_s^{(r)}$, where $r \geq 4$ indicates the length of the loop (r has to be even) and s labels different loops with the same length:

$$Z_N = e^{\sum_{i,j} \ln\cosh(\beta J_{ij})}\; 2^N \left(1 + \sum_{\{L_s^{(r)}\}} \prod_{\{i,j\} \in L_s^{(r)}} \tanh(\beta J_{ij})\right). \tag{5.3}$$

It follows that in a system with a finite number N of spins, the free energy of a disorder realization is given by

$$y_N = -\frac{1}{\beta}\left(B + \tilde{F}_N(t^{(4)}, \cdots, t^{(R)})\right) \tag{5.4}$$

where R is the maximum possible length of a loop in the system and $R < \text{const}\, N$, while

$$B = \ln 2 + \frac{1}{N}\sum_{i,j} \ln\cosh(\beta J_{ij}) \tag{5.4a}$$

and

$$\widetilde{F}_N = \frac{1}{N} \ln \left(1 + \sum_{\{L_s^{(r)}\}} t_s^{(r)} \right) . \tag{5.4b}$$

This function depends on the new set of variables

$$t_s^{(r)} = \prod_{\{i,j\} \in L_s^{(r)}} \tanh(\beta J_{ij}) \tag{5.4c}$$

given by products which run on the loops $L_s^{(r)}$, i.e. plaquettes of larger and larger perimeter at increasing r. For instance, the loop $L^{(4)}$ is the elementary plaquette of four bonds. Let us stress that a loop $L^{(r)}$ can be built up as a set of two (or more) distinct and separate loops $L^{(r_1)}$ and $L^{(r_2)}$ with $r_1 + r_2 = r$. In the following we often label a coupling by only one label instead of two, i.e. J_{kl} is denoted by J_k, since there is no risk of ambiguity on a closed line.

As the number of the nearest neighbour couplings in a hypercubic d-dimensional lattice is dN, in the thermodynamic limit the quenched free energy reads

$$f = -\frac{1}{\beta} \left[d \overline{\ln \cosh(\beta J_{ij})} + F(\{t_s^{(r)}\}) + \ln 2 \right] \tag{5.5}$$

where we have assumed that $\widetilde{F}_N$ in (5.4b) self-averages to the function

$$F(\{t_s^{(r)}\}) = \lim_{N \to \infty} \frac{1}{N} \overline{\ln \left[1 + \sum_{\{L_s^{(r)}\}} t_s^{(r)} \right]} . \tag{5.6}$$

Note that the reduction of the partition function (5.1) into (5.3) is the procedure used in the Kac and Ward solution of the $2d$ Ising model without disorder.

Equation (5.5) is fundamental since it shows that the quenched free energy depends only on the distributions of the variables $t^{(r)}$, apart from the mean value $\overline{\ln \cosh(\beta J_{ij})}$. We are allowed to classify the relevant variables α_N to be fixed to their mean value in a constrained annealed average of the free energy. With increasing difficulty we must consider the variables related to

(1) $\ln \cosh \beta J_{ij}$
(2) $t^{(4)} = \tanh(\beta J_1) \tanh(\beta J_2) \tanh(\beta J_3) \tanh(\beta J_4)$
(3) $t^{(r)}$ with $r \geq 6$ (considering only connected loops).

Fixing all the constraints corresponding to those quantities in an annealed average is equivalent to solving the system with quenched disorder. This can be understood noting that in (5.6) we have a sum over all the possible $t_s^{(r)}$. Because of the law of the large numbers the rescaled sum of the t-variables corresponding to the same topological kind of loops self-averages to its quenched mean value. Moreover, it is possible to prove that this result holds even in a finite volume system: the quenched

free energy can be obtained by a minimization of an annealed average of the Grand-partition function with respect to the Lagrange multipliers related to the variables $t_s^{(r)}$ over all the possible loops of the finite system.

In one dimension there are no loops, and we shall show in Sec. 5.2 that the constraint related to point (1) is sufficient to obtain the exact solution. In general, we expect that many physical aspects of a spin glass can already be reproduced by considering the first two points.

The case of random couplings $J_{ij} = \pm 1$ with equal probability is particularly simple in this respect, since the variable $\ln\cosh\beta J_{ij}$ is a constant and thus can be ignored. Moreover, one immediately sees that (5.4c) becomes

$$t_s^{(r)} = \tanh^r(\beta) \prod_{\{i,j\}\in L_s^{(r)}} J_{ij}\,, \tag{5.7}$$

implying that the quenched free energy only depends on the values of the product of J_{ij} on closed lines of bonds. The non-trivial part of the free energy y_N in (5.4) is thus

$$\widetilde{F}_N = \frac{1}{N}\ln(1 + x_4\tanh^4(\beta) + x_6\tanh^6(\beta) + \cdots)$$

where the coefficients $\{x_4, x_6, \cdots\}$ are non-independent random quantities. As a consequence, the quenched free energy is

$$f = -\frac{1}{\beta}\left(d\ln\cosh\beta + \ln 2 + \lim_{N\to\infty}\frac{1}{N}\overline{\ln(1 + x_4\tanh^4(\beta) + x_6\tanh^6(\beta) + \cdots)}\right).$$

In the annealed model, its last non-trivial term vanishes since $\overline{x_i} = 0$ (each x_i is a sum of products of independent random couplings). In the limit of high temperature ($\beta \to 0$), the difference between annealed free energy and quenched free energy thus is of the order $\tanh^4\beta \sim \beta^4$. More importantly, the simplest relevant constraint in the $\pm J$ spin glass is the frustration on the elementary plaquette, that is $\widetilde{J}_p = -J_1J_2J_3J_4$, since $t^{(4)} = -\widetilde{J}_p\tanh^4\beta$ (see Sec. 5.4).

Let us conclude by stressing that for all symmetric distributions the variable $\alpha_N = \frac{1}{N}\sum J_{ij}$ already assumes its mean value $\overline{J_{ij}} = 0$ in the annealed estimate so that no improvement is reached by fixing the related constraint. This is a straightforward generalization of the one-dimensional case where we have shown[19] that any variable constructed with an odd number of J's can be constrained without thermodynamic work. However, it is still meaningful to use such a constraint when $\overline{J_{ij}} \neq 0$, as in the case of diluted random Ising models where one gets results very close to those of the quenched system (see Sec. 5.3).

5.2. *Ising model with Gaussian random couplings*

A major example where the variable $\ln\cosh(\beta J_{ij})$ is a relevant constraint is given by the d-dimensional spin glass with i.i.d. couplings J_{ij} distributed according to a normal Gaussian ($\overline{J_{ij}} = 0$ and $\overline{J_{ij}^2} = 1$). In $1d$, the model can be exactly solved and

the free energy density reads

$$f = -\frac{1}{\beta}\overline{\ln\cosh(\beta J_{ij})} - \frac{1}{\beta}\ln 2\,. \tag{5.8}$$

In higher dimension d, it is not possible to find an exact solution for the quenched free energy, although it is easy to compute the annealed average $\overline{Z_N}$ since in (5.1) one has a product of dN (the number of bonds) i.i.d. random variables J_{ij}, so that

$$\overline{Z_N} = \sum_{\{\sigma\}} \left(\int_{-\infty}^{+\infty} e^{\beta J \sigma_i \sigma_j - J^2/2} \frac{dJ}{\sqrt{2\pi}} \right)^{dN} \tag{5.9}$$

and the annealed free energy density is

$$f_a = -\frac{1}{\beta}\ln 2 - \frac{d\beta}{2}\,. \tag{5.10}$$

This approximation is very poor at low temperatures, since f_a diverges as $\beta \to \infty$, while the quenched free energy tends to a finite value, e.g. in $1d$ the ground state energy $E_0 = \lim_{\beta\to\infty} f(\beta) = \overline{|J_{ij}|}$. Another unphysical feature is that at small temperature the entropy of the annealed system is negative and diverges like $S(\beta) \sim -d\beta^2/2$. However, there is a more intelligent way proposed by Toulouse and Vannimenus[29] to use the annealed free energy for obtaining an estimate of the ground state energy. The idea is that the entropy must be positive for a well-defined quenched model and, therefore, $f(T)$ must be a non-increasing function of $T = \beta^{-1}$. As a consequence, one can conclude that $E_0 = \sup_\beta f(\beta) \geq \sup_\beta f_a(\beta)$ and in our case one obtains

$$E_0 \geq \sup_\beta f_a(\beta) = -\sqrt{2d\ln 2}\,. \tag{5.11}$$

Both (5.10) and (5.11) can be improved using the constrained annealed approximations of the free energy. The natural choices for α_N have been discussed in the Sec. 5.1. In this case we can start by using the simplest one that is by the variable

$$\alpha_N = \frac{1}{N}\sum_{i,j} \psi(J_{ij}) \tag{5.12}$$

where

$$\psi(J_{ij}) = \ln\{\cosh(\beta J_{ij})\} - \overline{\ln\{\cosh(\beta J_{ij})\}}\,. \tag{5.13}$$

Note that this constraint automatically leads to the exact solution in $1d$ where it corresponds to consider an annealed average restricted to the disorder realizations which have free energy equal to the quenched free energy (5.8). In terms of the discussion of Sec. 5.1, this is a consequence of the fact that in $1d$ there are no loops, so that the free energy (5.5) is given by averaging only on the random variable $\ln\cosh(\beta J_{ij})$.

The Grand-partition function

$$\Omega_N(\mu) = \overline{\ln \sum_{\{\sigma\}} \prod_{i,j} \exp\{\beta J_{ij}\sigma_i\sigma_j - \mu\psi(J_{ij})\}} \tag{5.14}$$

can be easily computed since the product runs over independent variables. After taking the supremum with respect to μ of the related Gibbs potential, the free energy of the constrained annealed system reads

$$f_{ac} = -\frac{1}{\beta}\ln 2 - \frac{d}{\beta}\overline{\ln\cosh\{\beta J_{ij}\}}\,, \tag{5.15}$$

which is a better approximation than f_a, since at finite temperature strict inequality $f_{ac} > f_a$ holds. It is worth stressing that at zero temperature $f_{ac}(T=0) = -d\,\overline{|J_{ij}|}$ and does not diverge as in the annealed case. This is in qualitative agreement with the quenched model. However the zero temperature entropy $S_0 = (1-d)\ln 2$ is still negative for $d > 1$, indicating that the constraint (5.13) gives too crude an approximation and is not fully satisfactory. However it is possible to use it in order to improve the annealed estimate of the ground state energy by the method of Ref. 29:

$$E_0 \geq \sup_{\beta} f_{ac}(\beta)\,. \tag{5.16}$$

For example, in $3d$ the inequality (5.11) gives $E_0 \geq -2.04$ while the inequality (5.16) gives $E_0 \geq -1.82$. The free energy densities f_a and f_{ac} as functions of the temperature are shown in Fig. 14 for the $3d$ case.

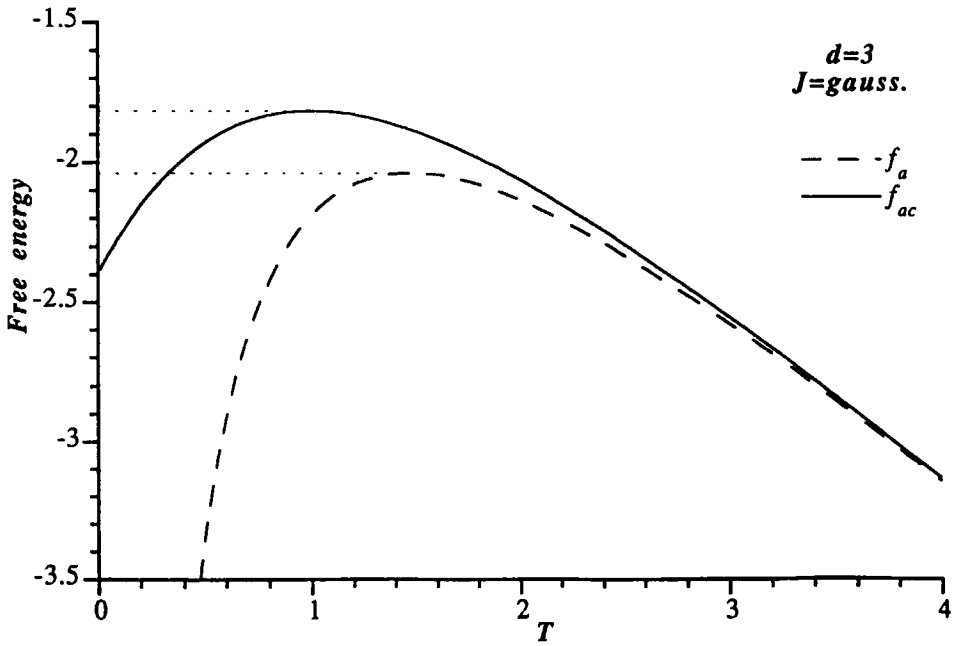

Fig. 14. $3d$ Ising model with Gaussian couplings (5.9): the annealed free energy f_a (dashed line) and constrained free energy f_{ac} (full line) versus $T = 1/\beta$. The maxima of the two functions (dotted lines) estimate the quenched ground state energy.

5.3. *Ising model with diluted disorder in two dimension*

A second, less trivial case, where constraints of the one-dimensional type give rather good results is the $2d$ nearest neighbour Ising model with i.i.d. couplings J_{ij} which assume the value 1 with probability $p > 1/2$ and the value -1 with probability $1-p$. The quenched density of positive coupling is p and, obviously, when p tends to 1 the ordinary ferromagnetic Ising model is recovered.

The annealed approximation of the free energy density is unsatisfactory because at low temperature the density of positive bond is higher than p. In fact, since the couplings are treated as *hot* variables which are allowed to flip, they preferably align in the same direction to minimize the annealed free energy.

It is then clear that the annealed approximation can be improved by fixing the density of positive bonds to the correct quenched value p. This constraint will be not sufficient to obtain the quenched free energy since the ordering of coupling is also altered by the annealing. Better results would be obtained by introducing further Lagrange multipliers associated to the loop variables $t^{(r)}$. However, frustration should not play an important role if p is not too close to $1/2$ so that we expect a considerable improvement with respect to the brute annealed approximation. The constrained model has been introduced and solved by Thorpe and Beeman.[27]

The intensive variable of the disorder to be considered is

$$\alpha_N = \frac{1}{2N}\sum_{i,j} J_{ij} - (2p-1) \tag{5.17}$$

where $\overline{J_{ij}} = 2p-1$ has been subtracted to $J_{i,j}$ to have a variable α_N with zero mean value. The Grand-partition function is given by

$$\Omega_N(\mu) = \overline{\sum_{\{\sigma\}}\prod_{i,j}\exp\{\beta J_{ij}\sigma_i\sigma_j - \mu(J_{ij}-2p+1)\}} \tag{5.18}$$

and it can easily be computed since the product runs over independent variables. One obtains

$$\Omega_N(\mu) = \sum_{\{\sigma\}}\prod_{i,j}(p\exp\{\beta\sigma_i\sigma_j - \mu(2-2p)\} + (1-p)\exp\{-\beta\sigma_i\sigma_j + 2\mu p\}). \tag{5.19}$$

Let us remark now that a function $\rho(x)$ of a dichotomic variable $x = \pm 1$ can always be written in an exponential form $\exp(A+Bx)$ where A and B are given by the solutions of the equation system $\exp(A+B) = \rho(1)$ and $\exp(A-B) = \rho(-1)$.

In our case, the dichotomic variable is the product $\sigma_i\sigma_j$, so that the Grand-partition function becomes

$$\Omega_N(\mu) = \sum_{\{\sigma\}}\prod_{i,j}\exp\{A + B\sigma_i\sigma_j\} \tag{5.20}$$

where $A = A(\beta, \mu, p)$ and $B(\beta, \mu, p)$ are the solution of the system

$$\exp\{A + B\} = p\exp\{\beta - \mu(2 - 2p)\} + (1 - p)\exp\{-\beta + 2\mu p\} \tag{5.21}$$

$$\exp\{A - B\} = (p\exp\{-\beta - \mu(2 - 2p)\} + (1 - p)\exp\{\beta + 2\mu p\}) . \tag{5.22}$$

We see that the Grand-partition function, a part a constant multiplicative term $\exp(2AN)$, is formally identical to the partition function of an Ising model. In other words, we have mapped our model into an ordinary ferromagnetic Ising model with an effective inverse temperature B. We then easily obtain

$$g(\beta, \mu, p) = -2A(\beta, \mu, p) - \Gamma(B(\beta, \mu, p)) \tag{5.23}$$

where $\Gamma(B)$ is the Onsager solution of the $2d$ Ising model

$$\Gamma(B) = +\ln\{2\cosh(2B)\} + \frac{1}{2\pi}\int_0^\pi d\theta \ln[\frac{1 + \sqrt{1 - (k\sin\theta)^2}}{2}] \tag{5.24}$$

with

$$k = \frac{2\sinh(2B)}{(\cosh(2B))^2} . \tag{5.25}$$

To obtain the constrained free energy it is sufficient to maximize g. We have in fact:

$$f_{ac}(\beta, p) = \frac{1}{\beta}\max_{\mu}\{g(\beta, \mu, p)\} . \tag{5.26}$$

In practice, one has to find the value $\mu^* = \mu^*(\beta, p)$ which satisfies the equation

$$2\frac{\partial A}{\partial \mu} + \frac{\partial \Gamma(B)}{\partial B}\frac{\partial B}{\partial \mu} = 0 \tag{5.27}$$

so that

$$f_{ac}(\beta, p) = \frac{1}{\beta}g(\beta, \mu^*, p) = -\frac{1}{\beta}[2A(\beta, \mu^*, p) + \Gamma(B(\beta, \mu^*, p))] . \tag{5.28}$$

For comparison, we recall that the annealed energy is simply

$$f_a(\beta, p) = \frac{1}{\beta}g(\beta, 0, p) = -\frac{1}{\beta}[2A(\beta, 0, p) + \Gamma(B(\beta, 0, p))] . \tag{5.29}$$

We can now study the qualitative behaviour of the two free energies in the phase space (β, p). In particular, one is interested in determining the transition line between paramagnetic and ferromagnetic phase.

We make the preliminary observation that we can limit ourselves to the region $p \geq 1/2$ since both f_a and f_{ac} are symmetric with respect to this value of p. Then we observe that both the free energies depend on Γ which is the Onsager solution of the Ising model. The function $\Gamma(B)$ has a singularity, corresponding to the ferro-para transition, when its argument equals the inverse Onsager critical temperature $\beta_c = \frac{1}{2}\ln(1 + \sqrt{2})$. The transition line $p(\beta)$ can thus be found by simply equating

B to β_c. For what concerns the annealed free energy the transition between the paramagnetic and ferromagnetic phases is obtained from the relation

$$B(\beta, \mu = 0, p) = \beta_c \,. \tag{5.30}$$

Using (5.21) and (5.22), which give B, one immediately gets

$$p_a(\beta) = \frac{1}{2} + \frac{\cosh(2\beta)}{(2\sqrt{2}+2)\sinh(2\beta)} \,. \tag{5.31}$$

Similarly, for the constrained annealed system the transition line is obtained from

$$B(\beta, \mu^*, p) = \beta_c \,. \tag{5.32}$$

In this case μ^* is a function of β and p given by (5.27) while B is again obtained from (5.21) and (5.22). After some lengthy but simple calculations one gets

$$p_{ac}(\beta) = \frac{1}{2} + \frac{\sqrt{2}\cosh(2\beta)}{4\sinh(2\beta)} \,. \tag{5.33}$$

Both the transition lines p_{ac} and p_a are shown in Fig. 15. In the annealed model, as expected, the ferromagnetic phase is larger than in the constrained model. This is a consequence of the fact that the couplings are allowed to flip and they arrange preferably in a positive ferromagnetic configuration. Nevertheless, even at vanishing temperature one has a paramagnetic phase below $p_a(T = 0) = 0.707\ldots$.

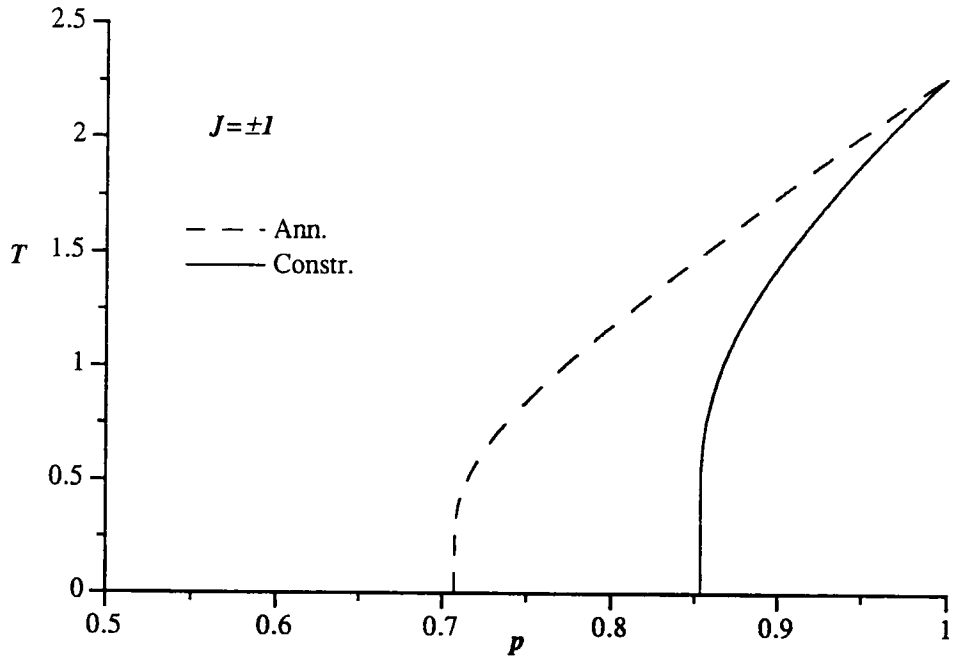

Fig. 15. $2d$ Ising model with diluted disorder (5.18): transition line between para and ferromagnetic phases in the space temperature (T)-probability (p) that $J_{ij} = +1$ for the annealed case (dashed line) and for the constrained case (straight line).

For the constrained model, at vanishing temperature, the paramagnetic phase appears below $p_{ac}(T = 0) = 0.853\ldots$. This value seems to coincide with the numerical results on the quenched model (see Ref. 27). In fact, we expect that the concentration of positive bonds is the only relevant intensive variable of the disorder for what concerns the transition from a paramagnetic and a ferromagnetic behaviour. This does not mean that other variables are not significant, but only that they probably do not influence the ferromagnetic order at large scale.

5.4. *Ising model with random dichotomic couplings*

Let us now consider a d-dimensional spin glass with i.i.d. nearest neighbour couplings which assume two possible values $J_{ij} = \pm 1$ with equal probability $p = \frac{1}{2}$. As discussed at the end of Sec. 5.1, the annealed free energy density is

$$f_a = -\frac{1}{\beta}\left(\ln 2 + d \ln \cosh \beta\right) . \tag{5.34}$$

In one dimension, it is equal to the quenched free energy density, since $\ln \cosh \beta J_{ij}$ is a constant, differing with the Gaussian case where that variable is random and has to be fixed by a Lagrange multiplier. In Sec. 5.1, we have discussed why it is not useful to consider intensive variable α_N of the type (5.12) with ψ depending on a single coupling J_{ij}, and we have shown that it is necessary to introduce variables α_N associated to the loop variable $t^{(4)}$, in order to improve (5.34). With this choice, we should consider the intensive variable

$$\alpha_N = \frac{1}{N} \sum_{\{L_s^{(4)}\}} \widetilde{J}_p \tag{5.35}$$

where $\widetilde{J}_p$ is minus the product of the four couplings on the elementary square plaquette $L^{(4)}$.

As usual, the Grand-partition function is given by the average

$$\Omega_N(\mu) = \overline{\sum_{\{\sigma\}} \exp\left\{ \sum_{i,j} \beta J_{ij} \sigma_i \sigma_j - \sum_{\{L_s^{(4)}\}} \mu \widetilde{J}_p \right\}} . \tag{5.36}$$

Taking into account that the couplings are dichotomic with equal probability we can perform the gauge transformation $J_{ij} \to J_{ij}\sigma_i\sigma_j$ which leaves unchanged the free energy of a disorder realization of the system as well as $\widetilde{J}_p$, i.e. $\widetilde{J}_p \to \widetilde{J}_p$. The Grand-partition function rewrites

$$\Omega_N(\mu) = 2^N \overline{\exp\left\{ \sum_{i,j} \beta J_{ij} - \sum_{\{L_s^{(4)}\}} \mu \widetilde{J}_p \right\}} . \tag{5.37}$$

This expression (which is meaningful for any dimension $d \geq 2$) cannot be computed exactly including all the possible square plaquettes $L_s^{(4)}$ of the d-dimensional lattice.

It can be shown that the problem is equivalent to solve a d-dimensional gauge model without disorder, whose solution is not known for $d \geq 2$.

However an exact solution can be achieved if we restrict the sum (5.35) only on a part (of order N) of the plaquettes. Let us consider the $2d$ model. It is convenient to limit ourselves to consider one half of the plaquettes, which must be chosen in such a way that they do not share any coupling, as it happens for the black squares of a chessboard. With this restriction (5.37) contains a sum over products of $N/2$ independent variables (corresponding to the $N/2$ black plaquettes) and the Grand-partition function can be rewritten as

$$\Omega_N(\mu) = 2^N \left(\overline{\exp\{\beta(J_1 + J_2 + J_3 + J_4) + \mu J_1 J_2 J_3 J_4\}}\right)^{\frac{N}{2}} \tag{5.38}$$

where J_1, J_2, J_3, J_4 are the couplings of one of the plaquettes. The average (5.38) can be easily computed and, after maximizing with respect to μ, one obtains

$$f_{ac}^{sq} = -\frac{1}{\beta}\left\{\frac{1}{2}\ln 2 + \frac{1}{4}\ln\cosh 2\beta + \frac{1}{4}\ln(3 + \cosh 4\beta)\right\}, \tag{5.39}$$

thus improving the annealed estimate (5.11). The functions f_a and f_{ac}^{sq} of the $2d$ spin glass are shown in Fig. 16. At difference with the annealed case the entropy of the constrained annealed system is never negative. We have thus obtained a real qualitative improvement.

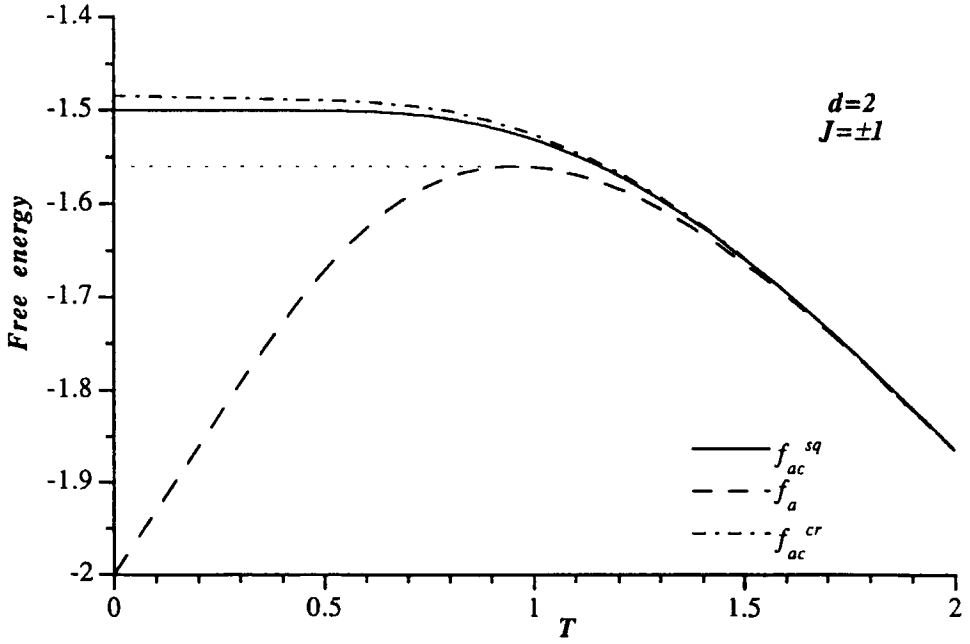

Fig. 16. $2d$ Ising model with dichotomic couplings: the annealed free energy f_a (5.34) (dashed line) and constrained free energy f_{ac} (5.39) (full line) versus temperature $T = 1/\beta$. The constraint is on $N/2$ independent plaquettes. The dotted line indicates the maximum of f_a.

The ground state energy of the constrained annealed system is $f_{ac}^{sq}(T=0) = -1.5$ and the residual entropy is zero. The $d=2$ estimate obtained from $\sup_\beta f_a(\beta)$ gives $E_0 \geq -1.56$ (see Fig. 16). These results can be usefully compared with the numerical or finite volume estimate $E_0 = -1.404 \pm 0.002$ and $S_0 = 0.075 \pm 0.004$ (see Ref. 25).

It is more interesting to consider the heat capacity C which can be explicitly computed. For $\beta \to \infty$ one has

$$C \sim \beta^2 e^{-4\beta}. \tag{5.40}$$

The argument of the exponential has been conjectured in Ref. 24 for the quenched model and it is different from the exact one-dimension result where the argument is -2β. Note that the annealed system also gives $C \sim \exp(-2\beta)$.

Finally, since we have fixed only one half of the single plaquette frustration, it is reasonable to ask what the residual frustration ϕ_{ac} is. Since the black plaquettes have zero frustration, this quantity will be one half of the frustration of a white plaquette. The four couplings forming a white plaquette are independent since they are not coupled by the Lagrange multiplier. Therefore, the white plaquette frustration can be written as $-\langle\langle J_i \rangle\rangle^4$, where $\langle\langle \cdot \rangle\rangle$ is an average on the Gibbs measure associated to the Grand-partition function (5.39). One therefore has

$$\langle\langle J_i \rangle\rangle = -\frac{1}{2}\frac{\partial g(\beta, \mu^*)}{\partial \beta} \tag{5.41}$$

where μ^* is the value which maximizes $g(\beta, \mu)$. Since

$$f_{ac}^{sq}(\beta) = \frac{1}{\beta} g(\beta, \mu^*(\beta)) \tag{5.42}$$

and since the derivative with respect to μ vanishes, one has

$$\langle\langle J \rangle\rangle = -\frac{1}{2}\frac{d(\beta f_{ac}^{sq})}{d\beta}. \tag{5.43}$$

Using the expression (5.39) for ϕ_{ac} we can compute the residual frustration

$$\phi_{ac} = -\frac{1}{2}\langle\langle J_i \rangle\rangle^4 = -\frac{1}{2}\left[\frac{\mathrm{th}(2\beta)}{4} + \frac{\mathrm{th}(2\beta)}{4 - 2(\mathrm{th}^2(2\beta))^2}\right]^4. \tag{5.44}$$

At zero temperature $\phi_{ac} = 3^4/2^9 \approx 0.158$. This result indicates the presence of a small but non-vanishing residual frustration. An analogous expression can be found from the annealed free energy f_a. In this case all plaquettes can have negative frustration and one has

$$\phi_a = -\langle\langle J_i \rangle\rangle^4 = -\mathrm{th}^4(\beta) \tag{5.45}$$

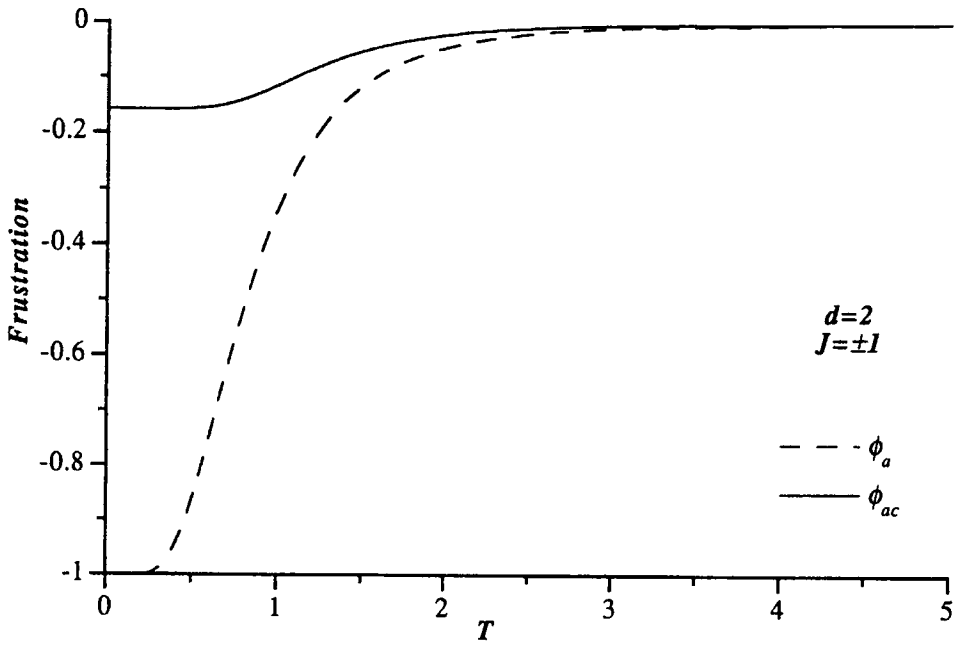

Fig. 17. $2d$ Ising model with dichotomic couplings: residual frustration for the $2d$ annealed model (ϕ_a) (5.45) and for the constrained model (ϕ_{ac}) (5.44). The constraint is on $N/2$ independent plaquettes.

were the average is in the Gibbs measure generated by the annealed average. The frustration $\phi_a(T = 0) = -1$ since at zero temperature the spins are aligned and the annealed system is completely ferromagnetic. Both ϕ_a and ϕ_{ac} are reported in Fig. 17. The fact that in the constrained annealed model the residual frustration does not vanish indicates that the free energy estimate could be improved by further Lagrangian multipliers as we shall discuss in the next section.

It is not immediately evident that our approach can be repeated in any dimension $d > 2$, i.e. that it is possible to find a way of grouping the couplings in independent elementary plaquettes. In three dimensions, after some reasoning and some experiment in a kindergarten with toy cubes, one get convinced that this is, in fact, possible. The only difference is that the number of plaquettes is 3/2 larger than in the $d = 2$ case This consideration leads to the expression for the $d = 3$ constrained free energy density

$$f_{ac}^{sq} = -\frac{1}{\beta}\left\{\frac{1}{4}\ln 2 + \frac{3}{8}\ln\cosh 2\beta + \frac{3}{8}\ln(3 + \cosh 4\beta)\right\} \tag{5.46}$$

which is shown together with f_a in Fig. 18. The estimate (5.46) is still unsatisfactory since f_{ac}^{sq} is unphysical at low temperature because the entropy becomes negative. The ground state energy is estimated by the supremum of (5.46) as $E_0(d = 3) \geq -1.939$.

In the next section we shall show that all these results can be improved in a systematic way by a new method of imposing the constraints.

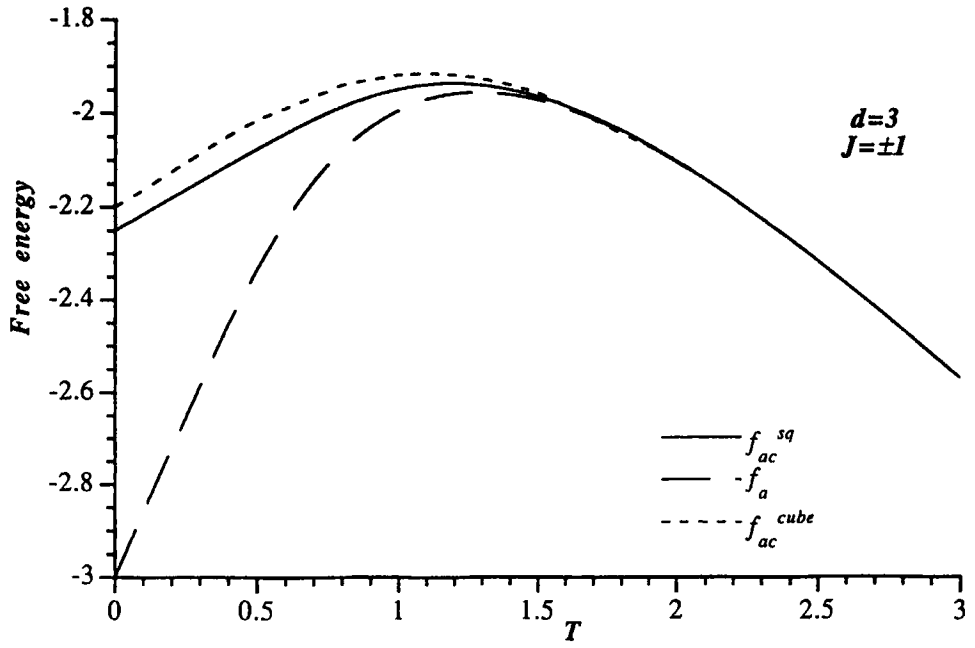

Fig. 18. $3d$ Ising model with dichotomic couplings: the annealed free energy f_a (dashed line) and constrained free energies f_{ac}^{sq} (full line) given by (5.46) and f_{ac}^{cube} (dotted line), obtained with the constraints on all the loops of alternated cubes, versus $T = 1/\beta$.

6. Constraints Without Lagrange Multipliers for Dichotomic Spin Glasses

6.1. *The method*

Up to now, we have performed annealed averages where the relevant constraints have been imposed by means of Lagrange multipliers. Actually, in most cases it is too difficult to derive analytically an annealed average using more than one constraint. The spin glass with dichotomic couplings (Sec. 5.4), however, presents some algebraic features which permit us to introduce an alternative method,[21] in order to obtain constrained annealed averages without Lagrange multipliers, avoiding in this way the problem of their maximization.

Let us consider a d-dimensional lattice of N points: the total number of the couplings is dN, but the system can be fully described by a lower number of independent random variables. In fact the non-trivial part of the quenched free energy (5.6) is a function of the variables $\{t_s^{(r)}\}$, see (5.4c), defined on the closed loops of the lattice. Indeed, from a topological point of view any closed loop $L_s^{(r)}$ can be thought as the union of some elementary square plaquettes, say a number k, so that the related variable $t_s^{(r)}$ can be written as

$$t_s^{(r)} = \tanh^r(\beta) \prod_{i=1}^{k} \tilde{J}_p^{(i)} \tag{6.1}$$

where we have introduced the frustrations $\widetilde{J_p} = \prod_{i=1}^{4} J_i = \pm 1$ with equal probability

of the k plaquettes (the index i runs over the four sites of each square). Let us stress that a loop can be either connected or disconnected, i.e. it is not necessary that the plaquettes are neighbours. Notice that the plaquette frustration was previously defined with a minus sign, but in this context we neglect it to simplify the notation. Moreover, we do not assume periodic boundary conditions in this section.

In other terms, we can consider the plaquette frustrations as the random variables of the system, instead of the couplings, with no loss of generality. We shall see that the number of frustrations necessary for a full description of the system is always lower than dN, the total number of couplings. For instance, in $2d$, there are $2N$ couplings, while the number of plaquette frustrations is of the order N when $N \to \infty$.

However, the $2d$ case exhibits a special feature, since all the possible plaquette frustrations turn out to be a set of independent random variables (let us recall that we do not assume periodic boundary condition). In the general case $d \geq 3$, this statement is no longer valid. Indeed, let us consider a cube in a d-dimensional lattice: a moment of reflection shows that the product of the related six plaquette frustrations has to be 1, so that only five frustrations are independent random variables. As a consequence, for $d \geq 3$ the change of random variables from the couplings to the plaquette frustrations involves only a subset of them, let us say N^*, chosen in such a way that they are independent random variables and that they can build up every possible closed loop on the lattice.

Let us illustrate this change of random variables for a $3d$ lattice of N points, where we can introduce three integer coordinates $\{x, y, z\}$. A square plaquette has three different orientations, belonging to a $x = const$ plane, or $y = const$, or $z = const$. In the thermodynamic limit one has $N^{\frac{1}{3}}$ planes for each orientation, and every plane is a set of $N^{\frac{2}{3}}$ plaquettes, for a total number of $3N$ plaquettes in the whole lattice.

Consider a cube of six plaquettes, with one of them on the $z = 0$ plane. As previously discussed, one of the $\widetilde{J}$'s cannot belong to the subset of plaquette frustrations that describe the whole system, let us say the frustration of the plaquette on the $z = 1$ plane. Let us now consider the other cube sharing this 'neglected' plaquette. By introducing as random variables of the system the frustrations which are related to the four plaquettes that share a coupling with the plaquette on the $z = 1$ plane, we have to neglect the sixth frustration of the plaquette on the $z = 2$ plane, since the product of all the plaquette frustrations of these two nearest neighbour cubes is fixed to 1 (the presence of the plaquette on $z = 1$ in the product is irrelevant, since it appears twice). These arguments can easily be repeated, so that the subset of plaquette frustrations that fully describes the whole system can be found as follows: one has to consider two distinct orientations and to take into account all the plaquettes with these orientations, for a total number of $2N$. Moreover, the third orientation gives only a plane of $N^{\frac{2}{3}}$ plaquettes (the $z = 0$ plane in our example), so that in the thermodynamic limit the number of independent random frustrations N^* that fully describe the systems grows as $2N$.

The generalization of this result is immediate: in the thermodynamic limit the d-dimensional lattice of N points (with $d \geq 3$) is built up by $N\binom{d}{2}$ distinct plaquettes, since $\binom{d}{2}$ is the number of different orientations for a plane in the lattice, but only a number $N^* = 2N$ of plaquette frustrations is necessary to describe the system. In fact the change of random variables involves only the frustrations of the plaquettes with two well-defined orientations ($2N$ plaquettes), together with a plane of plaquettes for each one of the other orientations, but in the thermodynamic limit the main contribution to N^* comes from the former term. Using this subset of plaquettes, whose related frustrations are all independent random variables, it is possible to build up every possible closed loop on the d-dimensional lattice.

In conclusion the described change of variables permits us to reduce the number of independent random variables necessary to describe the system, from dN couplings to N^* frustrations, with $N^* = N$ for d=2 or $N^* = 2N$ for $d \geq 3$. In the following, if not differently specified, the term 'frustration' always indicates one of the N^* random variables of the system.

Recalling (6.1), Eq. (5.6) becomes

$$F = \lim_{N\to\infty} \frac{1}{N} \overline{\ln \left[1 + \sum_{k=1}^{N^*} \sum_{i_1 \ldots i_k} \tilde{J}_p^{(i_1)} \ldots \tilde{J}_p^{(i_k)} \tanh^{r(i_1,\ldots,i_k)}(\beta) \right]} \qquad (6.2)$$

where one considers all the products of $k \in [1, N^*]$ elementary plaquettes (the indices $i_1 \neq i_2 \neq \ldots \neq i_k$ run from 1 to N^*) and $r(i_1, \ldots, i_k)$ is the length of the loop built up by the k plaquettes, while the overline represents now the average over the frustrations.

In the following paragraph, we consider the plaquettes of the lattice only from a topological point of view, i.e., the fact that a frustration of a plaquette is a random variable of the system (or not) is irrelevant. At this point, let us perform a decomposition of the set of plaquettes into subsets such that each coupling J_{ij} belongs to one and only one of these subsets. It follows that two distinct subsets can have in common only isolated lattice points. For instance, three different decompositions of the bonds of a $2d$ lattice (the black regions) are illustrated in Fig. 19. It is worth stressing that, after the decomposition, we get a collection of sub-systems which do not cover the whole original lattice, e.g. in Fig. 19(a) only one half of the 2d lattice is covered. This kind of decomposition divides the plaquettes into two classes: the 'black' ones, which are organized in groups corresponding to the sub-systems [e.g. the crosses in Fig. 19(b)], and the 'white' ones, which do not belong to any sub-system.

Coming back to the frustrations, note that in general there are many 'white' or 'black' plaquettes whose frustration is not a random variable of the system.

At this point we can derive an upper estimate F_{ac} of the function (6.2), by treating all the random variables of frustration associated to the 'white' plaquettes as annealed variables, i.e. by averaging over these variables only the argument of

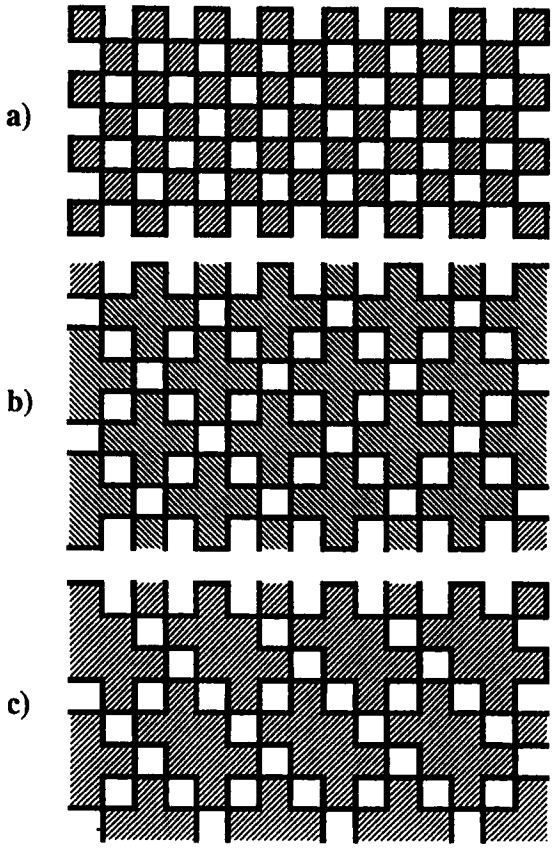

Fig. 19. Ising model with nearest neighbour interactions on a $2d$ lattice: decompositions of the set of the couplings in independent subsets (the black areas): (a) squares, (b) crosses and (c) elongated crosses.

the logarithm in (6.2), instead of the logarithm itself:

$$F_{ac} = \lim_{N\to\infty} \frac{1}{N} \ln \overline{\left[1 + \sum_{k=1}^{N^*} \sum_{i_1 \ldots i_k} \overline{\tilde{J}_p^{(i_1)} \ldots \tilde{J}_p^{(i_k)}}^{(w)} \tanh^{r(i_1,\ldots,i_k)}(\beta) \right]}^{(b)} \tag{6.3}$$

where $\overline{\bullet}^{(w)}$ and $\overline{\bullet}^{(b)}$ represent the averages over the frustrations related to, respectively, the 'white' and the 'black' squares. In (6.3) only the terms $\tilde{J}_p^{(i_1)} \ldots \tilde{J}_p^{(i_k)}$ with all 'black' frustrations do not vanish after performing the 'white' average:

$$F_{ac} = \lim_{N\to\infty} \frac{1}{N} \ln \overline{\left[1 + \sum_{k=1}^{N_b} \sum_{i_1 \ldots i_k} \tilde{J}_p^{(i_1)} \ldots \tilde{J}_p^{(i_k)} \tanh^{r(i_1,\ldots,i_k)}(\beta) \right]}^{(b)} \tag{6.4}$$

where now the indices $i_1, \ldots, i_k$ run only over the N_b 'black' frustrations. In general, a loop is now a set of 'sub-loops', each of them limited to a single sub-system. This implies that $\tanh^{r(i_1,\ldots,i_k)}(\beta)$ can be factorized, i.e., $r(i_1, \ldots, i_k)$ is the sum of the lengths of the various 'sub-loops'. As a consequence, the whole argument of the logarithm in (6.4) is factorized among all the sub-systems, and it immediately

follows:

$$F_{ac} = \frac{d}{n_j} \overline{\ln \left[1 + \sum_{k=1}^{n_p} \sum_{i_1 \ldots i_k} \tilde{J}_p^{(i_1)} \ldots \tilde{J}_p^{(i_k)} \tanh^{r(i_1,\ldots,i_k)}(\beta) \right]} \tag{6.5}$$

where the average is performed over the n_p 'black' plaquette frustrations that belong to a single sub-system, and where n_j is the number of couplings in the sub-system, so that $\frac{d}{n_j}$ represents the total number of sub-systems, rescaled with N in the thermodynamic limit. Recalling (5.5), from (6.5) the lower estimate f_{ac} of the quenched free energy yields

$$-\beta f_{ac} = d \ln \cosh(\beta) + \frac{d}{n_j} \overline{\ln \left[1 + \sum_{k=1}^{n_p} \sum_{i_1 \ldots i_k} \tilde{J}_p^{(i_1)} \ldots \tilde{J}_p^{(i_k)} \tanh^{r(i_1,\ldots,i_k)}(\beta) \right]} + \ln 2 \,. \tag{6.6}$$

It is preferable to choose a topological decomposition of the system such that the n_p 'black' frustrations fully describe the single sub-system. In this context F_{ac} is practically equivalent to the quenched free energy of the sub-system, a part a factor proportional to $\ln \cosh \beta$. If n_p is not too large, the computation of F_{ac} can be performed analytically or numerically.

The correct multiplicative factors to transform the quenched free energy of a sub-system to the global constrained free energy (6.6) can also be obtained by a simple argument. Indeed the partition function Z_{n_s} of a sub-system made of n_s spins and n_j couplings is the sum of 2^{n_s} terms which are given by the product of n_j exponentials while the global constrained Grand-partition function Ω_N of a system with N spins and dN couplings is the sum of 2^N terms which are given by the product of dN exponentials. In order to compare two quantities of order one, one has to write the following equality:

$$(2^{-N} \Omega_N)^{\frac{1}{dN}} = (2^{-n_s} Z_{n_s})^{\frac{1}{n_j}}$$

so that

$$f_{ac} = -\frac{1}{\beta} \left(1 - d \frac{n_s}{n_j} \right) \ln 2 + d \frac{n_s}{n_j} f_{\text{sub}} \,. \tag{6.7}$$

This formula is completely equivalent to (6.6), but has the advantage that the quenched free energy f_{sub} of the sub-system explicitly appears. The ground state energy can be estimated by the saddle point method when $\beta \to \infty$ in the quenched average of Z_{n_s}:

$$E_{ac}(T=0) = \frac{d}{n_j} \overline{\max_{\{\sigma\}} H(\{J\})} \tag{6.8}$$

where the maximum is taken over all the σ configurations for each disorder realization. One can also obtain the residual entropy as

$$S_{ac}(T=0) = \left(1 - \frac{n_s}{n_j} d \right) \ln 2 + \overline{\ln \deg(\{J\})} \tag{6.9}$$

where $\deg(\{J\})$ is the number of disorder configurations which have energy $H(\{J\})$ equal to $E_{ac}(T=0)$.

Let us briefly resume the discussion, in order to clarify the meaning of the result. Using a topological decomposition of the system into independent sub-systems, we have derived a lower estimate of the quenched free energy that depends from the quenched free energy of the single sub-system. As discussed in the appendix, the quenched free energy of a sub-system can be obtained by a minimization of an annealed average of the Grand-partition function with Lagrange multipliers over all its possible loops. Using our procedure, we are able to get the annealed average f_{ac} of the global system where the constraints are imposed over all the possible rescaled sums of frustration on all the loops that appear in the various sub-systems.

To illustrate our method, we apply it in the case of Sec. 3, i.e. the decomposition of a $2d$ spin glass in 'white' and 'black' square plaquettes, like in a chessboard, see Fig. 19(a). In this case all the N plaquette frustrations represent a set of independent random variables, i.e. $N^* = N$. One half of them is related to 'white' plaquettes, so that they are treated as annealed variables, while the other half related to 'black' plaquettes as quenched variables. We also have $n_j = 4$ and $n_p = 1$, so that (6.6) becomes

$$-\beta f_{ac}^{sq} = 2\ln\cosh(\beta) + \frac{1}{4}\ln[1-\tanh^8(\beta)] + \ln 2\,. \tag{6.10}$$

It is easy to check that after trivial algebraic manipulations one again gets (5.39).

The ground state energy and the zero temperature entropy can be directly computed from (6.10), and are respectively

$$E_{ac}^{sq}(T=0) = -1.5 \qquad S_{ac}^{sq}(T=0) = 0\,. \tag{6.11}$$

The method of constrained annealed averages without multipliers can be applied to non-elementary sub-systems in an easy numerical way, since one does not have to introduce a set of corresponding Lagrange multipliers. In order to improve (6.10) and (6.11), we have considered a partition of the $2d$ lattice into independent crosses of $n_p = 5$ square plaquettes, as shown in Fig. 19(b). We thus obtain a constrained annealed average where all the relevant variables on the non-connected loops inside the crosses are frozen. Since $n_j = 16$, from (6.6) the free energy is easily evaluated, and is shown in Fig. 16. The ground state energy and the residual entropy are respectively

$$E_{ac}^{cr}(T=0) = -1.484375 \qquad S_{ac}^{cr}(T=0) = 0.00882\ldots\,. \tag{6.12}$$

The next step is to consider the elongated cross of the type shown in Fig. 19(c) ($n_p = 8$ and $n_j = 24$). In this case we get

$$E_{ac}^{ecr}(T=0) = -1.477865 \qquad S_{ac}^{ecr}(T=0) = 0.0130\ldots\,. \tag{6.13}$$

At increasing the size of the subsets of the decomposition, the convergence to the quenched ground state energy E_0 is rather slow (the numerical result of Ref. 25 gives $E_0 = -1.404 \pm 0.002$ and $S_0 = 0.075 \pm 0.004$). However, the main qualitative feature (positive residual entropy) is reproduced by our approximations.

We have also applied our technique to the $\pm J$ spin glass in three dimensions. From a topological point of view, a $3d$ lattice with N points can be thought of as the union of distinct cubes distributed in such a way that two of them have only one lattice point in common (the $3d$ equivalent of the $2d$ chessboard). In the thermodynamic limit their number is $\frac{1}{4}N$. Let us stress that in this case the six plaquettes of each cube represent the 'black' regions for a total of $\frac{3}{2}N$, while there are $\frac{3}{2}N$ 'white' plaquettes between cubes.

It is easy to realize that there exists a different, but equivalent, change of random variables with respect to the one previously described, which involves $n_p = 5$ 'black' plaquette frustrations for each cube (the neglected plaquette must always be the same), together with other plaquette frustrations in the 'white' areas, for a total of $2N$ independent random variables. After performing the 'white' annealed average, the problem is reduced to the calculation of the quenched free energy of a single cube. In this case we obtain a constrained annealed average of the free energy of the global system f_{ac}^{cube}, where the constraints are imposed over the products of the J_{ij}'s on all the closed loops of bonds in the $\frac{1}{4}N$ independent cubes. The resulting f_{ac}^{cube} has a very long analytical expression, and it is shown in Fig. 18, together with the annealed free energy, for comparison. Unfortunately, the residual entropy is still negative, indicating that in three dimensions more constraints are necessary to get a fair approximation of the quenched system at low temperature. The ground state energy can be estimated using (5.16) as $E_0(d = 3) \geq -1.917$ which improves the lower bounds obtained by supremum of (5.46), $E_0 \geq -1.939$, and by the supremum of the annealed free energy, $E_0 \geq -1.956$.

6.2. *Constrained annealed averages on infinite loops*

The method introduced in Sec. 6.1 can be extended to obtain very accurate estimates (lower bounds) of the quenched free energy using products of random matrices. Indeed, the main limitation we have met is that the number of plaquettes n_p of the sub-system should be not too large in order to perform the calculation of its quenched free energy in a reasonable computer time. For instance in $2d$ we have stopped the estimates at the elongated cross with $n_p = 8$ plaquettes.

However, we can consider infinite loops by estimating the free energy of sub-systems of infinite size only in one direction, via the Lyapunov exponent of the product of random transfer matrices.

To simplify, let us consider again the two-dimensional Ising model with $J_{ij} = \pm 1$ with equal probability, although our discussion can be extended to three and higher dimensions. The idea is to find an independent decomposition of the $2d$ lattice in strips of size L. We then compute the quenched free energy of the strip as

the Lyapunov exponent of the infinite product of random transfer matrices of size $2^L \times 2^L$. In this way, we automatically obtain the annealed average where the products of J_{ij}'s on all the possible loops inside the strip are constrained to their quenched mean value. In particular, some of the loops are of infinite size in one direction. Our proposal requires a numerical calculation of the Lyapunov exponent, but it is superior to the direct application of the transfer matrix method,[6] since it allows one to obtain lower bounds of the free energy which become more and more accurate at increasing the size L of the strips, as a consequence of the standard inequalities satisfied by the constrained annealed averages.

In order to reduce the number of random variables, it is convenient to perform a gauge transformation, before computing the free energy of the spin glass. In two dimensions, we have chosen a gauge transformation which map the original system into a new one where the horizontal couplings are i.i.d. random variables ($J_{ij} = \pm 1$ with equal probability) and the vertical couplings are positive and constant ($J_{ij} = 1$). We consider strips of L spins which are parallel to the bisectrix of the lattice as shown in Fig. 20, in order to obtain a convenient independent decomposition of the lattice with the properties discussed at the beginning of Sec. 6.1. The basic cell of the strip is formed by three layers, the first and the third ones of length L and the intermediate layer of length $L+1$. We denote by σ the spins of the first layer, ξ the spins of the second layer and η the spins of the third layer. The particular form of the strip allows one to perform a preliminary analytic integration over the spins of intermediate layers. For instance, the case $L = 3$ is illustrated in Fig. 20. The partition function of that strip can be obtained by a product of transfer matrices between the first and the third spin layers:

$$\widetilde{Z}_N(L=3) = \mathrm{Tr} \prod_{i=1}^{N} \mathbf{T}(i; J_1, \cdots, J_6) \tag{6.14}$$

where there are 2^6 different random transfer matrix $\mathbf{T}$. As the Boltzmann factor $\exp(-\beta H)$ of three consecutive layers of the $L = 3$ strip is

$$\exp(\sigma_1\xi_1 + \sigma_2\xi_2 + \sigma_3\xi_3)\, \exp(J_1\sigma_1\xi_2 + J_2\sigma_2\xi_3 + J_3\sigma_3\xi_4)$$

$$\exp(\eta_1\xi_2 + \eta_2\xi_3 + \eta_3\xi_4)\, \exp(J_4\eta_1\xi_1 + J_5\eta_2\xi_2 + J_6\eta_3\xi_3)$$

one immediately sees that, after integrating out the intermediate lay (the four ξ spins), the elements of the 8×8 transfer matrices are

$$\begin{aligned} T_{\sigma_1,\sigma_2,\sigma_3,\eta_1,\eta_2,\eta_3} = {} & \cosh(\sigma_1 + J_4\eta_1)\cosh(J_1\sigma_1 + J_5\eta_2 + \sigma_2 + \eta_1) \\ & \times \cosh(J_2\sigma_2 + J_6\eta_3 + \sigma_3 + \eta_2)\cosh(J_3\sigma_3 + \eta_3)\,. \end{aligned} \tag{6.15}$$

The extension to a generic value of L is straightforward, and one has to deal with the product of 2^{2L} independent random matrices of size $2^L \times 2^L$.

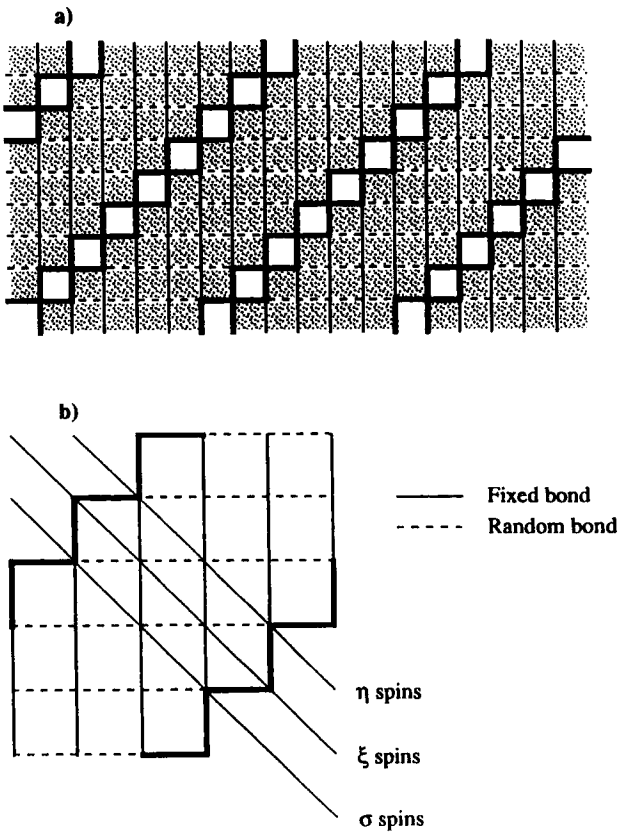

Fig. 20. Ising model with nearest neighbour interactions on a $2d$ lattice: (a) decompositions of the set of the couplings into independent subsets of infinite area (the black strips) in the case $L = 3$, (b) the basic cell of the strip ($L = 3$) is formed by three layers: σ, ξ and η spins; after the gauge transformation the vertical couplings are fixed ($J_{ij} = +1$, full lines) and the horizontal couplings are random ($J_{ij} = \pm 1$, dashed lines).

The Lyapunov exponent λ of the product of the transfer matrices is related to the quenched free energy $f_{\text{strip}}^{(L)}$ by

$$\lambda = -n_s \beta f_{\text{strip}}^{(L)} . \tag{6.16}$$

A moment of reflection shows that there are $n_s = 2L + 1$ spins and $n_j = 4L$ couplings, so that from (6.7) one obtains the constrained annealed free energy $f_{ac}^{(L)}$ of the total system:

$$-\beta f_{ac}^{(L)} = \frac{\lambda - \ln 2}{2L} . \tag{6.17}$$

The constrained free energy as a function of the temperature for $L = 1$, $L = 2$, $L = 3$ and $L = 8$ is shown in Fig. 21. It is evident that the difference between the constrained annealed free energy $f_{ac}^{(L)}$ and the quenched one increases at lowering the temperature, since the frustration effect becomes more and more important. In two dimensions at varying the size of the transversal length L we found a monotonous convergent sequence of approximations from below to the ground state energy

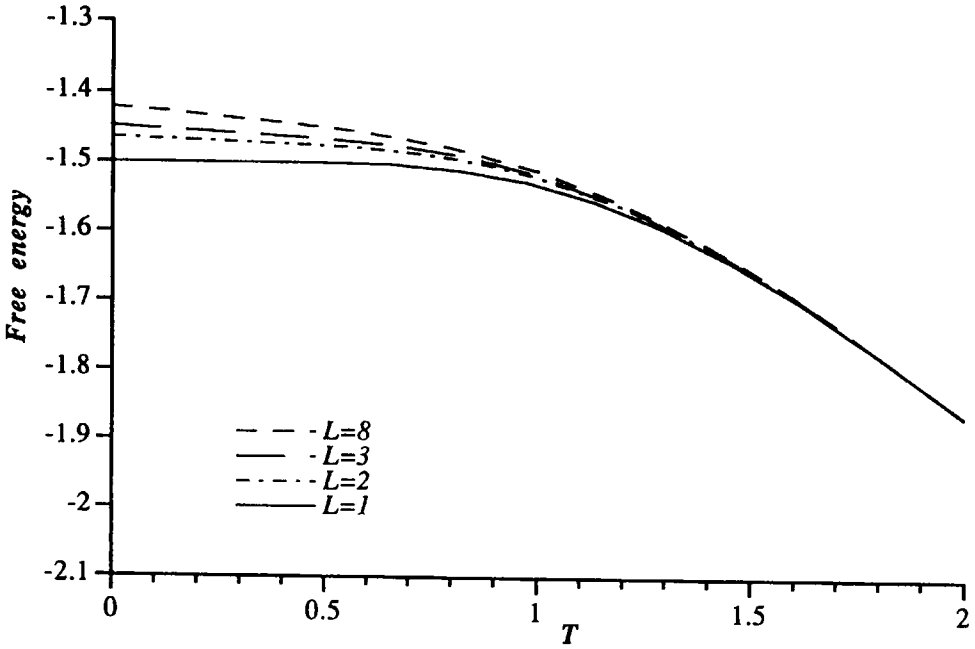

Fig. 21. $2d$ Ising model with nearest neighbour interactions: quenched free energy of the system obtained by a decomposition of the lattice into independent strips, as function of the temperature T at different widths $L = 1, 2, 3, 8$ of the strips.

and to the residual entropy:

$L = 1$	$E_0 = -1.500$	$S_0 = 0.00$
$L = 2$	$E_0 = -1.464$	$S_0 = 0.02$
$L = 3$	$E_0 = -1.448$	$S_0 = 0.03$
$L = 4$	$E_0 = -1.438$	$S_0 = 0.04$
$L = 5$	$E_0 = -1.429$	$S_0 = 0.05$
$L = 6$	$E_0 = -1.425$	$S_0 = 0.05$
$L = 7$	$E_0 = -1.423$	$S_0 = 0.05$
$L = 8$	$E_0 = -1.421$	$S_0 = 0.05$
$L = 9$	$E_0 = -1.419$	$S_0 = 0.06$

7. Conclusions

In this paper we have presented a general tool for estimating the free energy of disordered systems. Far from being a simple quantitative approach, the method leads to a deeper understanding of the role of the intensive variables of the disorder. Its range of applicability is extremely wide, varying from infinite range models to products of transfer random matrices and it can compete with the replica trick for practical purposes. The constrained annealing is able to transform a system with quenched disorder into a new model without disorder, although it depends on new

variables beside the temperature, the Lagrange multipliers. It is thus necessary to perform a maximization to choose the appropriate value of these variables.

The major problem is its application to Ising spin glasses in three or more dimensions where replica symmetry breaking is expected. In this case the ideas of Secs. 5 and 6 should be adapted to dimensions higher than two. The simplest generalization is to fix the frustration on the elementary plaquettes composed of four independent couplings. We have seen that this can be done in $2d$ by choosing the black plaquettes of a chessboard. In this way one obtains the result that any coupling belongs to one and only one of the plaquettes. This can be done in $3d$, too, although there is a physical breakdown at low temperature where the entropy becomes negative.

We have also shown that the constrained annealing in $\pm J$ spin glasses can be performed even without Lagrange multipliers, by performing quenched averages on sub-systems of small size. However, it is an open problem to extend it to other coupling distributions such as the Gaussian distribution. The equivalence between the two methods has rather deep consequences. Indeed, we do not expect to find a transition to a glassy phase in the approximations of Secs. 5 and 6.1, where the constrained annealed free energies are analytic functions of the inverse temperature β, since they can be expressed as the sum of a finite number of terms. Analyticity breaking might appear when there is an infinite number of terms in the sum, and one is tempted to conjecture that this corresponds to the necessity of considering infinite loops as we have done in Sec. 6.2, by infinite product of random transfer matrices.

In conclusion, more constraints have to be imposed. The first step will be to fix the frustration on interacting square plaquettes and not only on independent plaquettes. This is possible in an analytic way and the corresponding constrained annealed systems is given by a decorated Ising model without disorder. In our opinion the solution of such a model or even its numerical study is quite an interesting open problem. A more ambitious goal is to fix the frustration of larger plaquettes of 6, 8, ... and more couplings. In fact, fixing the frustration on plaquettes of all sizes corresponds to recover the exact quenched result. This program is now under study and we hope that it might become a new tool for the characterization of the glassy phase in disordered systems.

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Appendix A1. Large Deviations

A1.1. *Mathematical definition of the model*

We consider an Ising spin glass system on the d-dimensional lattice $\mathbb{Z}^d$ with nearest neighbour interactions. The Hamiltonian which defines the model is

$$H_\Lambda = -\sum_{i,j} J_{ij}\sigma_i\sigma_j \tag{A1.1}$$

where Λ is a finite subset of $\mathbb{Z}^d$ with cardinality $|\Lambda|$ and the sum is on nearest neighbours sites i, j in Λ. The $\{J_{ij}|\, i,j \in \mathbb{Z}^d\}$ are random couplings and the $\{\sigma_i|i \in \mathbb{Z}^d\}$ are spin variables which can take values $+1$ or -1. In the following the couplings J_{ij} will be chosen to be i.i.d. random variables.

The associated partition function is obtained as a sum over all the configurations of $\sigma = \{\sigma_i|i \in \Lambda\}$

$$Z_\Lambda(\beta) = \sum_{\{\sigma_i\}} \exp\{-\beta H_\Lambda\} \tag{A1.2}$$

where $\beta > 0$ is the inverse of the temperature. The free energy density is then defined by the limit

$$f(\beta) = -\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \ln\{Z_\Lambda(\beta)\}\,. \tag{A1.3}$$

The thermodynamic limit (A1.2), performed in the sense of Van Hove, was shown to exist with probability one for very general coupling distributions by different authors (see Refs. 32, 30, 18 and 13). Their proofs are not limited to nearest neighbour couplings but they extend to the case of generic short-range interactions. Moreover, the same authors rigorously proved the self-averaging property: the free energy density $f(\beta)$ equals the mean free energy density $\overline{f(\beta)}$ almost surely.

In this appendix we discuss the large deviation properties of the model. We refer to Refs. 4, 30 and 31 for some already established rigorous results on disordered systems and in particular on spin glasses.

A1.2. *Large deviation upper bounds*

Let us consider a sequence $\{\Lambda_N\}$ of regular cubes of increasing size with $|\Lambda_N| = N$ and such that $\Lambda_N \subset \Lambda_{N+1} \nearrow \mathbb{Z}^d$, and the associated sequences of random variables $\{Z_{\Lambda_N}\}$. We then define the moment-generating function

$$\phi_N(t) \equiv \frac{1}{N} \ln \overline{(Z_{\Lambda_N})^t}\,. \tag{A1.4}$$

It is well-known that if the moment-generating function

$$\phi(t) \equiv \lim_{N\to\infty} \phi_N(t) \tag{A1.5}$$

exists finite for all $t \in \mathbb{R}$, then the following large deviation upper bound

$$\limsup_{N\to\infty} \frac{1}{N} \ln Q_N(K) \leq - \inf_{y\in K} R(y) \tag{A1.6}$$

is verified for each closed set $K \subset \mathbb{R}$, where Q_N is the distribution of the random variable $-y_N \equiv -\frac{1}{N\beta} \ln(Z_{\Lambda_N})$ on $\mathbb{R}$. The random variable $y_N(\beta)$ is the finite volume free energy density and $R(y)$ is the rate function given by the Legendre–Fenchel transform (see Ref. 11)

$$R(y) \equiv \sup_{t\in\mathbb{R}}(-t\beta y - \phi(t)) . \tag{A1.7}$$

In the physical literature the moment-generating function and its Legendre transform are widely studied in the context of the so-called multifractal approach (see Ref. 22 for a review). We will prove the following result which, as remarked, implies (A.16):

Theorem A1.1: Assume there is a function λ: $\mathbb{R}^+ \to \mathbb{R}^+$ such that one has $\exp\{-\lambda(u)\} \leq \overline{\exp\{-u|J_{ij}|\}}$ and $\overline{\exp\{+u|J_{ij}|\}} \leq \exp\{+\lambda(u)\}$, then the function ϕ defined in (A1.5) exists finite on $\mathbb{R}$.

Proof: Let's subdivide the cube Λ_N in sub-cubes $\{\Lambda_M^i | i = 1, \ldots, k^d\}$ of cardinality $|\Lambda_M^i| = M \equiv m^d$ for all i so that $N = (km)^d = k^d M$. To any sub-cube of such decomposition it the partition function $Z^i \equiv Z_{\Lambda_M^i}$ is associated. From definitions (A1.1) and (A2.2) it is easy to state the following inequalities:

$$\left(\prod_{i=1}^{k^d} Z^i\right) \exp\left\{\beta \sum_{\langle i,j\rangle} -|J_{ij}|\right\} \leq Z_{\Lambda_N} \leq \left(\prod_{i=1}^{k^d} Z^i\right) \exp\left\{\beta \sum_{<i,j>} +|J_{ij}|\right\} \tag{A1.8}$$

where the sum runs over all the couples of nearest neighbour sites $\langle i, j\rangle$ with i and j in different sub-cubes. Since the couplings J_{ij} are independent random variables, we have from (A1.8)

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{k^d} \ln \overline{(Z^i)^t} + \frac{1}{N}\sum_{\langle i,j\rangle} \ln \overline{\exp\{-\beta|t||J_{ij}|\}} &\leq \frac{1}{N} \ln \overline{(Z_{\Lambda_N})^t} \\ \leq \frac{1}{N}\sum_{i=1}^{k^d} \ln \overline{(Z^i)^t} + \frac{1}{N}\sum_{\langle i,j\rangle} \ln \overline{\exp\{+\beta|t||J_{ij}|\}} & \end{aligned} \tag{A1.9}$$

moreover, since the couplings J_{ij} are identically distributed, we have $\overline{(Z^i)^t} = \overline{(Z^1)^t}$. This implies

$$\frac{1}{N}\sum_{i=1}^{k^d} \ln \overline{(Z^i)^t} = \frac{k^d}{N} \ln \overline{(Z^1)^t} = \frac{1}{M} \ln \overline{(Z^1)^t} . \tag{A1.10}$$

The total number of couples of nearest neighbour sites $\langle i, j\rangle$ with i and j in different sub-cubes is estimated from above by $dk^d m^{d-1}$. Therefore, using our hypotheses and inequality (A1.9) we get

$$\left|\frac{1}{N}\ln\overline{(Z_{\Lambda_N})^t} - \frac{1}{M}\ln\overline{(Z^1)^t}\right| \leq \frac{dk^d m^{d-1}}{N}\lambda(|t|\beta) = \frac{d}{m}\lambda(|t|\beta) \tag{A1.11}$$

i.e., with the notations of (2A1.4),

$$\left|\phi_N(t) - \frac{1}{M}\ln\overline{(Z^1)^t}\right| \leq \frac{c}{m} \tag{A1.12}$$

where $c = c(\beta, t, d) > 0$. The above estimate now implies

$$0 \leq \limsup_{N\to\infty}\phi_N(t) - \liminf_{N\to\infty}\phi_N(t) \leq \frac{2c}{m} \tag{A1.13}$$

for all m. The result follows. □

Remark A1.1: There is a well-known argument (e.g. [Ruelle]) which ensures that the existence of the limit for the sequence $\{\phi_N\}$ attached to the geometrical sequence of cubes $\{\Lambda_M\}$ implies the existence of of the thermodynamic limit for $\{\phi_N\}$ ($\Lambda \nearrow \mathbb{Z}^d$ in the sense of van Hove).

Remark A1.2: The result of the theorem 2.1 remains true, up to small technical modifications, for short-range interaction systems.

Remark A1.3: The hypothesis on the J_{ij}'s contained in the theorem defines a large class of distributions which includes all distributions with finite support and the Gaussian distribution.

Before concluding this section we give an alternative proof of the existence of the mean free energy density.

Theorem A1.2: Assume the hypothesis of theorem 2.1 and moreover assume $\lambda(u) = \mathcal{O}(u)$ as $u \to 0$ then the derivative of $\phi(t)$ at $t = 0$ exists and equals $-\beta\overline{f(\beta)}$.

Proof: by (2.11) and the hypothesis on λ we have

$$\left|\frac{\phi_N(t)}{t} - \frac{1}{Mt}\ln\overline{(Z^1)^t}\right| \leq \frac{c}{m} \tag{A1.14}$$

with $c = c(\beta, d) > 0$ and $|t|$ small enough. Letting $N \to \infty$ (as before M is kept fixed), by theorem A1.1 we have that (A1.14) also holds with $\phi(t)$ replacing $\phi_N(t)$.

By a first order Taylor expansion of the exponential and logarithmic functions we obtain

$$\ln\overline{(Z^1)^t} = (t + o(t))\overline{\ln(Z^1)} \tag{A1.15}$$

for $|t|$ sufficiently small.

Therefore

$$\left|\frac{\phi(t)}{t} - \left(\frac{1}{M} + \frac{o(t)}{t}\right) \ln \overline{(Z^1)}\right| \leq \frac{c}{m} \tag{A1.16}$$

which implies

$$0 \leq \limsup_{t\to\infty} \frac{\phi(t)}{t} - \liminf_{t\to\infty} \frac{\phi(t)}{t} \leq \frac{2c}{m} \tag{A1.17}$$

for all m. Since $\phi(0) = 0$ the differentiability of ϕ at $t = 0$ follows. Now we take in (A1.16) the limit for $t \to 0$ and we get

$$\left|\phi'(0) - \overline{\frac{1}{M} \ln(Z_{\Lambda_M^1})}\right| \leq \frac{c}{m} \tag{A1.18}$$

(in order to stress the dependence on M we avoided here the use of the short notation Z^1). Then for $m = M^{\frac{1}{d}} \to \infty$ we obtain the result. □

Remark A1.4: Since $\phi(t)$ is a convex function vanishing at the origin one has $f(\beta) \geq -\frac{1}{\beta}\frac{\phi(t)}{t}$ for any positive t. In particular one has the well-known lower bound $f(\beta) \geq f_a(\beta) \equiv -\frac{1}{\beta}\phi(1)$.

A1.3. *Large deviation lower bounds*

The problem of the large deviation upper bound is still open in dimension $d > 1$, while in one dimension has a trivial answer. Let us consider, in fact, the partition function of the one-dimensional Ising spin glass

$$Z_N = \sum_{\{\sigma_i\}} \prod_{\langle i\rangle} \exp\{\beta J_i \sigma_i \sigma_{i+1}\} = 2^N \prod_{i,j} \cosh\{\beta J_{ij}\} \tag{A1.19}$$

since the variables J_i are i.i.d. we immediately obtain

$$\phi(t) = \lim_{N\to\infty} \frac{1}{N} \ln \overline{(Z_N)^t} = t \ln 2 + \ln\{\overline{(\cosh\{\beta J_i\})^t}\} \tag{A1.20}$$

where J_i is any one of the couplings. Let us recall that Eq. (A1.7) gives the entropy function $R(y)$ as a Legendre–Fenchel transform of $\phi(t)$ computed in (A1.20). Assume now a distribution of the couplings J_i such that the expectation in (A1.20) is differentiable for all real t. In this case, the function $\phi(t)$ is also differentiable for all real t and the following inequality holds:

$$\liminf_{N\to\infty} \frac{1}{N} \ln Q_N(G) \geq -\inf_{y\in G} R(y) \tag{A1.21}$$

for each open set G in $\mathbb{R}$. This inequality is the large deviation lower bound for Ising spin glasses in one dimension. The extension of this result to higher dimensions is an open problem. The large deviation upper bound previously stated, on the contrary, holds for Ising spin glasses in arbitrary dimension.

Appendix A2. Large Deviations for Generalized Free Energies

In this section we consider again an Ising spin glass system on the d-dimensional lattice $\mathbb{Z}^d$ with first neighbours interaction. The random couplings $\{J_{ij}\}$ are assumed to satisfy the condition of theorem A1.1.

Let ψ: $\mathbb{R} \to \mathbb{R}$ be a function such that the $\{\psi(J_{ij})\}$ are i.i.d. random variables which satisfy $\overline{\psi(J_{ij})} = 0$. We then define a class of finite volume generalized free energy densities:

$$y_\Lambda^\mu \equiv y_\Lambda + \frac{\mu}{\beta}\alpha_\Lambda \tag{A2.1}$$

$$= -\frac{1}{\beta N} \ln \left[\sum_{\{\sigma_i n\}} \exp\{-\beta H_\Lambda - \mu \alpha_\Lambda N\} \right] \tag{A2.2}$$

where

$$\alpha_\Lambda \equiv \frac{1}{N} \sum_{i,j} \psi(J_{ij}) \,. \tag{A2.3}$$

As before, Λ is a finite subset of $\mathbb{Z}^d$ with cardinality $|\Lambda| = N$ and the sum is on nearest neighbours sites i, j in Λ. The real variable μ has to be seen as a Lagrange multiplier, we have.

Proposition A2.1: For any real μ the following equality holds:

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \overline{y_\Lambda^\mu} = \overline{f(\beta)} \,. \tag{A2.4}$$

Proof: Only one has to take into account that $\overline{\alpha_\Lambda} = 0$ for any subset Λ. □

Proposition A2.2: Assume that the i.i.d. random variables $\{\psi(J_{ij})\}$ satisfy the condition $\mathbb{E}[|\psi(J_{ij})|] < \infty$, then the following limit holds almost surely:

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} y_\Lambda^\mu = f(\beta) \,. \tag{A2.5}$$

Proof: It is sufficient to remark that the strong law of large numbers implies

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \alpha_\Lambda = 0 \tag{A2.6}$$

almost surely. □

We consider now, as before, a sequence $\{\Lambda_N\}$ of regular cubes of increasing size with $|\Lambda_N| = N$ s.t. $\Lambda_N \subset \Lambda_{N+1} \nearrow \mathbb{Z}^d$. We can define:

$$\alpha_N \equiv \alpha_{\Lambda_N} \tag{A2.7}$$

and

$$y_N^\mu \equiv y_{\Lambda_N}^\mu \,. \tag{A2.8}$$

By virtue of propositions 3.1 and 3.2, the generalized free energy densities y_N^μ, are equivalent to ordinary free energy density in the thermodynamic limit. The finite volume fluctuations, on the contrary, depend on μ.

We are now interested to give estimates on the large deviations of the y_N^μ. Define

$$\phi_N^\mu(t) \equiv \frac{1}{N} \ln \overline{(Z \exp\{-\mu\alpha_N N\})^t}. \tag{A2.9}$$

The following theorem shows that under suitable conditions the function

$$\phi^\mu(t) \equiv \lim_{N\to\infty} \phi_N^\mu(t) \tag{A2.10}$$

exists finite for all $t \in \mathbb{R}$.

Theorem A2.1: Let ϵ be a real number for which there is a function λ^ϵ: $\mathbb{R}^+ \to \mathbb{R}^+$ such that the following inequalities: $\exp\{-\lambda^\epsilon(u)\} \leq \overline{\exp\{-u|J_{ij}| + \epsilon u\alpha(J_{ij})\}}$ and $\overline{\exp\{+u|J_{ij}| + \epsilon u\alpha(J_{ij})\}} \leq \exp\{+\lambda^\epsilon(u)\}$ hold. Then for $\mu = \epsilon\beta$ the function ϕ^μ defined in () exists finite on $\mathbb{R}$.

Proof: The proof is a trivial modification of the proof of Theorem 2.1. □

Theorem A2.2: Assume the hypothesis of theorem 3.1 and moreover assume that $\lambda^\epsilon(u) = \mathcal{O}(u)$ as $u \to 0$, then the derivative of $\phi^\mu(t)$ at $t = 0$ exists and equals $-\beta f(\beta)$.

Proof: The proof is the same as in Theorem 2.2. □

The existence of the function $\phi^\mu(t)$ for any real t (notice that $\phi^0(t) = \phi(t)$ where $\phi(t)$ is defined in the previous appendix) allows us to estimate the large deviations of the variables $y^\mu - N$. In fact, the following large deviation upper bound is verified

$$\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^\mu(K) \leq -\inf_{y\in K} R^\mu(y) \tag{A2.11}$$

for each closed set $K \subset \mathbb{R}$, (where Q_N^μ is the distribution of y_N^μ) on $\mathbb{R}$. The rate function $R^\mu(y)$ is given by the Legendre–Fenchel transform

$$R^\mu(z) \equiv \sup_{t\in\mathbf{R}}(-t\beta y - \phi^\mu(t)) \tag{A2.12}$$

We can give to our results the following interpretation: the multiplicative term $\exp\{\mu\alpha_N N\}$ has the effect of a constraint on the disorder which modifies the distribution of the finite volume free energy.

Remark A2.1: Since $\phi^\mu(t)$ is a convex function of t, vanishing at $t = 0$, one has $f \geq -\frac{1}{\beta}\frac{\phi^\mu(t)}{t}$ for any $t \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$.

Remark A2.2: Notice that if the $\{J_{ij}\}$ have distribution of the type described in remark 2.3 then theorem 2.1 holds and Γ includes the origin.

Remark A2.3: It is in general possible to extend the previous results to the case of short-range interactions and to the case where the function ψ depends not only on the variable J_{ij} but also on the variables $J_{i'j'}$ with $i'j'$ in a short range around ij.

Remark A2.4: The results can also be easily extended to the case where a vectorial function α_N and a vectorial Lagrange multiplier μ are considered. In this case the exponent $\mu\alpha$ will be the scalar product $\mu\alpha = \mu_1\alpha_1 + \mu_2\alpha_2 + \ldots + \mu_n\alpha_n$.

Appendix A3. Constraints on n-ples of Matrices

In this appendix, with reference to Sec. 3, we give a proof that the calculation of annealed averages where the frequency of the different n-ples of matrices is fixed to the law of large numbers can be reduced to the determination of the largest eigenvalue of a $2^n \times 2^n$ matrix $\mathbf{G}_n$. It is possible to show that in order to impose the constraints on the n-ples, we have to compute quantities of the type

$$\mathrm{Tr} \sum_{\eta_1=\pm1,\cdots\eta_N=\pm1} \prod_{i=1}^{N} \mathbf{g}(\eta_i, \cdots, \eta_{i+n-1}) \tag{A3.1}$$

where $\mathbf{g}(\eta_i, \cdots, \eta_{i+n-1})$ is a 2×2 matrix depending on n parameters $\eta_i \cdots \eta_{i+n-1}$ and η_i are random independent variables which can assume 2 different values ± 1 and with periodic boundary conditions $\eta_i = \eta_{i+N}$. The generalization to variables η_i which can assume M different values and to a matrix $\mathbf{g}$ of size $D \times D$ is straightforward.

Our goal is to write (A3.1) as the trace of a transfer matrix $\mathbf{G}^N$. For $n = 1$ (the case of one constraint) the result is trivial with

$$\mathbf{G}_{n=1} = \sum_{\eta=\pm1} \mathbf{g}(\eta) = \frac{1}{2}\mathbf{A}_1 e^{\mu_1} + \frac{1}{2}\mathbf{A}_{-1} e^{-\mu_1} . \tag{A3.2}$$

$\mathbf{G}_1$ is thus a matrix of the same size of $\mathbf{g}$ and L_1 is the logarithm of its largest eigenvalue. The second equality refers to the matrices considered in Sec. 3 and there used to compute $f_{ac}^{(1)}$. The key point is that when $\mathbf{g}$ depends on more than one variable η_i, we can write a transfer matrix $\mathbf{G}$ of size larger than $\mathbf{g}(\eta_i, \cdots, \eta_{i+n-1})$, although many entries of $\mathbf{G}$ are zero. The matrix $\mathbf{G}$ should be built up as a block matrix at starting from the different matrices $\mathbf{g}(\eta_i, \cdots, \eta_{i+n-1})$. From the parameters $(\eta_i \cdots \eta_{i+n-1})$ we get the arrow and column index of the single block, in such a way that the former becomes the latter when $i \to i+1$. We thus obtain a 2×2 block of $\mathbf{G}$ as

$$\mathbf{G}_{(\eta_i \cdots \eta_{i+n-2}),(\eta_{i+1} \cdots \eta_{i+n-1})} = \mathbf{g}(\eta_i \cdots \eta_{i+n-1}) . \tag{A3.3}$$

The remaining blocks are 2×2 zero matrix, so that one has ($e_i = \pm 1$)

$$\mathbf{G}_{(\eta_i \cdots \eta_{i+n-2}),(e_i \cdots e_{i+n-2})} = \mathbf{g}(\eta_i \cdots \eta_{i+n-2},\, e_{i+n-2}) \prod_{s=i+1}^{i+n-2} \delta_{\eta_s, e_{s-1}} \tag{A3.4}$$

where $\delta_{a,b}$ is the 2×2 identity matrix if $a = b$ and the zero matrix otherwise. In conclusion $\mathbf{G}$ is a matrix of size 2^n, and each arrow has only 4 non-zero entries (only for $n = 2$ all the elements of $\mathbf{G}$ are non-zero).

Using (A3.3) and (A3.4), the trace of $\mathbf{G}^N$ is given by ($\xi_i = \pm 1$)

$$\mathrm{Tr} \sum_{\xi_1^{(1)} \cdots, \xi_{n-1}^{(1)}} \cdots \sum_{\xi_1^{(N)}, \cdots, \xi_{n-1}^{(N)}} \left(\prod_{i=1}^{N} \prod_{s=2}^{k-1} \delta_{\xi_s^{(i)}, \xi_{s-1}^{(i+1)}} \right) \left(\prod_{i=1}^{N} \mathbf{g}(\xi_1^{(i)} \cdots \xi_{n-1}^{(i)},\, \xi_{n-1}^{(i+1)}) \right). \tag{A3.5}$$

It is easy to see that in (A3.5) there are N groups of $n-1$ indices ξ which are equal, because of the various δ matrices. In particular one has

$$\xi_m^{(k)} = \xi_1^{(k+m-1)} \tag{A3.6}$$

and, as a consequence,

$$\mathrm{Tr}\, \mathbf{G}^N = \mathrm{Tr} \sum_{\xi_1^{(1)} \cdots \xi_1^{(N)}} \prod_{i=1}^{N} \mathbf{g}(\xi_1^{(i)} \cdots \xi_1^{(i+n-1)}) \tag{A3.7}$$

which is equal to (A3.1). This concludes the proof.

Let us finally write the form of $\mathbf{G}$ for $n = 2$ (constraint on the couples of matrices) and $n = 3$ which are used in the paper to determine $f_{ac}^{(2)}$ and $f_{ac}^{(3)}$. In the former case we have the 4×4 matrix

$$\mathbf{G}_{n=2} = \begin{pmatrix} \mathbf{g}(+1,+1) & \mathbf{g}(+1,-1) \\ \mathbf{g}(-1,+1) & \mathbf{g}(-1,-1) \end{pmatrix} \tag{A3.8}$$

with the 2 block matrices

$$\mathbf{g}(\eta_1, \eta_2) = \frac{1}{2} \mathbf{A}_{\eta_1} e^{-\mu_1 \eta_1 - \mu_2 \eta_1 \eta_2} \tag{A3.9}$$

while in the latter case we have the 8×8 matrix

$$\mathbf{G}_{n=3} = \begin{pmatrix} \mathbf{g}(+1,+1,+1) & \mathbf{g}(+1,+1,-1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & g(+1,-1,+1) & \mathbf{g}(+1,-1,-1) \\ \mathbf{g}(-1,+1,+1) & \mathbf{g}(-1,+1,-1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{g}(-1,-1,+1) & \mathbf{g}(-1,-1,-1) \end{pmatrix} \tag{A3.10}$$

with the 2×2 block matrices

$$\mathbf{g}(\eta_1, \eta_2, \eta_3) = \frac{1}{2} \mathbf{A}_{\eta_1} e^{-\mu_1 \eta_1 - \mu_2 \eta_1 \eta_2 - \mu_3 \eta_1 \eta_3 - \mu_4 \eta_1 \eta_2 \eta_3}. \tag{A3.11}$$

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