RELATIVISTIC QUANTUM MECHANICS AND PATH INTEGRAL FOR KLEIN-GORDON EQUATION

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Abstract. The subject of the paper is a path integral rapresentation for the semigroup $\{e^{-tH_1}\}_{t\geq 0}$ generated by the quantum Hamiltonian H_1 of a relativistic spinless particle in an external electromagnetic field. The result is compared with the "Feynman-Kac" formula which holds for relativistic Schrödinger operators.

1. Introduction

A sensible physical model depicts ordinary bodies as collections of pointlike nuclei and electrons attracting or repelling each other by Coulomb forces. According to classical physics atoms should be unstable against collapse because their classical energy is not bounded from below but according to quantum mechanics they have a ground state energy $E_{\min} > -\infty$. This result doesn't solve the stability problem for matter in bulk. Due to the long range of Coulomb interaction why two separated pieces of matter do not feel each other's influence ? The point is that even if bodies are globally neutral it is not obvious that they cannot become polarized in such a way as to attract or repel each other. An important facet of stability is the saturation property of the binding energy per particle i.e. the requirement that for a system of N bodies not only the ground state energy $E_{\min}(N)$ is finite (stability of the first kind) but also that $E_{\min}(N) \sim -C\,N$ for some positive constant C (stability of the second kind). In order to appreciate this point let suppose for instance that $E_{\min}(N) \sim -C\,N^r$ with $r > 1$. Then bringing

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together two large pieces of matter each containing say 10^{23} particles, would release a huge amount of energy $\Delta E = 2C(2^{r-1} - 1)10^{23r}$ therefore bodies should collapse under Coulomb interaction! The problem of the stability for a system of point charges was raised about forty years after the birth of quantum mechanics by Fisher and Ruelle[10] and the answer was given only in the late sixties and early seventies beginning with two papers of Dyson and Lenard[8,9,14]. One can simplify the theory of N electrons interacting with K nuclei by assuming (Born-Oppenheimer approximation) that nuclei have an infinite mass and are located at fixed but arbitrary space points $\mathbf{R}_1, \ldots, \mathbf{R}_K$. Due to the large mass ratio between nuclei and electrons such an assumption is not a bad one, moreover it gives an exact lower bound because allowing a finite mass can only raise the energy. Now the system consists of N (Coulomb interacting) electrons in the static electric field

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{K} \frac{Z_i|e|}{|\mathbf{r} - \mathbf{R}_i|^3}(\mathbf{r} - \mathbf{R}_i) \tag{1.1}$$

generated by K positive charges $Z_1|e|, \ldots, Z_K|e|$ and its Hamiltonian H_N is given by

$$H_N = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i - \sum_{i=1}^{N}\sum_{j=1}^{K}\frac{Z_j e^2}{|\mathbf{r}_i - \mathbf{R}_j|} + \sum_{1\le i<j\le N}\frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|} + \sum_{1\le i<j\le K}\frac{Z_i Z_j e^2}{|\mathbf{R}_i - \mathbf{R}_j|} \tag{1.2}$$

Dyson and Lenard demonstrated that the ground state energy of (1.2) satisfies the inequality $E_{\min}(N) \ge -C\,N$ for all $\mathbf{R}_1, \ldots, \mathbf{R}_K$ provided that electrons behave as fermions because if they obeyed Bose-Einstein statistics matter definitely wouldn't be stable as $E_{\min}(N) \sim -C'\,N^{7/5}$ in that case. The key point for the stability[14] rests on the fact that the kinetic energy $K(\rho)$ of a system of N fermions grows with its density $\rho(\mathbf{r}) = N\int |\psi(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_N)|^2\, d\mathbf{r}_2 \ldots d\mathbf{r}_N$ according to the law $K(\rho) \ge A\int_{\mathbb{R}^3} \rho(\mathbf{r})^{5/3} d^3\mathbf{r}$ for each antisymmetric N-particles wave function $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)$, a result which depends in a critical way on the nonrelativistic relation $K(\mathbf{p}) = |\mathbf{p}|^2/2m$ between kinetic energy and momentum embodied in Schrödinger's theory which is notoriously unphysical in presence of nuclei with high atomic number because relativistic corrections are not negligible in that case. In the spirit of Dyson and Lenard, stability of matter should proceed from Quantum Electrodynamics which, however, is beyond an effective mathematical control in the sense of constructive field theory. The main present approach to the stability of relativistic matter[15] rests upon the replacement of the operator $-\frac{\hbar^2}{2m}\Sigma_{i=1}^{N}\Delta_i$ by its relativistic version $\Sigma_{i=1}^{N}\sqrt{-\hbar^2c^2\Delta_i + m^2c^4}$

which defines the "relativistic Schrödinger operators" and, in particular, the one-body Hamiltonian $\tilde{H}_1 = \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} + e\,A^0$ for just one "electron" in the electric field $\mathbf{E} = -\nabla A^0$. We remark that the relevant relativistic corrections to one- electron energy levels in a Coulomb field proceed not only from kinematics but also from spin-orbit coupling and Darwin terms, therefore the adopted solution is physically suspect. Relativistic Schrödinger operators differ in many aspects from usual Schrödinger Hamiltonians. For instance, in the Coulomb field of a nucleus, the ground state energy of $\tilde{H}_1 = \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} - Ze^2/r$ collapses[12] to $-\infty$ when Z exceeds a critical value Z_c close to the inverse fine constant α^{-1} but there is[15] a theorem which relates the (second kind) stability of the many body Hamiltonian

$$\tilde{H}_N = \sum_{i=1}^{N} \sqrt{-\hbar^2 c^2 \Delta_i + m^2 c^4} - \sum_{i=1}^{N} \sum_{j=1}^{K} \frac{Z_j e^2}{|\mathbf{r}_i - \mathbf{R}_j|} + \sum_{1 \le i < j \le N} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|} + \sum_{1 \le i < j \le k} \frac{Z_i Z_j e^2}{|\mathbf{R}_i - \mathbf{R}_j|} \tag{1.3}$$

to the (first kind) stability of one "electron" in the Coulomb field of one atomic nucleus. In the latter case, the spectrum of $\tilde{H}_1$ is bounded from below[12] provided that $Z < Z_c = 2/\pi\alpha$ and the theorem says that if $Z_j < Z_c$ for all $j = 1, \ldots, K$, then $\tilde{H}_N$ exibits stability of second kind for fermions of spin 1/2 when $2\alpha \le 1/47$. Since the actual value of α is $\approx 1/137$, at least in our corner of Universe, it would seem that the theory of relativistic Schrödinger operators is in a good shape but what about their physical status? In other words is there any physically reasonable roughening of Q.E.D. leading to Hamiltonians (1.2)? In the Furry picture of Q.E.D. in presence of an external field A^{μ}_{ext} (for instance the Coulomb field of one or several infinite massive nuclei), the unperturbed Hamiltonian $H_{0,M}$ of "matter" is that of the quantized Dirac field or (for spinless particles) of the Klein-Gordon one interacting with A^{μ}_{ext}. Such "external field approximation" of Q.E.D. is a rough picture since accounts only for the interaction of each particle with A^{μ}_{ext} but not with the quantized electromagnetic field which is responsible for their mutual Coulomb repulsion. Besides that, the unperturbed theory may not one which conserves the number of bodies as an external electric field could create pairs[16] at non-zero rate in the static limit but when the field strenght is below some critical value $E_c \approx \frac{m^2 c^3}{e\hbar}$ the "Klein Paradox" can't occur and the Dirac (or Klein-Gordon) equation provides a well defined linear quantum field theory[17,18] and the only relevant operators are the one-particle and one-antiparticle Hamiltonians H_1, $\overline{H}_1$. They are not trivial as include the interaction with the external field and one can construct from them the "unperturbed" matter Hamiltonian $H_{0,M}$ by second quantization. We want to compare one-body relativistic Schrödinger operators $\tilde{H}_1 = \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} + V$ with one-particle Hamiltonians in the external field approximation of Q.E.D. We choose to do that precisely in the Klein-Gordon case because, for trivial reasons, the Klein-Gordon H_1 doesn't include any spin-orbit coupling exactly as $\tilde{H}_1$ and therefore one may believe that the two relativistic theories coincide at one-body level. Besides that, Klein-Gordon quantum mechanics in an external field is far less known than the Dirac one which, of course, is much more interesting from a physical point of view. What is the Hamiltonian of a relativistic spinless particle in an external electromagnetic field? Is it true that $H_1 = \sqrt{c^2(-i\hbar\nabla - e/c\,\mathbf{A})^2 + m^2c^4} + e\,A^0$ when $\mathbf{E} = -\nabla A^0 \neq \mathbf{0}$? A convenient approach to the problem is to study the semigroup $\{e^{-tH_1}\}_{t\geq 0}$, indeed we shall be able to give an explicit path integral representation of Euclidean Klein-Gordon propagators which, moreover, exploits ideas and techniques useful in the Dirac case[7]. We remark that relativistic Schrödinger semigroups $\{e^{-t\tilde{H}_1}\}_{t\geq 0}$ admit[2,3] the path- integral ($\hbar = c = m = 1$)

$$(e^{-t\tilde{H}_1}\psi)(\mathbf{x}) = e^{-t}\,\mathbb{E}_{\mathbf{x}}\left(\psi(\xi_t)\exp - \int_0^t V(\xi_s)\,ds\right) \tag{1.4}$$

where e^{-t} accounts for the rest energy while $t \mapsto \boldsymbol{\xi}_t$ is a Markov process in the three-dimensional space with generator $L = 1 - \sqrt{-\Delta + 1}$, a jump process at variance with the well known Feynman-Kac formula of nonrelativistic quantum mechanics where paths are those of a three- dimensional Brownian motion $t \mapsto \mathbf{X}_t$. It may be interesting to observe that $t \mapsto \boldsymbol{\xi}_t$ can be constructed[2,4] in terms of space-time diffusions. Indeed $\boldsymbol{\xi}_t = \mathbf{X}_{\tau_t}$, $\{\tau_t\}_{t\geq 0}$ being the family of Markov times defined by

$$\tau_t = \inf\{\, s \geq 0 \,:\, s + W_s^0 = t\,\} \tag{1.5}$$

where $s \mapsto W_s^0$ is an extra one- dimensional Brownian motion independent from $s \mapsto \mathbf{X}_s$ and starting from 0. In other words, if $X_s^0 = s + W_s^0$ and $s \mapsto X_s^\mu$ is the four-dimensional diffusion $X_s^\mu = (X_s^0, \mathbf{X}_s)$ (which starts from the space- time point $(0, \mathbf{x})$ when $s \mapsto \mathbf{X}_s$ starts from $\mathbf{x} \in \mathbb{R}^3$), then τ_t is the first hitting time of the hyperplane $\Sigma_t = \{(x^0, \mathbf{x}) \in \mathbb{R}^4 : x^0 = t\}$ by $s \mapsto X_s^\mu$ while the three-dimensional random variable $\boldsymbol{\xi}_t$ represents the space-coordinates of the point $X_{\tau_t}^\mu$ on the wordline $s \mapsto X_s^\mu$. The different physical status of H_1 and $\tilde{H}_1$ can be grasped by inspecting the path integrals of the corresponding semigroups which, as we shall see in the next section, are subtly different when the electric field is not trivial. We add the fact that (1.4) can be generalized[5] in a gauge-invariant way when a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is superimposed to the electric one $\mathbf{E} = -\nabla A^0$. By minimal coupling, $\tilde{H}_1 = \sqrt{c^2(-i\hbar\nabla - e/c\mathbf{A})^2 + m^2c^4} + e\,A^0$ and (1.4) becomes

$$(e^{-t\tilde{H}_1}\psi)(\mathbf{x}) = e^{-t}\,\mathbb{E}_{(0,\mathbf{x})}\Big(\psi(\boldsymbol{\xi}_t)exp - \Big\{e\int_0^t A^0(\boldsymbol{\xi}_s)\,ds + ie\int_0^{T_t}\mathbf{A}(\mathbf{X}_s)\cdot d\mathbf{X}_s\Big\}\Big) \tag{1.6}$$

if the vector potential $\mathbf{A}$ satifies $\nabla\cdot\mathbf{A} = 0$. One can notice at once that it displays a remarkable asimmetry between electric and magnetic contributions which doesn't fit relativistic covariance and may be a case against relativistic Schrödinger operators.

2. Klein-Gordon semigroups and their path integrals

According to the best tradition of constructive field theory in this section we take any $d \geq 2$ space-time dimensions as $d = 4$ plays no critical role. The corresponding dimension $d-1$ of space will be $n \geq 1$. We assume natural units $\hbar = c = m = 1$ and we consider the Klein- Gordon equation

$$\Box_d\varphi + 2ie\,A_\mu^{ext}\partial^\mu\varphi + (1 - e^2A_\mu^{ext}A^{ext\,\mu})\varphi + ie(\partial_\mu A_{ext}^\mu)\varphi = 0 \tag{2.1}$$

in an external electromagnetic field $F_{ext}^{\mu\nu} = \partial^\mu A_{ext}^\nu - \partial^\nu A_{ext}^\mu$ which we always take static in order to preserve invariance under time-translation. As it is well known, (2.1) cannot be interpreted as a simple relativistic version of the Schrödinger equation because the conserved current

$$J_\varphi^\mu = i\,(\overline{\varphi}(\partial^\mu + ieA_{ext}^\mu)\varphi - \varphi(\partial^\mu - ieA_{ext}^\mu)\overline{\varphi}) \tag{2.2}$$

has no probabilistic meaning since the charge density

$$\rho_\varphi = i\,(\overline{\varphi}(\partial^0 + ieA_{ext}^0)\varphi - \varphi(\partial^0 - ieA_{ext}^0)\overline{\varphi}) \tag{2.3}$$

is not positive. It is better to look at (2.1) as a field equation which, together with the canonical commutation relations for φ, defines an "external field problem"[17,18], the accepted name[18] for a general class of theories in which one or more quantized fields interact with classical "external sources" or "external fields". As they are given c-number functions of space-time coordinates, one neglects the back reaction of the quantum field upon the source itself, a sensible assumption for macroscopic sources but not too bad for a microscopic one much larger than field quanta, for instance an atomic nucleus with its surrounding Coulomb field. The linear quantum field theory (2.1) breaks[17] when the external electric field has a strength above the critical threshold for pair creation because a static field (which acts from $t = -\infty$) has been created an infinite number of particles at any time and the

Fock space formalism can't cope with that. Nevertheless, (2.1) makes sense for weak fields and there will be a (symmetric) Fock space $\mathcal{F}_s(\mathcal{D}_1) \otimes \mathcal{F}_s(\overline{\mathcal{D}}_1)$ built upon the one-particle and one-antiparticle Hilbert spaces $\mathcal{D}_1$, $\overline{\mathcal{D}}_1$. We shall give an explicit construction of $\mathcal{D}_1$ and of the one-particle Hamiltonian H_1 acting on $\mathcal{D}_1$ through a path-integral representation of its semigroup $P^t = e^{-tH_1}$. We begin with the free equation

$$(\Box_d + 1)\varphi = 0 \tag{2.4}$$

in order to make clear its quantum meaning. In that case $\mathcal{D}_1$ can be identified in a standard way[17] with the linear space of normalizable positive frequency solutions. They are the tempered distributions

$$\varphi(x) = (2\pi)^{-\frac{n}{2}} \int_{H^+} \psi(p) e^{-ip_\mu x^\mu} \mu(dp) \tag{2.5}$$

where $H^+ = \{p = (p^0, \mathbf{p}) \in \mathbb{R}^d : p^\mu p_\mu = 1, p^0 \geq 1\}$ is the positive mass shell equipped with the standard Lorentz-invariant measure $\mu(dp) \equiv d^n\mathbf{p}/2p^0$ and $\int_{H^+} |\psi(p)|^2 \mu(dp) < +\infty$. For such solutions the conserved charge $\int_{\{x^0=t\}} \rho_\varphi \, d^n\mathbf{x}$ defines a Lorentz-invariant Hilbert norm since

$$\|\varphi\|^2_{\mathcal{D}_1} = \int_{\{x^0=t\}} \rho_\varphi \, d^n\mathbf{x} = \int_{H^+} |\psi(p)|^2 \mu(dp) \tag{2.6}$$

The resulting Hilbert space carries an irreducible unitary representation of the Poincaré group with spin $s = 0$, therefore it describes[19] the quantum states of a relativistic spinless particle on which the free one-particle Hamiltonian group $\{e^{-itH_1}\}_{t\in\mathbb{R}}$ acts by time-translation

$$(e^{-itH_1}\varphi)(x^0, \mathbf{x}) = \varphi(x^0 + t, \mathbf{x})$$

We want to exibit another representation of $\mathcal{D}_1$ which is especially fit for the free semigroup $\{e^{-tH_1}\}_{t\geq 0}$ obtained from $\{e^{-itH_1}\}_{t\in\mathbb{R}}$ by the Wick rotation $t \to -it, t \geq 0$. We remark that positive frequency solutions are boundary values of holomorphic functions because, by looking at (2.5), it is easy to see that for $z = (z^0, z^1, \ldots, z^n) \in \mathbb{C}^d$, $\tilde{\varphi}(z) = (2\pi)^{-n/2} \int_{H^+} \psi(p) e^{-ip_\mu z^\mu} \mu(dp)$ is holomorphic when $\mathrm{Im}\, z$ belongs to the past open light cone $V_- = \{\eta \in \mathbb{R}^d : \eta_\mu \eta^\mu > 0, \eta^0 < 0\}$. The complex domain $\Omega = \{ z \in \mathbb{C}^d : \mathrm{Im}\, z \in V_- \}$ contains all points $z = (ix_d, x_1, \ldots, x_n)$ where $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $x_d < 0$. From now on $D \subset \mathbb{R}^d$ will be the domain $D = \{ (x_1, \ldots, x_d) = (\mathbf{x}, x_d) \in \mathbb{R}^d : x_d < 0\}$, with a boundary $\partial D = \{(\mathbf{x}, 0), \mathbf{x} \in \mathbb{R}^n\}$ which can be identified with the n- dimensional Euclidean space $\mathbb{R}^n$ and there is a one to one map between positive frequency solutions φ and Euclidean wave functions u defined through

$$u(\mathbf{x}, x_d) = (2\pi)^{-\frac{n}{2}} \int_{H^+} \psi(p) e^{(p^0 x_d + i\mathbf{p}\cdot\mathbf{x})} \mu(dp) \tag{2.7}$$

by restricting $\tilde{\varphi}(z)$ to the Schwinger points inside D. They are exponentially vanishing when $x_d \downarrow -\infty$ and satisfy the Euclidean free equation

$$(-\Delta_d + 1)u = 0 \tag{2.8}$$

by transition from the hyperbolic to the elliptic case through complex domains[13]. It easy to check that Euclidean wave functions belong to the Sobolev space $H^1(D)$ of square-integrable functions on D with square-integrable first derivatives which is the natural one to be considered[1] in searching for weak solutions of (2.8). By Plancherel's theorem and elementary integrations, it turns out that

$$\|\varphi\|^2_{\mathcal{D}_1} = \int_D (\Sigma^d_{i=1}|\partial_i u|^2 + |u|^2)dx \tag{2.9}$$

Since the left hand side of (2.9) is the standard squared norm in $H^1(D)$, we got a new representation of free one-particle states in which $\mathcal{D}_1$ is identified with the closed linear subspace of $H^1(D)$ consisting of all weak solutions of (2.8) normed according to

$$\|u\|^2_{\mathcal{D}_1} = \int_D (\Sigma^d_{i=1}|\partial_i u|^2 + |u|^2)\, dx \tag{2.10}$$

The free semigroup $\{P^t\}_{t\geq 0} = \{\exp -tH_1\}_{t\geq 0}$ acts on $\mathcal{D}_1$ by

$$(P^t\, u)(\mathbf{x}, x_d) = u^t(\mathbf{x}, x_d) = u(\mathbf{x}, x_d - t) \tag{2.11}$$

as it follows from Wick rotation. Formula (2.11) defines a reality preserving contractive semigroup on the whole $H^1(D)$ which, however, is *not* a self-adjoint one. It becomes self- adjoint only when *restricted* to the subspace of weak solutions which is invariant under translations. In order to check this important point we bound ourselves to real functions as $\{P^t\}_{t\geq 0}$ is reality preserving. Let u^t and v^τ be the time-translated of u and v by t and τ and

$$F(t,\tau) = (\, P^t u, P^\tau v\,)_{H^1} = \int_D (\,\Sigma^d_{i=1}\partial_i u^t \partial_i v^\tau + u^t v^\tau\,)\, dx \tag{2.12}$$

One can check that

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial \tau} = \int_D \Big((\,-\Delta_d u + u\,)^t \partial_d v^\tau - (\,-\Delta_d v + v\,)^\tau \partial_d u^t \Big)\, dx$$

therefore

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial \tau} = 0 \tag{2.13}$$

when u and v solve (2.8). It follows that $F(t,\tau) = G(t+\tau) \Rightarrow (\, P^t u, P^\tau v\,)_{H^1} = (\, P^\tau u, P^t v\,)_{H^1}$ for all $t, \tau > 0$ namely that $\{P^t\}_{t\geq 0}$ is self-adjoint. Because

$\{P^t\}_{t\geq 0}$ is a strongly continuous contractive semigroup, it has a positive self-adjoint generator H_1 which is the free one- particle Hamiltonian by definition. Now it is important to remark that when u_0 belongs to the half-integer Sobolev space $H^{1/2}(\mathbb{R}^n)$, an existence and uniqueness theorem holds[1] for the Dirichlet problem

$$\begin{cases} (-\Delta_d + 1)u = 0 & \text{in } D \\ u|_{\partial D} = u_0 \end{cases} \tag{2.14}$$

namely for each $u_0 \in H^{1/2}(\partial D)$ there exists one and only one weak solution of (2.14). By exploiting this one to one correspondence, finally we identify the one-particle space $\mathcal{D}_1$ with the Hilbert $H^{1/2}(\mathbb{R}^n)$ of boundary data equipped with the norm $\|u_0\|_{\mathcal{D}_1} = \|u\|_{H^1(D)}$ where u is the unique solution corresponding to u_0 and the free semigroup acts on $H^{1/2}(\mathbb{R}^n)$ by the obvious rule

$$P^t : u_0 \in H^{1/2}(\partial D) \mapsto u^t|_{\partial D}$$

In a more explicit way

$$(P^t u_0)(\mathbf{x}) = u(\mathbf{x}, -t) \tag{2.15}$$

By the previous discussion, it follows that $\{P^t\}_{t\geq 0}$ is a strongly continous, reality preserving, contractive self-adjoint semigroup therefore its generator H_1 is self-adjoint, reality preserving and positive, indeed $H_1 = \sqrt{-\Delta_n + 1}$. Since there exist path-integral formulas for solving Dirichlet problems, this approach provide a path integral representation of the free Euclidean propagator but it is better to consider the more interesting case of an external field where positive frequency solutions are not explicitly given and it is expedient to define them through the Euclidean strategy. Therefore we consider the imaginary-time version of (2.1) namely

$$-\Delta_d u + 2e\mathcal{A}^\beta \partial_\beta u + (1 - e^2 \mathcal{A}_\beta \mathcal{A}^\beta) u + e(\partial_\beta \mathcal{A}^\beta) u = 0 \tag{2.16}$$

where $\mathcal{A}_\beta$ is the Euclidean electromagnetic "four" potential defined by $\mathcal{A}_d = \mathcal{A}^d = A^0_{ext}$, $\mathcal{A}_\beta = \mathcal{A}^\beta = iA^\beta_{ext}$ for $\beta = 1, \ldots, n$. Of course $\partial_d \mathcal{A}^\beta = 0$ for $\beta = 1, \ldots, d$ by our assumption of a static external field. The elliptic equation (2.16) provides a Dirichlet problem in the domain D from which we construct the one-particle Hilbert space $\mathcal{D}_1$ and the one-particle Hamiltonian H_1. The analogous Dirichlet problem in the upper half-space $\overline{D}$ is related to the one- antiparticle structure but it is clear (by time-reversal and charge conjugation) that it is enough to make the change $e\mathcal{A}_\beta \to -e\mathcal{A}_\beta$. For the sake of simplicity, we shall consider purely electric external fields as the effect of magnetic ones is quite trivial. If $V = eA^0_{ext} = e\mathcal{A}^d$, our Dirichlet problem looks

$$\begin{cases} -\Delta_d u + 2V\partial_d u + (1 - V^2)u = 0 & \text{in } D \\ u|_{\partial D} = u_0 \end{cases} \tag{2.17}$$

but now we must explain when an external electric field $\mathbf{E}_{ext}$ is "weak". In general the strenght of $\mathbf{E}_{ext}$ should be tested[6] by inspecting the quadratic form

$$\phi \in H^1(\mathbb{R}^n) \mapsto E(\phi, \phi) = \int_{\mathbb{R}^n} (|\nabla \phi|^2 + (1 - V^2)|\phi|^2)\, d^n\mathbf{x} \tag{2.18}$$

which is related to the classical energy functional of the Klein- Gordon field. Roughly speaking, a weak field is one for which (2.18) is positive definite because the fullfillement of this condition provides a positive energy gap between vacuum and one- pair states. In $d = 4$, when $V(\mathbf{x}) = Ze^2/|\mathbf{x}|$, by exploiting Hardy's inequality

$$\int_{\mathbb{R}^3} \frac{|\phi(\mathbf{x})|^2}{|\mathbf{x}|^2}\, d^3\mathbf{x} \leq 4 \int_{\mathbb{R}^3} |\nabla\phi|^2 d^3\mathbf{x}$$

one can see that the positivity condition holds if $Z < 1/2\alpha$ but it is violated when $Z \geq 1/2\alpha$. In order to simplify the matter, we shall assume that V be bounded with the supremum $\|V\|_\infty$ of $|V|$ strictly lesser than one. Accordingly, we shall equip $H^1(D)$ with the new norm

$$\|u\|^2_{H^1} = \int_D (\Sigma^d_{i=1} |\partial_i u|^2 + (1 - V^2)\, |u|^2)dx \tag{2.19}$$

which is equivalent to (2.10) by $\|V\|_\infty < 1$. We start by defining the one-particle space as the closed subspace of $H^1(D)$ consisting of all its elements which are weak solutions of

$$-\Delta_d u + 2V\partial_d u + (1 - V^2)u = 0 \tag{2.20}$$

Of course, $\mathcal{D}_1$ is invariant under translation in the time direction and will be normed according to (2.19). In order to check the soundness of the new V-depending Hilbert structure, we remark that in a purely electric field the charge density ρ_φ of a Klein-Gordon wave function is

$$\rho_\varphi = i(\bar\varphi\partial_0\varphi - \varphi\partial_0\bar\varphi + 2iV\bar\varphi\varphi) \tag{2.21}$$

therefore the squared norm of a positive frequency solution should be

$$\|\varphi\|^2 = \int_{\{x^0=t\}} (i\bar\varphi\partial_0\varphi - i\varphi\partial_0\bar\varphi - 2V|\varphi|^2)\, d^n\mathbf{x} \tag{2.22}$$

The right hand side of (2.22) doesn't depend on time and it can be evaluated for $t = 0$. On the other hand, by the theorem of divergence applied to the domain $D \subset \mathbb{R}^d$, it easy to check that for Euclidean wave functions u

$$\|u\|_{H^1}^2 = \lim_{x_d \uparrow 0} \int_{\mathbb{R}^n} (\,\bar{u}\partial_d u + u\partial_d \bar{u} - 2V\,|u|^2\,)\, d^n\mathbf{x} \tag{2.23}$$

a formula which coincide with (2.22) by formal analytic continuation. Having got our one-particle space, we pick up the right one-particle Hamiltonian by defining its semigroup $P^t = \exp -tH_1$ still through (2.11). By the same procedure of the free case, it turns out that $\{P^t\}_{t\geq 0}$ is a self- adjoint, strongly continuous contractive semigroup on $\mathcal{D}_1$. Indeed, when u and v are weak real solutions of (2.20), the continous function $F : (0, +\infty) \times (0, +\infty) \mapsto \mathbb{R}$ defined by

$$F(t, \tau) = \int_D (\,\Sigma_{i=1}^d (\partial_i u)^t (\partial v)^\tau + (\,1 - V^2\,)\, u^t v^\tau\,)dx$$

satisfies the partial differential equation (2.13) in the distributional sense. Therefore $(\,P^t u, P^\tau v\,)_{H^1} = (\,P^\tau u, P^t v\,)_{H^1}$, namely $\{P^t\}_{t\geq 0}$ is self-adjoint and has a self-adjoint generator H_1. Because $\{P^t\}_{t\geq 0}$ is still contractive, H_1 is positive. Physically this means that a positive mass-gap survives the perturbation of the free theory by the external field. Now we come to the path integral representation of $P^t = \exp -tH_1$. In order to do that, we exploit once again the one to one correspondence between boundary data and solutions of (2.17) because, by standard theorems[1], it turns out that for each $u_0 \in H^{1/2}(\partial D)$ there is one and only one weak solution $u \in H^1(D)$ of (2.17) when $\|V\|_\infty < 1$. Finally we identify $\mathcal{D}_1$ with the space $H^{1/2}(\mathbb{R}^n)$ endowed with the V-depending norm $\|u_0\|_{\mathcal{D}_1} = \|u\|_{H^1}$ where u is the unique solution of (2.20) issuing from u_0. The semigroup $\{P^t\}_{t\geq 0}$ is carried on $H^{1/2}(\partial D)$ in the same way as in the free case, namely $P^t u_0$ is the trace of u^t on the boundary. Usually semigroups come out from the Cauchy problem for parabolic partial differential equations (as the imaginary-time Schrödinger one) but they can be generated by an elliptic operator L in some half d-dimensional space provided that L has "time-independent" coefficients in which case the semigroup act on boundary data for the corresponding Dirichlet problem. There are well known path-integral formulas[11] for solving such elliptic problems. Let D be a domain of the Euclidean space $\mathbb{R}^d$ with boundary ∂D and $L = -a^i_j \partial_i \partial^j - 2b^i \partial_i$ a second order partial differential elliptic operator, namely one for which the real and symmetric matrix valued function $x \in D \mapsto (a^i_j)(x)$ is positive definite in each point of the domain and, therefore, may be represented as $a^i_j = (\sigma\,\sigma^T)^i_j$. Let us consider the d- dimensional diffusion $s \mapsto Y_s =$

$(Y_s^1, \ldots, Y_s^d) = (\mathbf{Y}_s, Y_s^d)$ defined by Itô's stochastic differential equations

$$dY_s^i = b^i(Y_s)\, ds + \sum_{j=1}^{d} \sigma_j^i(Y_s)\, dW_s^j \quad i = 1, \ldots, d \tag{2.24}$$

where $s \to (W_s^1, \ldots, W_s^d)$ is a d-dimensional Brownian motion. Let τ_D be the first hitting time of the boundary ∂D by the diffusion $s \mapsto Y_s$, then the solution u of the Dirichlet problem

$$\begin{cases} Lu + cu = 0 & \text{in the domain } D \\ u|_{\partial D} = u_0 \end{cases} \tag{2.25}$$

is given by[11]

$$u(x) = \mathbb{E}_x \left(u_0(Y_{\tau_D}) \exp -\frac{1}{2} \int_0^{\tau_D} c(Y_s)\, ds \right) \tag{2.26}$$

where $\mathbb{E}_x(\cdot)$ means the expectation value when the process starts from the point $x \in D$. Therefore

$$\mathbb{E}_x(\, F[Y.]\,) = \int_{\Omega_x} F[Y.]\, \mathcal{D}\mu_Y^x$$

if $F[Y.]$ is some functional on the space Ω_x of continous paths $s \geq 0 \mapsto Y_s \in \mathbb{R}^d$ starting from x and $\mathcal{D}\mu_Y^x$ the (functional) probability measure which weights such paths according to (2.24). When D is unbounded, it is important[11] that $\tau_D^x < +\infty$ a.s. for all $x \in D$ which, unfortunately, is not automatically guaranteed by the drift b^i in (2.17). In order to pick up a more convenient Dirichlet problem, it is better to define auxiliary Euclidean wave functions v by

$$u(\mathbf{x}, x_d) = e^{x_d} v(\mathbf{x}, x_d) \tag{2.27}$$

They have the same traces as the old ones on the boundary $x_d = 0$ of the half-space $x_d < 0$ but they satisfy the new equation

$$-\Delta_d v - 2(1 - V)\partial_d v + V(2 - V)\, v = 0 \tag{2.28}$$

The advantage of (2.28) over (2.20) is due to the fact that now the drift $b^i = (0, \ldots, 0, 1 - V)$ points towards ∂D because we assumed $\|V\|_\infty < 1$. Since $b^d \geq 1 - \|V\|_\infty > 0$, by comparing the hitting time τ_D of $s \mapsto Y_s$ with the corresponding $\tilde{\tau}_D$ of the diffusion $s \mapsto \tilde{Y}_s$ defined by $d\tilde{Y}_s^d = (1 - \|V\|_\infty)\, ds + dW_s^d$, $d\tilde{Y}_s^i = dW_s^i, i = 1, \ldots, d-1$, it is easy to see, by standard result, that $\tau_D^x < +\infty$ a.s. $\forall x \in D$ (moreover $\mathbb{E}_x(\tau_D) < +\infty$). Therefore we can confidently apply (2.26) and we get

$$\begin{aligned} &u(\mathbf{x}, x_d) = \\ &e^{x_d}\mathbb{E}_{(\mathbf{x},x_d)}\left(u_0(\mathbf{Y}_{\tau_D}) \exp -\tfrac{1}{2}\int_0^{\tau_D} V(\mathbf{Y}_s)(2 - V(\mathbf{Y}_s))\, ds\right) \end{aligned} \tag{2.29}$$

Now we make a "change of variables" inside the path integral (2.29) by Girsanov's formula[11]. When two diffusions in $\mathbb{R}^d$, $s \in [0,\tau] \mapsto X_s^i = (\mathbf{X}_s, X_s^d)$ and $s \in [0,\tau] \mapsto Y_s^i = (\mathbf{Y}_s, Y_s^d)$ share the same Brownian noise and start from the same point x but have different drifts

$$dY_s^i = (\, b^i(Y_s) + \delta b^i(Y_s)\,)ds + \sum_{j=1}^{d} \sigma_j^i(Y_s)\, dW_s^j \quad i = 1, \ldots, d \tag{2.30}$$

$$dX_s^i = b^i(X_s)\, ds + \sum_{j=1}^{d} \sigma_j^i(X_s)\, dW_s^j \quad i = 1, \ldots, d \tag{2.31}$$

then the functional measure $\mathcal{D}\mu_Y$ which weights the paths of the first process is absolutely continous with respect to the corresponding measure $\mathcal{D}\mu_X$, moreover

$$\mathcal{D}\mu_Y = e^{\{\Sigma_{i=1}^d \int_0^\tau \phi^i(X_s)\, dW_s^i \frac{1}{2}\Sigma_{i=1}^d \int_0^\tau (\phi^i(X_s))^2\, ds\}}\, \mathcal{D}\mu_X \tag{2.32}$$

where ϕ^i and δb^i are related by

$$\delta b^i(x) = \sum_{j=1}^{d} \sigma_j^i(x)\phi^j(x) \tag{2.33}$$

As reference diffusion we choose the free one $s \mapsto X_s = (\mathbf{X}_s, X_s^d)$ defined by

$$\begin{aligned} dX_s^i &= dW_s^i \quad i = 1, \ldots, n \\ dX_s^d &= ds + dW_s^d \end{aligned} \tag{2.34}$$

and we obtain the new path integral

$$u(\mathbf{x}, x_d) = e^{x_d}\mathbb{E}_{(\mathbf{x},x_d)}\left(u_0(\mathbf{X}_{\tau_D}) \exp - \int_0^{\tau_D} V(\mathbf{X}_s)\, dX_s^d\right) \tag{2.35}$$

where τ_D is now the first hitting time of ∂D by $s \mapsto X_s$. By its definition

$$(P^t u_0)(\mathbf{x}) = u(\mathbf{x}, -t) = e^{-t}\mathbb{E}_{(\mathbf{x},-t)}\left(u_0(\mathbf{X}_{\tau_D}) \exp - \int_0^{\tau_D} V(\mathbf{X}_s)\, dX_s^d\right)$$

It is expedient to translate the free process forward in time in order to display the dependence of P^t on t in a more explicit way. By this device we get

$$(e^{-tH_1} u_0)(\mathbf{x}) = e^{-t}\, \mathbb{E}_{(\mathbf{x},0)} \left(u_0(\mathbf{X}_{\tau_t}) \exp - \int_0^{\tau_t} V(\mathbf{X}_s)\, dX_s^d \right) \tag{2.36}$$

where $\tau_t = \inf\{ s \geq 0 : s + W_s^d = t\}$ is the first hitting time of the hyperplane $\Sigma_t = \{x \in \mathbb{R}^d : x^d = t\}$. Formula (2.36) remind us of the usual Feymann- Kac one but with the difference that the path-dependent ordinary integral $\int_0^t V(\mathbf{X}_s)\, ds$ is replaced by the stochastic integral with a random upper limit $\int_0^{\tau_t} V(\mathbf{X}_s) dX_s^d$. The jump Markov process $t \mapsto \boldsymbol{\xi}_t$ (appearing in the path integral of the one- body relativistic Schrödinger semigroup $\{e^{-t\tilde{H}_1}\}_{t\geq 0}$) can be represented[2,4] as $\boldsymbol{\xi}_t = \mathbf{X}_{\tau_t}$ and therefore (2.36) looks

$$(e^{-tH_1} u_0)(\mathbf{x}) = e^{-t}\, \mathbb{E}_{(\mathbf{x},0)} \left(u_0(\xi_t) \exp - \int_0^{\tau_t} V(\mathbf{X}_s)\, dX_s^d \right) \tag{2.37}$$

which is subtly different from

$$(e^{-t\tilde{H}_1} \psi)(\mathbf{x}) = e^{-t}\, \mathbb{E}_{(\mathbf{x},0)} \left(\psi(\xi_t) \exp - \int_0^{t} V(\xi_s)\, ds \right) \tag{2.38}$$

except, of course, in the free case $V = 0$. In presence of a magnetic field, (2.37) must be modified according to

$$(e^{-tH_1} u_0)(\mathbf{x}) = e^{-t}\, \mathbb{E}_{(\mathbf{x},0)} \Big(u_0(\xi_t) exp - \Big\{ e \int_0^{\tau_t} \mathcal{A}_\beta(\mathbf{X}_s)\, dX_s^\beta + \frac{e}{2} \int_0^{\tau_t} (\,\partial_\beta \mathcal{A}^\beta\,)(\mathbf{X}_s)\, ds \Big\}\Big) \tag{2.39}$$

a more elegant and covariant looking path-integral than that one which holds[5] for relativistic Schrödinger operators in a magnetic field. By dimensional analysis

$$\tau_t = \inf \left\{ s \geq 0 \, : \, s + \sqrt{\frac{\hbar}{mc^2}}\, W_s^d = t \right\}$$

in c.g.s. units and it is not difficult to check[4] that $t \mapsto \tau_t$ converges in probability (uniformly on bounded intervals) to the deterministic time t in the formal nonrelativistic limit $c \uparrow +\infty$. Therefore $t \mapsto \boldsymbol{\xi}_t$ approaches the $\tilde{n}$-dimensional Brownian motion in the same limit. By subtracting the rest energy mc^2, it is clear that the nonrelativistic limit of (2.37) is the right one, namely the Feynman-Kac formula of nonrelativistic quantum mechanics in n-space dimensions.

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