COMMENTS

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Reply to "Comment on 'Repeated measurements in stochastic mechanics'"

Philippe Blanchard
BiBoS and Facultät der Physik, Universität Bielefeld, 4800 Bielefeld 1, Germany

Maurizio Serva
Dipartimento di Matematica and INFN, Università dell'Aquila, 67010 Coppito, L'Aquila, Italy
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In a previous paper we showed that predictions about the results of repeated measurements coincide in quantum and stochastic mechanics when wave-packet collapse is taken into account. Our result removed some apparent pardoxes pointed out by Grabert, Hänggi, Talkner, and Nelson. Recently, Wang and Liang argued that our analysis was incorrect and they produced an explicit counterexample in order to confute our results. We think that their comment is due to a misunderstanding of our paper. For this reason we discuss here, in more detail, the controversial point of our work, concerning the regularization of the quantum-mechanical propagator. After stating more clearly our point, we show that the criticism of Wang and Liang is supported by an improer treatment of the divergences in their counterexample.

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In our paper [1] we consider an ideal experiment where the position of a system is measured at two different times and we show that previsions about this experiment are the same in quantum mechanics and in stochastic mechanics, provided that wave-packet collapse is taken into account. In practice, we show that the quantum-mechanical correlation for two consecutive ideal measurements of the position and the stochastic correlation are the same.

Let us recall that the quantum-mechanical correlation $E_q(X_tX)$ for two consecutive ideal measurements of the position, according to the standard theory of wave-packet reduction, is

$$E_q(X_tX) \equiv \lim_{\epsilon\to 0}\int\int dx\, dx_0 x|K_\epsilon(x,t;x_0,0)|^2 \times x_0|\psi(x_0,0)|^2 \ . \tag{1}$$

In fact, the first measurement at time $t=0$ gives an output x_0 with probability $|\psi(x_0,0)|^2$ and, at the same time, projects the system into the state α_ϵ which is peaked around x_0 with a vanishing variance ϵ. After that, the conditional probability of having x as a result of the second measurement at time t is simply $|K_\epsilon(x,t;x_0,0)|^2$ where

$$K_\epsilon(x,t;x_0,0) \equiv \int du\, K(x,t;u,0)\alpha_\epsilon(u-x_0) \tag{2}$$

is a regularized quantum-mechanical propagator and $K(x,t;u,0)$ is the usual propagator.

This is, indeed, the content of formula (34) of [1] where we state that the quantum propagator K should be regularized in terms of L^2 functions. This is certainly the case if $\alpha_\epsilon(u-x_0)$ is itself a smooth L^2 wave function. Unfortunately, in (34) of [1] we do not show explicitly this regularization, and this is, probably, one of the sources of the misunderstandings.

On the other hand, we point out that the quantum correlation

$$\langle X_tX\rangle = \int\int dx\, dx_0 \psi^*(x,t)xK(x,t;x_0,0)x_0\psi(x_0,0) \tag{3}$$

has no direct experimental meaning connected with two repeated ideal mesurements of the position and, in general, it is even not real. This fact has been clearly stressed in [1] when we introduced the "ideal experiment" correlation (1) (Equation (34) of [1]). Nevertheless, it is easy to show that (3) equals (1) whenever the Heisenberg operators X_t and X commute. The authors of [2] start their paper claiming that in this case (3) is real but the equality with (1) does not hold. This is incorrect as we show below.

Let us preliminarily remark that in the case in which X_t and X commute they have the same eigenstates and one has

$$\int dx\, xK(x,t;x_1,0)K^*(x,t;x_2,0) \equiv \langle x_1|X_t|x_2\rangle = g_t(x_1)\delta(x_1-x_2) \tag{4}$$

with $g_0(x_1) = x_1$. With the aid of (4) we can now derive our result. From (1) and (2) we have

$$E_q(X_tX) = \lim_{\epsilon\to 0} \int\int\int\int dx dx_0 dx_1 dx_2 x K(x,t;x_1,0) \times K^*(x,t;x_2,0)\alpha_\epsilon(x_1 - x_0)\alpha^*_\epsilon(x_2 - x_0) \times x_0\psi(x_0,0)\psi^*(x_0,0) \ . \tag{5a}$$

Since in the limit $\epsilon \to 0$ the wave function $\alpha^*_\epsilon(x_2 - x_0)$ vanishes for $x_2 \neq x_0$ we can replace $\psi^*(x_0, 0)$ with $\psi^*(x_2, 0)$. Moreover, by virtue of (4), we can replace $\alpha^*_\epsilon(x_2 - x_0)$ with $\alpha^*_\epsilon(x_1 - x_0)$. We can then integrate with respect to x_2 and obtain

$$E_q(X_tX) = \lim_{\epsilon\to 0} \int\int\int dx\, dx_0 dx_1 \psi^*(x,t) x K(x,t;x_1,0) \times |\alpha_\epsilon(x_1 - x_0)|^2 x_0\psi(x_0,0) \ . \tag{5b}$$

Finally, since $|\alpha_\epsilon(x_1 - x_0)|^2$ reduces to a δ function when $\epsilon \to 0$, the integration with respect to x_1 gives

$$E_q(X_tX) = \int\int dx\, dx_0 \psi^*(x,t) x K(x,t;x_0,0) x_0\psi(x_0,0) = \langle X_tX\rangle \ , \tag{5c}$$

whenever X_t and X commute. In conclusion we can state the following.

(a) When X_t and X commute, contrary to what is claimed in [2], not only the quantum correlation (3) is real but it coincides with (1).

After having recalled this basic fact, [1] follows by incorporating the effects of the wave-packet collapse into stochastic mechanics. Without entering into the details, our point is that a measurement has not only the effect of establishing new initial conditions for the process but it also has the effect to modify the drift. This can be simply understood since the drift is constructed from the wave function which changes because of the wave-packet reduction. Our reasoning easily brings us (formulas (36)–(38) of [1]) to the conclusion that the stochastic mechanical probability density of the initial position is simply $\rho_0(x_0) = |\psi(x_0, 0)|^2$ while the transition probability density $p(x,t;x_0,0)$ equals $|K_\epsilon(x,t;x_0,0)|^2$ in the limit of vanishing ϵ. With our choice,

$$E_s(\zeta(t)\zeta(0)) \equiv \int\int dx\, dx_0 x P(x,t;x_0,0) x_0\rho_0(x_0) \tag{6}$$

is identical to the "ideal experiment" correlation (1). We can thus state the following.

(b) The correlations (1) and (6) are equal by construction. This fact and the previous statement (a) imply that the quantum correlation (3) and the stochastic correlation (6) are equal whenever X_t and X commute.

At this point the question is why the authors of [2] find a counterexample with commuting X_t and X? Let us try to answer this question. They consider a two-dimensional problem. The first measurement gives the value (x_0, y_0) and, therefore, the quantum propagator is (6) of [2]. According to our general prescription, they write down the first-order stochastic differential equations (Eq. (7) of [2])

$$\begin{aligned} d\zeta_x &= \frac{\omega}{2}\left(\cot\left\{\frac{\omega t}{2}\right\}(\zeta_x - x_0) + (\zeta_y - y_0)\right) dt + dW_x \ , \\ d\zeta_y &= \frac{\omega}{2}\left(\cot\left\{\frac{\omega t}{2}\right\}(\zeta_y - y_0) - (\zeta_x - x_0)\right) dt + dW_y \ . \end{aligned} \tag{7}$$

These equations, according to the ideas of our paper, must be solved with the initial conditions (x_0, y_0). In fact, according to our construction, the first measurement establishes at the same time a new drift and new initial conditions. One immediately realizes that serious problems may arise from the singularity of the drift at initial time $t = 0$. For the moment we just ignore this fact and we introduce the conditional expectations $\bar{x}(t)$ and $\bar{y}(t)$ with respect to the initial condition (x_0, y_0). The evolution equations of these quantities are obtained from the linear stochastic equations (7) by taking the expectations of both sides. One has

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \frac{\omega}{2}\left(\cot\left\{\frac{\omega t}{2}\right\}(\bar{x} - x_0) + (\bar{y} - y_0)\right) \ , \\ \frac{d\bar{y}}{dt} &= \frac{\omega}{2}\left(\cot\left\{\frac{\omega t}{2}\right\}(\bar{y} - y_0) - (\bar{x} - x_0)\right) \ , \end{aligned} \tag{8}$$

with the initial conditions $\bar{x}(0) = x_0$ and $\bar{y}(0) = y_0$.

It can be verified by direct substitution that a solution of (8) is

$$\begin{aligned} \bar{x}(t) &= x_0 \cos^2\frac{\omega t}{2} + y_0 \sin\frac{\omega t}{2}\cos\frac{\omega t}{2} \ , \\ \bar{y}(t) &= y_0 \cos^2\frac{\omega t}{2} - x_0 \sin\frac{\omega t}{2}\cos\frac{\omega t}{2} \ . \end{aligned} \tag{9}$$

These conditional expectations allow us to perform a simple computation of the correlations

$$\begin{aligned} E_s(\zeta_x(t)\zeta_x(0)) &= \int\int dx_0 dy_0 \bar{x}(t) x_0 \rho(x_0,y_0) = \sigma^2\cos^2\frac{\omega t}{2} \ , \\ E_s(\zeta_y(t)\zeta_y(0)) &= \int\int dx_0 dy_0 \bar{y}(t) y_0 \rho(x_0,y_0) = \sigma^2\cos^2\frac{\omega t}{2} \ , \end{aligned} \tag{10}$$

where σ^2 is the variance of the initial density $\rho(x_0, y_0) = |\psi(x_0, y_0, t = 0)|^2$ with ψ given in Eq. (4) of [2]. At $t = 2n\pi/\omega$ and at $t = (2n + 1)\pi/\omega$ they are in complete agreement with the quantum correlations (Eq. (5) of [2]).

At this point one would conclude that the theory works perfectly and ask himself where is the problem. The answer is simple: consider again Eq. (8), one sees immediately that they have another solution that is

$$\bar{x}(t) = x_0, \quad \bar{y}(t) = y_0 \tag{11}$$

which leads to the correlations

$$E_s(\zeta_x(t)\zeta_x(0)) = \sigma^2, \quad E_s(\zeta_y(t)\zeta_y(0)) = \sigma^2 . \tag{12}$$

This is the result found by the authors of [2] (Eq. (10b) of [2]) and used to criticize our paper [1].

It is now clear that the initial time divergence of the drift gives rise to serious problems since the solution of (8) is not unique. The authors of [2] do not consider at all solution (9) and (10) and claim that the solution (11) and (12) (Eq. (10b) of [2]) is the correct solution. To prove this fact they introduce an extra time s and they reformulate the problem in a complicated form. Unfortunately their considerations are invalid since they improperly deal with diverging terms. To illustrate this point it is necessary to follow their calculations.

(a) The authors of [2] solve the stochastic equations (8) giving initial conditions at a generic time $s > 0$. These solutions (Eq. (8) of [2]) are not defined for $s = 2n\pi/\omega$, since they diverge for these values of s. The divergence at $s = 2n\pi/\omega$ appears not only in the expectations but also in the stochastic integrals. This is a first problem since one is interested at the solution when $s = 0$. Nevertheless, until the forbidden values of s are chosen, one can go on.

(b) From (8) of [2], by taking the expectation, the authors find (9) of [2] which is incorrect. There is, in fact, a missing term which comes from the correlation $E_s(\zeta_y(s)\zeta_x(0))$. For solution (9) the missing correlation takes the value $-\sigma^2 \sin(\omega s/2)\cos(\omega s/2)$ while it is vanishing for solution (11). The corrected Eq. (9) of [2] is satisfied both by (9) and (10) and by (11) and (12). Let us ignore the fact that there is a missing term in (9) of [2] and further follow the author's calculations.

(c) Notice that Eq. (9) of [2] is also not defined for $s = 2n\pi/\omega$ but the authors of [2] exactly choose the forbidden values of s to state that (9) of [2] implies (10a) in [2]. This last implication is the most manifestly illegitimate. In fact, even a limiting procedure does not help since the cancellation of the diverging terms in the limit of $s \to 2n\pi\omega$ is not at all assured by the fact that the correlation at the right side of the equation tends to σ^2 in the same limit. For example, the cancellation does not hold if the correlation tends to σ^2 linearly.

In conclusion, the arguments in [2] do not solve the ambiguities due to the divergences at time $t = 0$ and have the only effect to obtain by a complicated and partially wrong formalism the simple (11) and (12) which is one of the possible solutions.

The singularity of the drift originates from the propagator (6) of [2] and can be easily eliminated by substituting the propagator with a regularized one. In this case the drift also regularizes and one can solve Eq. (8) with the proper initial conditions. This solution will be unique. The regularization of (6) of [2] can be done using a real positive wave function $\alpha_\epsilon(u - x_0, v - y_0)$ such that $|\alpha^\epsilon|$ is a Gaussian of variance $\epsilon\hbar/m\omega$ with respect to both variables x_0 and y_0. Since both the propagator (6) of [2] and α are of Gaussian type it is easy to compute the normalized propagator using the convolution (2) and obtain the drift. The pathological equations (8) are replaced by the regularized equations

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \frac{\omega}{2}\left(\cot\left\{\frac{\omega t}{2}\right\}[a(t)\bar{x} - b(t)x_0] + [\bar{y} - b(t)y_0]\right) , \\ \frac{d\bar{y}}{dt} &= \frac{\omega}{2}\left(\cot\left\{\frac{\omega t}{2}\right\}[a(t)\bar{y} - b(t)y_0] - [\bar{x} - b(t)x_0]\right) , \end{aligned} \tag{13}$$

where

$$\begin{aligned} a(t) &= \frac{1 + \epsilon\,\cot(\omega t/2) + \epsilon/\cot(\omega t/2) - \epsilon^2}{1 + [\epsilon\,\cot(\omega t/2)]^2} , \\ b(t) &= \frac{1 + \epsilon\,\cot(\omega t/2)}{1 + [\epsilon\,\cot(\omega t/2)]^2} , \end{aligned} \tag{14}$$

which formally reduces to (8) in the limit $\epsilon \to 0$. As can be verified by direct substitution, these equations are (uniquely) satisfied by solution (9) for any value of ϵ. Therefore, (9) is the solution obtained by regularizing (8) and taking the limit $\epsilon \to 0$, while solution (11) is unstable with respect to the regularization. This fact can be better seen considering Eq. (13) at the time $t = 0$ where they are

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \frac{\omega}{2}\left(\frac{1}{\epsilon}(\bar{x} - x_0) + \bar{y}\right) , \\ \frac{d\bar{y}}{dt} &= \frac{\omega}{2}\left(\frac{1}{\epsilon}(\bar{y} - y_0) - \bar{x}\right) . \end{aligned} \tag{15}$$

Independent of ϵ, one has the initial time solution

$$\bar{x} \approx x_0 + \frac{\omega}{2}y_0 t, \quad \bar{y} \approx y_0 - \frac{\omega}{2}x_0 t , \tag{16}$$

in agreement with (9). We would like to stress that it is not astonishing that we find agreement with quantum mechanics in this particular case since, by virtue of (5), our result holds in general.

In conclusion, the criticisms to our paper should be rejected. In fact, our formula (34) of [1] coincides with the quantum correlation $\langle X_t X\rangle$ whenever X_t and X commute. Moreover, the counterexample in [2] is based on an illegitimate manipulation of the associated stochastic equations [3,4].

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