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Bethe–Peierls approximation for the 2D random Ising model

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Abstract. The partition function of the 2D Ising model with random nearest-neighbour coupling is expressed in the dual lattice made of square plaquettes. The dual model is solved in the mean field and in different types of Bethe–Peierls approximations, using the replica method.

1. Introduction

The application of methods of mean-field type to Ising models allows one to obtain very accurate approximations of the thermodynamic quantities. However, in the presence of quenched disorder this approach is difficult to implement. In this paper, we show that rather good results can be obtained after performing a duality transformation of the Ising model with random nearest-neighbour coupling that assumes the values $J_{ij} = \pm 1$ with equal probability. The model is thus defined on a dual lattice where the spin variables are attached to the square plaquettes. The advantage is that the quadratic term of the dual Hamiltonian has constant coefficients instead of random ones. It is therefore possible to use the standard methods of the mean field to estimate the quenched free energy. Our results can be generalized to higher dimensions, although the approximations become rougher, because the number of plaquette spins over the number of interaction links increases with the dimensionality [1]. In particular, we obtain an extremely good estimate of the ground-state energy of the random Ising model, by applying the Bethe–Peierls approximation where part of the short-range order is taken into account.

In section 1, we introduce the dual lattice made of elementary square plaquettes. On this lattice the partition function can be expressed as a function of the inverse temperature $\widetilde{\beta} = -\frac{1}{2} \ln \tanh(1/T)$ where $T = \beta^{-1}$ is the temperature of the original lattice. In section 2, we apply the mean-field approximation to the dual model, using the replica method. In section 3, we introduce the Bethe–Peierls approximation. This allows us to obtain a very precise estimate of the ground-state energy of the two-dimensional Ising model with random coupling. In section 4, we show that it is possible to improve the Bethe–Peierls approximation by introducing an interaction between different replicas. In section 5, the reader will find some remarks and conclusions.

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2. Duality transformation

The partition function of the Ising models on a lattice of N sites with nearest-neighbour couplings J_{ij} which are independent identically distributed random variables, in the absence of an external magnetic field, is

$$Z_N(\beta, \{J_{ij}\}) = \sum_{\{\sigma\}} \prod_{(i,j)} \exp(\beta J_{ij} \sigma_i \sigma_j) \tag{1}$$

where the sum runs over the 2^N spin configurations $\{\sigma\}$, and the product over the $2N$ nearest-neighbour sites (i, j). One is interested in computing the quenched free energy

$$f = - \lim_{N\to\infty} \frac{1}{\beta N} \overline{\ln Z_N} \tag{2}$$

where $\overline{A}$ indicates the average of an observable A over the distribution of the random coupling. The quenched free energy is a self-averaging quantity, i.e. it is obtained in the thermodynamic limit for almost all realizations of disorder [2].

On the other hand, it is trivial to compute the so-called annealed free energy

$$f_a = - \lim_{N\to\infty} \frac{1}{\beta N} \ln \overline{Z} \tag{3}$$

corresponding to the free energy of a system where the random coupling is not quenched but can thermalize with a relaxation time comparable to that of the spin variables. In our model, where the couplings are independent dichotomic random variables $J_{ij} = \pm 1$ with equal probability, one has

$$f_a = -\beta^{-1} \ln(2 \cosh^2 \beta\,) . \tag{4}$$

However, f_a is a very poor approximation of the quenched free energy, and is not able to capture the qualitative features of the model.

In order to estimate (1), it is convenient to use the link variable $x_{ij} = \sigma_i\, \sigma_j$, since only terms corresponding to products of the variables x_{ij} on closed loops survive after summing over the spin configurations: on every closed loop of the lattice $\prod x_{ij} = 1$, while $\prod x_{ij} = \sigma_a\, \sigma_b$ for a path from site a to site b. A moment of reflection shows that it is sufficient to fix $\prod x_{ij} = 1$ on the elementary square plaquettes $\mathcal{P}$ to automatically fix it on all the closed loops. The partition function thus becomes

$$Z_N(\beta, \{J_{ij}\}) = \sum_{\{x_{ij}\}} \prod_{i=1}^{N_p} \frac{1 + \widetilde{x}_i}{2} \prod_{(i,j)} e^{\beta J_{ij} x_{ij}} \tag{5}$$

where the number of plaquettes is

$$N_p = N$$

and we have introduced the plaquette variable $\widetilde{x}_i = \prod_{\mathcal{P}_i} x_{ij}$.

For dichotomic random coupling $J_{ij} = \pm 1$ with equal probability, the free energy of the model is invariant under the gauge transformation $x_{ij} \to J_{ij}\, x_{ij}$, so that one has

$$Z_N = \sum_{\{x_{ij}\}} \prod_{i=1}^{N} \frac{1 + \widetilde{x}_i \widetilde{J}_i}{2} \prod_{(i,j)} e^{\beta x_{ij}} \tag{6}$$

where $\widetilde{J}_i = \prod_{\mathcal{P}_i} J_{ij}$ is again a dichotomic random variable (the 'frustration' [3] of the plaquette $\mathcal{P}_i$). It is worth remarking that (6) gives the partition function in terms of a sum over the 2^{2N} configurations of the independent random variables $x_{ij} = \pm 1$ with probability

$$p_{ij} = \frac{e^{\beta x_{ij}}}{2\cosh\beta}\,. \tag{7}$$

In this section we shall indicate the average of an observable A over such a normalized weight by

$$\langle A \rangle \equiv \sum_{\{x_{ij}\}} \prod_{(i,j)} p_{ij}\, A$$

e.g. one has $\langle x_{ij} \rangle = \tanh\beta$ and $\langle \widetilde{x}_i \rangle = \tanh^4\beta$. With such a notation, the partition function assumes the compact form

$$Z_N = 2^N \cosh^{2N}(\beta) \left\langle \prod_{i=1}^{N} (1 + \widetilde{x}_i \widetilde{J}_i) \right\rangle. \tag{8}$$

In order to estimate the average in (8), let us introduce the dual lattice [4] where the sites are located at the centres of each square of the original lattice. A dual-spin variable is attached to each square plaquette and can assume only the values $\widetilde{\sigma}_i = \pm 1$ with equal probability, so that one has the identity

$$1 + \widetilde{x}_i \widetilde{J}_i = \sum_{\widetilde{\sigma}_i = \pm 1} (\widetilde{x}_i\, \widetilde{J}_i)^{(1+\widetilde{\sigma}_i)/2}. \tag{9}$$

Since there is a one-to-one correspondence between links on the original and on the dual lattice, we can estimate the link average noting that

$$\left\langle \prod_{i=1}^{N} \widetilde{x}_i^{(1+\widetilde{\sigma}_i)/2} \right\rangle = \left\langle \prod_{(i,j)} x_{ij}^{(1+\widetilde{\sigma}_i)/2+(1+\widetilde{\sigma}_j)/2} \right\rangle = \prod_{(i,j)} (\tanh\beta)^{(1-\widetilde{\sigma}_i\widetilde{\sigma}_j)/2}\,. \tag{10}$$

The last equality in (10) follows from the fact that

$$x_{ij}^{(1+\widetilde{\sigma}_i)/2+(1+\widetilde{\sigma}_j)/2} = \begin{cases} \langle x_{ij} \rangle = \tanh\beta & \text{if} \quad \widetilde{\sigma}_i \neq \widetilde{\sigma}_j \\ 1 & \text{if} \quad \widetilde{\sigma}_i = \widetilde{\sigma}_j\,. \end{cases} \tag{11}$$

Inserting (10) and (9) into (8) one has

$$Z_N = 2^N \cosh^{2N}(\beta)\, e^{-N\,2\widetilde{\beta}} \sum_{\{\widetilde{\sigma}\}} \prod_{i=1}^{N} \widetilde{J}_i^{(1+\widetilde{\sigma}_i)/2} \prod_{(i,j)} e^{\widetilde{\beta}\,\widetilde{\sigma}_i\widetilde{\sigma}_j} \tag{12}$$

where we have introduced the variable

$$\widetilde{\beta} = -\tfrac{1}{2} \ln\tanh\beta \tag{13}$$

which is the inverse temperature of the dual model vanishing as $e^{-2\beta}$ when the temperature $T = \beta^{-1} \to 0$. The quenched free energy (2) thus becomes

$$-\,\beta\, f(\beta) = \ln\sinh(2\beta) - \widetilde{\beta}\,\widetilde{f}(\widetilde{\beta}) \tag{14}$$

where $\widetilde{f}$ is the free energy of the dual model, defined as

$$\widetilde{f}(\widetilde{\beta}) = -\lim_{N\to\infty} \frac{1}{\widetilde{\beta} N} \ln \mathcal{Z}_N \tag{15}$$

in terms of the partition function

$$\mathcal{Z}_N = \sum_{\{\widetilde{\sigma}_i\}} \mathrm{e}^{\widetilde{\beta} \sum_{(i,j)} \widetilde{\sigma}_i \widetilde{\sigma}_j} \prod_{i=1}^{N} \widetilde{J}_i^{(1+\widetilde{\sigma}_i)/2} . \tag{16}$$

From equation (16) the Hamiltonian of the dual model can be defined via the relation $\mathcal{Z}_{N_p} = \sum_{\{\widetilde{\sigma}\}} \mathrm{e}^{-\widetilde{\beta} H}$, as

$$H = -\sum_{(i,j)} \widetilde{\sigma}_i \widetilde{\sigma}_j - \sum_{i=1}^{N_p} \ln(\widetilde{J}_i) \frac{(1+\widetilde{\sigma}_i)}{2\widetilde{\beta}} . \tag{17}$$

Let us stress that the quadratic term of the dual Hamiltonian is independent of the random coupling and the randomness enters via a random complex magnetic field that can assume the two values 0 and $\mathrm{i}\pi/(2\widetilde{\beta})$ with equal probability. In fact, the weight $\exp(-\widetilde{\beta} H)$ does not define a standard Gibbs probability measure on the dual lattice: it defines a signed measure, differing from that of the pure Ising model only by the presence of the random sign related to the frustrations of the square plaquettes $\{\widetilde{J}_i\}$.

3. Replica trick and mean-field approximation

The introduction of the dual model allows one to apply the mean-field approximation, since one can easily linearize the Hamiltonian (17) by neglecting fluctuations. A similar method has been introduced in the framework of field theory in statistical systems without disorder, such as lattice-gauge theories or spin models [5, 6].

For our purposes, it is convenient to use the replica method in order to get the quenched free energy of the dual model as

$$\widetilde{f}(\widetilde{\beta}) = -\lim_{n\to 0} \lim_{N\to\infty} \frac{1}{\widetilde{\beta} n N} \ln \overline{(\mathcal{Z}_N)^n} . \tag{18}$$

Let us thus consider n non-interacting replicas of our disordered system labelled by $\alpha = 1, \ldots, n$. Now, the $\widetilde{J}$ are are independent random variables in 2D. Indeed, one can easily verify that $\overline{\prod_i \widetilde{J}_i} = \prod_i \overline{\widetilde{J}_i}$ because $\widetilde{J}_i = \pm 1$ with equal probability. It is worth noting that this is not true in 3D, where the $\widetilde{J}_i$ of the ith square plaquette of a cube can be obtained as a product of the remaining five $\widetilde{J}$'s of the cube, implying $\prod_{\text{cube}} \widetilde{J}_k \equiv 1$, so that $\overline{\prod_{\text{cube}} \widetilde{J}_k} = 1$ while $(\overline{\widetilde{J}_k})^6 = 0$. In contrast, a plaquette frustration $\widetilde{J}_i$ cannot be expressed as a product of the other ones in 2D. As a consequence, from (16) the partition function of n replicas becomes

$$\overline{(\mathcal{Z}_N)^n} = \sum_{\{s\}} \mathrm{e}^{\widetilde{\beta} \sum_{\alpha=1}^{n} \sum_{(i,j)} \widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_j^{(\alpha)}} \prod_{i=1}^{N} \overline{\prod_{\alpha=1}^{n} \widetilde{J}_i^{(1+\widetilde{\sigma}_i^{(\alpha)})/2}} \tag{19a}$$

where the sum in (19*a*) runs over the 2^{Nn} spin configurations $\{s\}$ of the replicas, and we use the compact notation:

$$\{s\} \equiv \{\widetilde{\sigma}^{(1)}\}, \ldots, \{\widetilde{\sigma}^{(n)}\} .$$

One can easily perform the disorder average in (19*a*) and get

$$\overline{(\mathcal{Z}_N)^n} = \sum_{\{s\}} \mathrm{e}^{\widetilde{\beta} \sum_\alpha \sum_{(i,j)} \widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_j^{(\alpha)}} \prod_{i=1}^{N} \frac{1}{2}\left(1 + (-1)^n \prod_\alpha \widetilde{\sigma}_i^{(\alpha)}\right) . \tag{19b}$$

As the free energy is invariant under the gauge transformation $\widetilde{\sigma}_i^{(\alpha)} \to -\widetilde{\sigma}_i^{(\alpha)}$, equation (19*b*) assumes the simpler form

$$\overline{(\mathcal{Z}_N)^n} = \sum_{\{s\}} \mathrm{e}^{\widetilde{\beta} \sum_\alpha \sum_{(i,j)} \widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_j^{(\alpha)}} \prod_{i=1}^{N} \frac{1}{2}\left(1 + \prod_\alpha \widetilde{\sigma}_i^{(\alpha)}\right) . \tag{19c}$$

It is worth stressing that the above expression differs from the partition function of a collection of n non-interacting Ising systems *without disorder* only because of the factor $\prod_i (1 + \prod_\alpha \widetilde{\sigma}_i^{(\alpha)})/2$. Such a term introduces an 'effective' interaction between replicas: a configuration contributes to the annealed partition function $\overline{\mathcal{Z}^n}$ only if $\prod_\alpha \widetilde{\sigma}_i^{(\alpha)} = 1$ on each site of the dual lattice (plaquette of the original lattice).

Now we can use the mean-field approximation to estimate (19), by introducing the magnetizations

$$m_\alpha = \lim_{N\to\infty} \frac{1}{N} \sum_{i}^{N} \widetilde{\sigma}_i^{(\alpha)} \qquad \alpha = 1, \ldots, n . \tag{20}$$

Indeed, if we neglect the fluctuations, the quadratic term of (19*b*) can be estimated as $\widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_j^{(\alpha)} = m_\alpha^2$ so that (16) becomes

$$\overline{(\mathcal{Z}_N)^n} = \sum_{\{s\}} \mathrm{e}^{N\, 2\widetilde{\beta} \sum_\alpha m_\alpha^2} \prod_{i=1}^{N} \left(1 + \prod_\alpha \widetilde{\sigma}_i^{(\alpha)}\right) \frac{1}{2} . \tag{21}$$

The mean-field solution can be found by the introduction of n auxiliary fields $\Phi_1, \ldots, \Phi_n$. Using the saddle-point method, one has, in the limit $N \to \infty$,

$$\mathrm{e}^{N\, 2\, \widetilde{\beta}\, m_\alpha^2} \sim \int_{-\infty}^{\infty} \mathrm{d}\Phi_\alpha \exp\left(N\, 2\, \widetilde{\beta}\, (2\, m_\alpha \Phi_\alpha - \Phi_\alpha^2)\right) . \tag{22}$$

As a consequence, $\overline{(\mathcal{Z}_N)^n}$ is given by the maximum over $\{\Phi_1, \ldots, \Phi_n\}$ of

$$\begin{aligned} &\sum_{\{s\}} \mathrm{e}^{-N\, 2\widetilde{\beta} \sum_\alpha \Phi_\alpha^2} \prod_{i=1}^{N} \frac{1}{2}\left(1 + \prod_\alpha \widetilde{\sigma}_i^{(\alpha)}\right) \mathrm{e}^{4\widetilde{\beta} \sum_\alpha \Phi_\alpha \widetilde{\sigma}_i^{(\alpha)}} \\ &\quad = \sum_{\{s\}} \mathrm{e}^{-N\, 2\widetilde{\beta} \sum_\alpha \Phi_\alpha^2} \prod_{i}^{N} \frac{1}{2}\left[\prod_\alpha^n \mathrm{e}^{4\widetilde{\beta} \widetilde{\sigma}_i^{(\alpha)} \Phi_\alpha} + \prod_\alpha^n \widetilde{\sigma}_i^{(\alpha)} \mathrm{e}^{4\widetilde{\beta}\, \Phi_\alpha \widetilde{\sigma}_i^{(\alpha)}}\right] . \end{aligned} \tag{23}$$

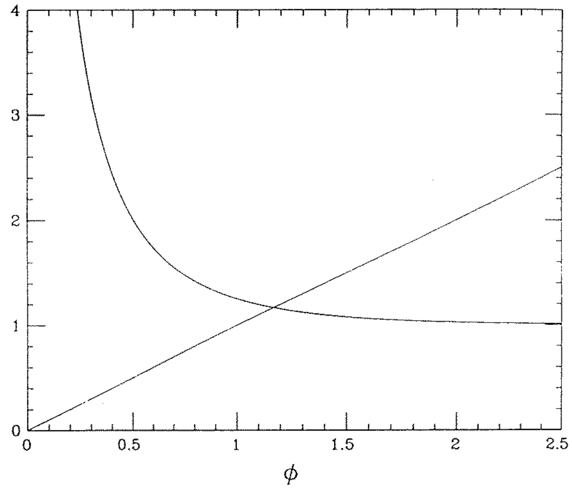

Figure 1. Graphical solution of the implicit equation (25) at $T = \beta^{-1} = 1$ corresponding to $\widetilde{\beta} = 0.136\ldots$. The full curves are $\coth(8\widetilde{\beta}\Phi)$ versus Φ and the straight line $\Phi = \Phi$.

Now, we can explicitly carry out the sum over the 2^{Nn} spin configurations in (23) and obtain

$$\overline{(Z_N)^n} = \max_{\Phi_1,\ldots,\Phi_n} \mathrm{e}^{-2\widetilde{\beta}N\sum_{\alpha=1}^{n}\Phi_\alpha^2} \prod_i^N \frac{1}{2}\left(\prod_\alpha^n 2\cosh(4\widetilde{\beta}\Phi_\alpha) + \prod_\alpha^n 2\sinh(4\widetilde{\beta}\Phi_\alpha)\right). \tag{24}$$

In 2D, it is commonly believed that there is no glass transition and no replica symmetry breaking, so that we expect that the maximum of (24) is realized at the same $\Phi_\alpha = \Phi^*$ for all the replicas. As a consequence, using the replica trick (18), the quenched free energy in the mean-field approximation reads as

$$\widetilde{f}(\widetilde{\beta}) = -\widetilde{\beta}^{-1} \max_\phi \left(\tfrac{1}{2}\ln(2\sinh(8\,\widetilde{\beta}\Phi)) - 2\widetilde{\beta}\,\Phi^2\right) \tag{24}$$

where the maximum of (24) is realized by the value Φ^*, solution of the self-consistency equation

$$\coth(8\widetilde{\beta}\Phi) = \Phi\,. \tag{25}$$

The graphical solution of this implicit equation is showed in figure 1. One sees that Φ^* should always be larger than unity and at $\widetilde{\beta} \to \infty$ (infinite temperature $T = \beta^{-1}$ limit) $\Phi^* = 1$. It can appear rather odd that in the dual model the magnetization $\Phi^* \geqslant 1$. This stems from the fact that the Gibbs measure $\exp(-\widetilde{\beta}H)$ is a signed measure because the random coupling is transformed into a complex random magnetic field in (17). From figure 1, it is also clear that the mean-field solution does not exhibit phase transitions at finite temperature. However, there is an essential singularity at $T = 0$, since inserting (24) into (15) and (14) one sees that $f \sim \exp(1/T)$ for $T \to 0$.

It is possible to explicitly solve the self-consistency equation when $\widetilde{\beta} \to 0$ since (25) becomes

$$\Phi^* = (8\,\widetilde{\beta})^{-1/2}\left(1 + 4\widetilde{\beta}/3 + \mathrm{O}(\widetilde{\beta}^2)\right). \tag{26}$$

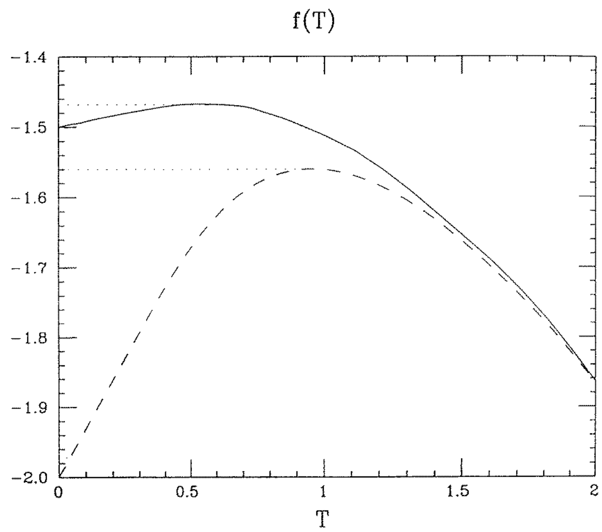

Figure 2. Annealed free energy f_a given by (4) (broken curve) and the mean-field solution (full curve) versus temperature $T = \beta^{-1}$. The broken lines are the Maxwell constructions obtained by imposing that the free energy is a monotonic non-decreasing function of T. The annealed solution estimates a ground-state energy $E_0 \geqslant -1.559$; the mean-field solution gives $E_0 \geqslant -1.468$; the numerical result of [5] is $E_0 = -1.404 \pm 0.002$.

The zero-temperature energy of the mean-field solution is $E_0 = -1.5$ while the numerical simulations [7] give $E_0 = -1.404 \pm 0.002$. In figure 2, we show the free energy as a function of T. One sees that entropy is negative at low temperature, thus indicating that the solution is unphysical. As a consequence, a better estimate of the ground-state energy is given by the maximum of $f(\beta)$, following a standard argument of Toulouse and Vannimenus [8], and one has $E_0 \geqslant \max_\beta f(\beta) = -1.468$.

4. Bethe–Peierls approximation

The mean-field approximation neglects the short-range order, that can be taken into account by the so-called Bethe–Peierls approximation [9, 10]. It is still useful to consider the model on the dual lattice and, moreover, it is convenient to work on the internal energy

$$U(\beta) = \frac{\partial}{\partial \beta}(\beta f) \tag{27}$$

instead of the free energy. From equations (14) and (15) one thus has

$$U(\beta) = -2\coth(2\beta) - \frac{1}{\sinh(2\beta)}\mathcal{U}(\widetilde{\beta}) \tag{28}$$

with

$$\mathcal{U}(\widetilde{\beta}) = \frac{\partial}{\partial \widetilde{\beta}}(\widetilde{\beta}\widetilde{f}(\widetilde{\beta})) \equiv \lim_{n \to 0} \mathcal{U}_n(\widetilde{\beta})$$

where $\widetilde{f}(\widetilde{\beta})$ is given by (18) so that the internal energy of n replicas is

$$U_n(\widetilde{\beta}) = \left(\overline{(Z_N)^n}\right)^{-1} \sum_{\{s\}} \widetilde{\sigma}_k^{(\gamma)} \widetilde{\sigma}_l^{(\gamma)} \prod_i^N \frac{1}{2}\left(1 + \prod_\alpha^n \widetilde{\sigma}_i^{(\alpha)}\right) \prod_\alpha^n \mathrm{e}^{\widetilde{\beta} \sum_{(i,j)} \widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_j^{(\alpha)}} \tag{29}$$

that in the limit $n \to 0$ gives $\mathcal{U}(\widetilde{\beta})$.

Now comes the key step. Let us sum over all the spin couples except the $4n$ nearest-neighbour spins $\widetilde{\sigma}_1^{(\alpha)}$, $\widetilde{\sigma}_2^{(\alpha)}$, $\widetilde{\sigma}_3^{(\alpha)}$, $\widetilde{\sigma}_4^{(\alpha)}$ around the n spin $\widetilde{\sigma}_0^{(\alpha)}$ in the numerator of (29). In order to simplify the notation, in the following we shall indicate all these $5n$ spins as $\boldsymbol{s}_{(5)}$ and the $4n$ lateral ones as $\boldsymbol{s}_{(4)}$

Noting that the prefactor $\overline{(Z_N)^n}$ is a constant depending on n, the expression (29) becomes

$$\mathcal{U}_n(\widetilde{\beta}) = \left(\overline{(Z_N)^n}\right)^{-1} \sum_{\{\boldsymbol{s}_{(5)}\}} \widetilde{\sigma}_k^{(\gamma)} \widetilde{\sigma}_l^{(\gamma)}\, \Psi_n(\boldsymbol{s}_{(4)}) \left(\tfrac{1}{2}\right)^5 \prod_{i=0}^4 \left(1 + \prod_\alpha^n \widetilde{\sigma}_i^{(\alpha)}\right) \prod_{i=1}^4 \prod_\alpha^n \mathrm{e}^{\widetilde{\beta} \widetilde{\sigma}_0^{(\alpha)} \widetilde{\sigma}_i^{(\alpha)}} \tag{30}$$

where Ψ_n is a function of the $4n$ lateral spins $\boldsymbol{s}_{(4)}$. In practice, we are considering the $5n$ free spins $\boldsymbol{s}_{(5)}$ on replicated crosses which are interacting with the mean field generated by the other $n(N-5)$ spins.

The ansatz of the Bethe–Peierls approximation consists in assuming that any function Ψ_n of the spin configurations such as (30) might be expressed as

$$\Psi_n = \left(\prod_{\alpha=1}^n \prod_{i=1}^4 \mathrm{e}^{\mu \widetilde{\sigma}_i^{(\alpha)}}\right) \frac{\overline{(Z_N)^n}}{z(n, \widetilde{\beta}, \mu)} \tag{31}$$

where we have introduced the normalization factor

$$z(n, \widetilde{\beta}, \mu) = \sum_{\{\boldsymbol{s}_{(5)}\}} W_n(\boldsymbol{s}_{(5)}) \tag{32}$$

related to the weight of the $\boldsymbol{s}_{(5)}$ configurations

$$W_n(\boldsymbol{s}_{(5)}) = \left(\frac{1}{2}\right)^5 \left(1 + \prod_\alpha^n \widetilde{\sigma}_0^{(\alpha)}\right) \prod_{i=1}^4 \left(1 + \prod_\alpha^n \widetilde{\sigma}_i^{(\alpha)}\right) \prod_\alpha^n \mathrm{e}^{(\widetilde{\beta}\, \widetilde{\sigma}_0^{(\alpha)} + \mu) \widetilde{\sigma}_i^{(\alpha)}}\,. \tag{33}$$

The parameter μ is a sort of chemical potential representing the energy cost necessary to flip the lateral spins in the opposite direction of $\widetilde{\sigma}_0$, destroying the short-range order. In fact, the Bethe–Peierls approximation is also indicated as the *quasi-chemical* approximation.

As a consequence, the internal energy becomes

$$\mathcal{U}_n(\widetilde{\beta}) = -2 \langle \widetilde{\sigma}_0\, \widetilde{\sigma}_1 \rangle_n\,. \tag{34}$$

Here and in the following $\langle A \rangle_n$ indicates the average of an observable A,

$$\langle A \rangle_n \equiv z^{-1}(n, \widetilde{\beta}, \mu) \sum_{\{\boldsymbol{s}_{(5)}\}} A\, W_n(\boldsymbol{s}_{(5)}) \tag{35}$$

over the 2^{5n} configurations of the spins on the replicated crosses weighted by W_n. The chemical potential μ depends on the replica number n and can be determined through a self-consistency equation given by the requirements that the average value of the dual spin is invariant under translations, i.e.

$$\langle\widetilde{\sigma}_0\rangle_n = \langle\widetilde{\sigma}_i\rangle_n \qquad i = 1, \ldots, 4 \quad \alpha = 1, \ldots, n\,. \tag{36}$$

One is interested in the limit $n \to 0$, as usual. In order to write the self-consistency equation in a simpler form, it is convenient to introduce the generating function

$$\phi_n(h, \mu, \widetilde{\beta}) = \ln \sum_{\{s_{(5)}\}} W_n(s_{(5)})\, \mathrm{e}^{h \sum_\alpha \widetilde{\sigma}_0^{(\alpha)}} \tag{37}$$

so that (36) corresponds to requiring

$$\left.\frac{\partial\phi}{\partial h}\right|_{h=0} = \left.\frac{1}{4}\frac{\partial\phi}{\partial\mu}\right|_{h=0} \tag{38}$$

where ϕ is the quenched generating function,

$$\phi(h, \mu, \widetilde{\beta}) = \lim_{n\to 0} \frac{\phi_n}{n}\,.$$

The solution of this implicit equation gives the value of the chemical potential $\mu^*(\widetilde{\beta})$ as a function of the temperature. The internal energy can then be expressed in terms of the generating function as

$$\mathcal{U}(\widetilde{\beta}) = -\frac{1}{2}\left.\frac{\partial\phi}{\partial\widetilde{\beta}}\right|_{h=0,\,\mu^*(\widetilde{\beta})}\,. \tag{39}$$

In order to obtain the quenched generating function ϕ, we should perform some algebraic manipulations. After performing the sum over the 2^{4nN} configurations $\{s_{(4)}\}$ in (37) we remain with a sum over the configurations $s_0 \equiv \widetilde{\sigma}_0^{(1)}, \ldots, \widetilde{\sigma}_0^{(n)}$

$$\begin{aligned}
2^5\,\mathrm{e}^{\phi_n} &= \sum_{\{s_0\}}\left(1 + \prod_\alpha \widetilde{\sigma}_0^{(\alpha)}\right)\left(\prod_\alpha 2\cosh\eta^{(\alpha)} + \prod_\alpha 2\sinh\eta^{(\alpha)}\right)^4 \mathrm{e}^{h\sum_\alpha \widetilde{\sigma}_0^{(\alpha)}} \\
&= \sum_{\{s_0\}}\sum_{k=0}^{4}\binom{4}{k}\left(1 + \prod_\alpha \widetilde{\sigma}_0^{(\alpha)}\right)\prod_\alpha\left(2\cosh\eta^{(\alpha)}\right)^{4-k}\prod_\alpha\left(2\sinh\eta^{(\alpha)}\right)^k \mathrm{e}^{h\widetilde{\sigma}_0^{(\alpha)}} \\
&= \sum_{\{s_0\}}\sum_{k=0}^{4}\sum_{j=\pm1}\binom{4}{k}\prod_\alpha (\widetilde{\sigma}_0^{\alpha)})^{(1+j)/2}\left(2\cosh\eta^{(\alpha)}\right)^{4-k}\left(2\sinh\eta^{(\alpha)}\right)^k \mathrm{e}^{h\widetilde{\sigma}_0^{(\alpha)}}
\end{aligned} \tag{40}$$

where we have introduced the variable

$$\eta^{(\alpha)} \equiv \mu + \widetilde{\beta}\,\widetilde{\sigma}_0^{(\alpha)} \tag{41}$$

for simplifying the notation. Now, the previous sum has been obtained by an annealed average over the disorder, i.e.

$$\sum_{k=0}^{4}\sum_{j=\pm1} 2^{-5}\binom{4}{k} A_{k,j} = \overline{A} \tag{42}$$

where one has

$$A_{k,j} = \sum_{\{s_0\}} \prod_{\alpha} (\widetilde{\sigma}_0^{(\alpha)})^{(1+j)/2} \big(2\cosh\eta^{(\alpha)}\big)^{4-k} \big(2\sinh\eta^{(\alpha)}\big)^k \, \mathrm{e}^{h\widetilde{\sigma}_0^{(\alpha)}}$$

$$= \Bigg(\sum_{\{\widetilde{\sigma}_0\}} \widetilde{\sigma}_0^{(1+j)/2} \, (2\cosh\eta)^{4-k} \, (2\sinh\eta)^k \, \mathrm{e}^{h\widetilde{\sigma}_0}\Bigg)^n \equiv (a_{k,j})^n \tag{43}$$

with $\eta = \mu + \beta\widetilde{\sigma}_0$. Noting that

$$\lim_{n\to 0} \frac{1}{n} \ln \overline{a^n} = \overline{\ln a}$$

eventually one can write the quenched generating function as

$$\phi = \frac{1}{2^5} \sum_{j=\pm 1} \sum_{k=0}^{4} \binom{4}{k} \ln \sum_{\widetilde{\sigma}_0=\pm 1} \mathrm{e}^{h\widetilde{\sigma}_0} \, \widetilde{\sigma}_0^{(1+j)/2} \, \cosh^{4-k}\eta \, \sinh^k\eta \, . \tag{44}$$

Here and in the following we omit writing the constant additive term $4\ln 2$ in ϕ. Note that such a term disappears in the derivatives. The first sum over j in (44) can be performed by a trick. Let us use an auxiliary spin $\widetilde{\sigma}'_0 = \pm 1$ with equal probability so that

$$\phi = \frac{1}{2^5} \sum_{k=0}^{4} \binom{4}{k} \ln \sum_{\widetilde{\sigma}_0,\, \widetilde{\sigma}'_0=\pm 1} \mathrm{e}^{h(\widetilde{\sigma}_0+\widetilde{\sigma}'_0)} \, \widetilde{\sigma}_0 \, \big(\cosh\eta \, \cosh\eta'\big)^{4-k} \, \big(\sinh\eta \, \sinh\eta'\big)^k \tag{45}$$

with $\eta' \equiv \mu + \widetilde{\beta}\widetilde{\sigma}'_0$. It is easy to realize that if $\widetilde{\sigma}_0$ and $\widetilde{\sigma}'_0$ have opposite sign, the contribution to the sum vanishes. We can limit ourselves to consider the case of equal sign, so that (45) becomes

$$\phi = \frac{1}{2^5} \sum_{k=0}^{4} \binom{4}{k} \ln \sum_{\widetilde{\sigma}_0=\pm 1} \widetilde{\sigma}_0 \, \mathrm{e}^{2h\widetilde{\sigma}_0} \, \cosh^{2(4-k)}\eta \, \sinh^{2k}\eta \, . \tag{46}$$

Moreover, in the limit $h \to 0$, one has

$$\widetilde{\sigma}_0 \, \mathrm{e}^{2h\widetilde{\sigma}_0} \sim \widetilde{\sigma}_0 + 2\,h \, . \tag{47}$$

It follows that $\langle\widetilde{\sigma}_0\rangle$ is

$$\frac{\partial\phi}{\partial h}\bigg|_{h=0} = \frac{1}{2^5} \sum_{k=0}^{4} \binom{4}{k} S_k^{-1} \sum_{\widetilde{\sigma}_0=\pm 1} 2\cosh^{2(4-k)}\eta \, \sinh^{2k}\eta \tag{48}$$

with

$$S_k = \sum_{\widetilde{\sigma}_0=\pm 1} \widetilde{\sigma}_0 \, \cosh^{2(4-k)}\eta \, \sinh^{2k}\eta \tag{49}$$

while $\sum_{i=1}^{4} \langle\widetilde{\sigma}_i\rangle$ is

$$\frac{\partial\phi}{\partial\mu}\bigg|_{h=0} = \frac{1}{2^5} \sum_{k=0}^{4} \binom{4}{k} S_k^{-1} \sum_{\widetilde{\sigma}_0=\pm 1} \widetilde{\sigma}_0 Y_k(\eta) \tag{50}$$

with

$$Y_k(\eta) = \cosh^{2(4-k)}\eta\ \sinh^{2k}\eta\ [2(4-k)\tanh\eta + 2k\ \coth\eta]\ . \tag{51}$$

Inserting (45) and (46) into the self-consistency equation (38), the chemical potential μ^* can be obtained as a function of the dual inverse temperature $\widetilde{\beta}$. Once we have determined the value of μ^*, the internal energy is given by a derivative of the generating function. In particular, one has

$$\mathcal{U}(\widetilde{\beta}) = -\frac{1}{2}\left.\frac{\partial\phi}{\partial\widetilde{\beta}}\right|_{h=0,\,\mu^*(\widetilde{\beta})} = -\frac{1}{2^6}\sum_{k=0}^{4}\binom{4}{k}S_k^{-1}\sum_{\widetilde{\sigma}_0=\pm1}Y_k(\eta)\,. \tag{52}$$

In figure 3 we show $\mu^*/\widetilde{\beta}$ as a function of $\widetilde{\beta}$, where one observes that for $\widetilde{\beta}\to\infty$ (limit of high temperature T of the original lattice), $\mu^*/\widetilde{\beta}\to 3$. This can be understood noting that each one of the lateral free spins of the cross interacts with three other spins, so that at zero dual temperature $\widetilde{\beta}^{-1}$, the energy lost in a flip is exactly equal to 3. Figure 4 illustrates the graphical solution of (38) by plotting

$$\lim_{n\to0}\left(\langle\widetilde{\sigma}_0\rangle_n - \frac{1}{4}\sum_{i=1}^{4}\langle\widetilde{\sigma}_i\rangle_n\right) \tag{53}$$

as a function of μ at three different temperatures. The solution μ^* of the self-consistency equation is given by the intersection of the function with the horizontal axes. We look only for real solutions. At large $\widetilde{\beta}$, there exists only one solution. However, figure 4 shows that at low $\widetilde{\beta}$ two solutions appear, and for $\widetilde{\beta} < 0.031$ there is no real positive solution. The internal energy U given by the Bethe–Peierls approximation together with the annealed energy $U_a = -2\tanh(\beta)$ and the mean-field result are plotted as function of the temperature in figure 5. One sees that for $\widetilde{\beta} < 0.05$, i.e. $T < 0.667\ldots$, the energy increases

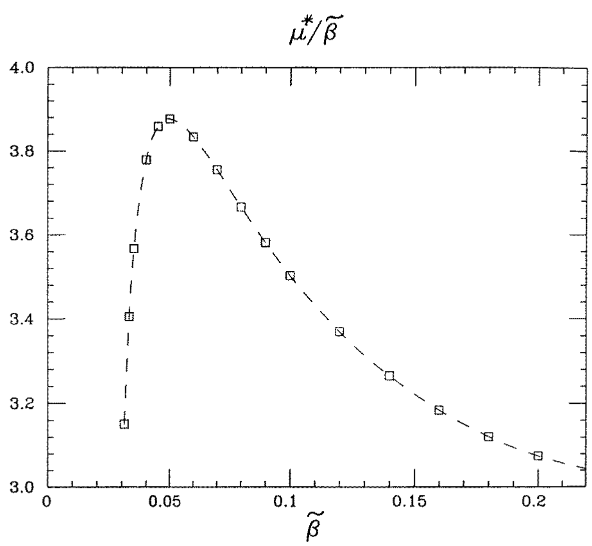

Figure 3. Chemical potential $\mu^*/\widetilde{\beta}$ as a function of the dual inverse temperature $\widetilde{\beta}$.

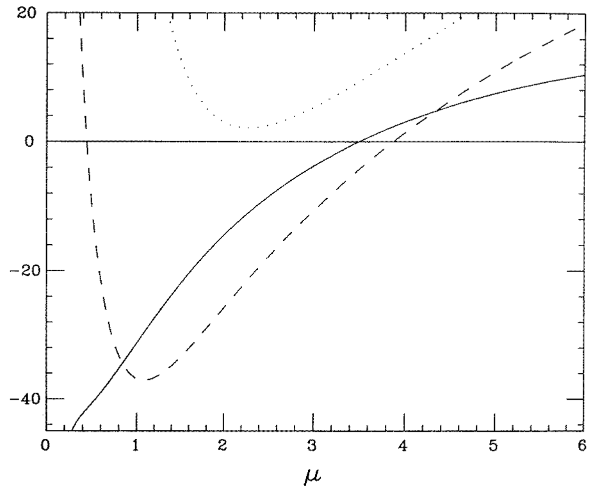

Figure 4. The graphical solution of the self-consistency equation (36) is given by the intersection of (53) with the horizontal axes. We illustrate three different cases: $\widetilde{\beta} = 0.1$ (full curve), $\widetilde{\beta} = 0.05$ (broken curve) and $\widetilde{\beta} = 0.028$ (dotted curve).

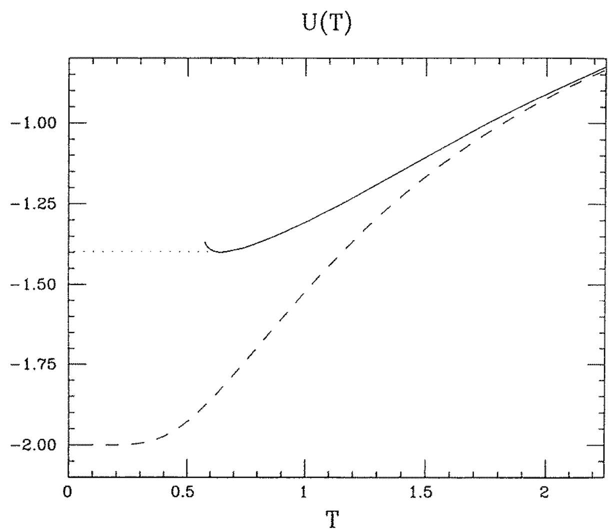

Figure 5. Internal energy given by the annealed approximation, i.e. $U_a = -2\tanh\beta$ (broken curve), and by the Bethe–Peierls solution (full curve) $U(T)$ versus temperature $T = \beta^{-1}$. The dotted line is obtained by imposing that the Bethe–Peierls internal energy is a monotonic non-increasing function of T.

with decreasing the temperature while the chemical potential μ^* decreases, indicating that the Bethe–Peierls solution becomes unphysical. The ground-state energy can be estimated by the minimum value assumed by the internal energy, i.e.

$$E_0 \approx \min_{\widetilde{\beta}} U(\widetilde{\beta}) = U(\widetilde{\beta} = 0.05\ldots) = -1.3975\,.$$

It is extremely close to the numerical estimate of [7] $E_0 = -1.404 \pm 0.002$.

Finally, we want to mention that the problem remains open to understand whether, with an appropriate ansatz of replica symmetry breaking, one can obtain a solution of the self-consistency equation for $\widetilde{\beta} < 0.03$. It is indeed well known that the mean-field approach can give phase transitions, even when they are absent in the original model.

5. Improved Bethe–Peierls approximation

In order to improve the Bethe–Peierls approximation in the framework of the replica method, we introduce a second variational parameter γ, beyond the chemical potential μ, that puts in interaction the different replicas. In other words, we replace the (standard) ansatz (31) with

$$\Psi_n = \prod_\alpha \prod_{i=1}^4 \mathrm{e}^{\mu \sum_\alpha \widetilde{\sigma}_i^{(\alpha)} + \gamma \sum_{\alpha>\beta} \widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_i^{(\beta)}} \frac{\overline{(Z_N)^n}}{g(n, \widetilde{\beta}, \mu, \gamma)} \tag{54}$$

where we have introduced the normalization factor

$$g(n, \widetilde{\beta}, \mu, \gamma) = \sum_{\{\boldsymbol{s}_{(5)}\}} P_n(\boldsymbol{s}_{(5)}) \tag{55}$$

related to the weight of the $\boldsymbol{s}_{(5)}$ configurations

$$P_n(\boldsymbol{s}_{(5)}) = W_n\, \mathrm{e}^{\gamma \sum_{\alpha>\beta} \widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_i^{(\beta)}} \tag{56}$$

with W_n given by (33). We shall indicates the average of an observable A over this new normalized weight by $\langle\langle A \rangle\rangle_n$.

The two variational parameters are determined by the coupled self-consistency equations obtained in the limit $n \to 0$ by

$$\langle\langle \widetilde{\sigma}_0 \rangle\rangle_n = \langle\langle \widetilde{\sigma}_i \rangle\rangle_n \tag{57a}$$
$$\langle\langle \widetilde{\sigma}_0^{(\alpha)}\, \widetilde{\sigma}_0^{(\beta)} \rangle\rangle_n = \langle\langle \widetilde{\sigma}_i^{(\alpha)}\, \widetilde{\sigma}_i^{(\beta)} \rangle\rangle_n \tag{57b}$$

with $i = 1, \ldots, 4$ and $\alpha, \beta = 1, \ldots, n$.

The ansatz proposed here is related to a hypothesis of existence of a glassy phase. In fact, one can apply this approximation scheme to the solution of the random-coupling Ising model directly on the original lattice in d dimensions [11]. In this case, after performing the limit $d \to \infty$, one obtains the Parisi solution [2] of the Sherrington–Kirkpatrick model.

Following the same idea as in the previous section, let us introduce the generating function

$$\psi_n(h, \ell, \mu, \gamma, \widetilde{\beta}) = \ln \sum_{\{\boldsymbol{s}_{(5)}\}} P_n(\boldsymbol{s}_{(5)}) \mathrm{e}^{h \sum_\alpha \widetilde{\sigma}_0^{(\alpha)} + \ell \sum_{\alpha>\beta} \widetilde{\sigma}_i^{(\alpha)} \widetilde{\sigma}_i^{(\beta)}} \tag{58}$$

so that (57a) and (57b) correspond to requiring

$$\left.\frac{\partial \psi}{\partial h}\right|_{h=0,\ell=0} = \frac{1}{4} \left.\frac{\partial \psi}{\partial \mu}\right|_{h=0,\ell=0} \tag{59a}$$

and

$$\left.\frac{\partial \psi}{\partial \ell}\right|_{h=0,\ell=0} = \frac{1}{4} \left.\frac{\partial \psi}{\partial \gamma}\right|_{h=0,\ell=0} \tag{59b}$$

where ψ is the quenched generating function,

$$\psi(h, \mu, \gamma, \widetilde{\beta}) = \lim_{n \to 0} \frac{\psi_n}{n} \,. \tag{60}$$

The solution of this implicit equation gives the values $\mu^*(\widetilde{\beta})$ and γ^* as functions of the temperature. The internal energy can then be expressed in terms of the generating function as

$$\mathcal{U}(\widetilde{\beta}) = -\frac{1}{2}\left.\frac{\partial\psi}{\partial\widetilde{\beta}}\right|_{h=0,\ell=0\,,\,\mu^*(\widetilde{\beta}),\gamma^*(\widetilde{\beta})}. \tag{61}$$

Let us now simplify as much as possible the self-consistency equations. It is convenient to use the standard Gaussian identity

$$\begin{aligned}\exp\left(\ell\sum_{\alpha>\beta}\widetilde{\sigma}_0^{(\alpha)}\widetilde{\sigma}_0^{(\beta)}\right) &= \exp\left(\frac{\ell}{2}\left(\sum_{\alpha}\widetilde{\sigma}_0^{(\alpha)}\right)^2-\frac{\ell n}{2}\right)\\ &= \int\frac{\mathrm{d}x_0}{\sqrt{2\pi}}\exp\left(-\frac{\ell n}{2}+\sqrt{\ell}x_0\sum_{\alpha}\widetilde{\sigma}_0^{(\alpha)}-\frac{x_0^2}{2}\right)\end{aligned} \tag{62}$$

in order to write the generating function as

$$\begin{aligned}\psi_n = -\frac{(4\gamma+\ell)\,n}{2} + \ln\Bigg(2^{-5}\int\prod_{i=0}^{4}\frac{\mathrm{d}x_i}{\sqrt{2\pi}}\,\mathrm{e}^{-x_i^2/2}\\ \times\sum_{\{s_0\}}\left(1+\prod_{\alpha}\widetilde{\sigma}_0^{(\alpha)}\right)\left(\prod_{\alpha}2\cosh\omega_i^{(\alpha)}+\prod_{\alpha}2\sinh\omega_i^{(\alpha)}\right)\mathrm{e}^{(h+x_0\sqrt{\ell})\sum_\alpha\widetilde{\sigma}_0^{(\alpha)}}\Bigg)\end{aligned} \tag{63}$$

with

$$\omega_i^{(\alpha)} = \eta^{(\alpha)} + x_i\sqrt{\gamma}\,. \tag{64}$$

Using the same 'algebraic' strategy that in the previous section leads to (44), we obtain the quenched generating function as

$$\begin{aligned}\psi = -2\gamma - \frac{\ell}{2} + \frac{1}{2^5}\int\prod_{i=0}^{4}\frac{\mathrm{d}x_i}{\sqrt{2\pi}}\,\mathrm{e}^{-x_i^2/2}\sum_{j_1,j_2,j_3,j_4=\pm1}\\ \times\ln\left(\sum_{\widetilde{\sigma}_0=\pm1}\widetilde{\sigma}_0\,\mathrm{e}^{2\,(h+\sqrt{\ell}x_0)\widetilde{\sigma}_0}\prod_{i=1}^{4}(\cosh\omega_i)^{1+j_i}\,(\sinh\omega_i)^{1-j_i}\right)\end{aligned} \tag{65}$$

with

$$\omega_i = \eta + x_i\sqrt{\gamma} = \widetilde{\beta}\,\widetilde{\sigma}_0 + \mu + x_i\sqrt{\gamma}\,.$$

The constant additive term $4\ln 2$ is again omitted.

By derivating ψ one has the self-consistency equations (57) for μ^* and γ^* in terms of the sum of five Gaussian integrals. A careful analytic and numerical study of these equations might give enlightenment as to the nature of spin glasses in low-dimensional systems.

6. Conclusions

Let us briefly summarize our main results.

(i) Formulation of the random coupling Ising model on the dual lattice made of square plaquettes. The dual model has signed Gibbs measure as the random coupling is transformed into a random complex magnetic field.

(ii) Application of the mean-field approximation to the two-dimensional dual model in the framework of the replica method. We find the solution using a replica-symmetry ansatz, obtaining a good estimate of the ground-state energy: $E_0 = -1.468$, compared with the numerical result of [7] $E_0 = -1.404 \pm 0.002$.

(iii) Application of the Bethe–Peierls approximation. It gives a very accurate estimate of the ground-state energy ($E_0 = -1.3975$) although it becomes unphysical below $\widetilde{\beta} = 0.05$ and there is no real solution of the self-consistency equation for $\widetilde{\beta} < 0.03$.

(iv) Improvement scheme of the Bethe–Peierls approximation by considering a second variational parameter that puts in interaction different replicas of the dual model.

There are still many problems that remain open in our approach. As a major issue, it would be interesting to understand whether the ansatz proposed in section 4, or other similar assumptions, lead to a solution of the self-consistency equations of the Bethe–Peierls approximation at low $\widetilde{\beta}$, that would allow one to decide whether a transition to a glassy phase is present at low dimension.

The dual transformation is indeed a very powerful tool for determining the critical temperature in non-disordered systems and our method might give some results in this direction. Last but not least, we plan to find a cluster expansion scheme that permits improvement of the mean field in a systematic way.

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