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ON THE CONCEPT OF COMPLEXITY OF RANDOM DYNAMICAL SYSTEMS*

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We show how to introduce a characterization the "complexity" of random dynamical systems. More precisely we propose a suitable indicator of complexity in terms of the average number of bits per time unit necessary to specify the sequence generated by these systems. This indicator of complexity, which can be extracted from real experimental data, turns out to be very natural in the context of information theory. For dynamical systems with random perturbations, it coincides with the rate K of divergence of nearby trajectories evolving under two different noise realizations. In presence of strong dynamical intermittency, the value of K is very different from the standard Lyapunov exponent λ_σ computed through the consideration of two nearby trajectories evolving under the same realization of the random perturbation. However, the former is much more relevant than the latter from a physical point of view as illustrated by some numerical examples of noisy and random maps.

1. Introduction

In deterministic dynamical systems there exist well established ways to define the complexity of a temporal evolution in terms of the Lyapunov exponents and the Kolmogorov–Sinai entropy.

The situation is much more ambiguous with random perturbations, which are always present in physical systems as a consequence of thermal fluctuations or hidden changes of control parameters, and, in numerical experiments, because of the roundoff errors.[1]

In the literature, a first crude conclusion is that the presence of a small noise does not change the qualitative behavior of the dynamics.[2] In the case of a regular (stable) system, the random perturbation just changes the very long time behavior by introducing the possibility of jumps among different attractors (stable fixed points, stable limit cycles or tori). A familiar example is a Langevin equation describing the motion of an overdamped particle in a double well.

Even in the opposite extreme of chaotic dissipative systems, the presence of noise is expected not to change the qualitative behavior in a dramatic way.

*This paper is dedicate to the memory of our friend Giovanni Paladin.

The typical situation is the following:

(a) the strange attractor maintains the fractal structure at larger scales, although it is smoothed at small scales of $O(\sigma)$, where σ is the strength of the noise;
(b) the value of the Lyapunov exponent differs from the unperturbed one by a quantity of $O(\sigma)$.

However, the combined effects of the noise and the deterministic part of the evolution law can produce highly nontrivial, and often intriguing, behaviors.[3–8] Let us mention the stochastic resonance where there is a synchronization of the jumps between two stable points[9–12] and the phenomena of the so called noise-induced order[7] and of the noise-induced instability.[5,6]

One of the main problems is the lack of a well defined method to characterize the "complexity" of the trajectories. Usually,[2,5,7] the degree of chaoticity is measured by treating the random term as a usual time-dependent term, and therefore, considering the separation of two nearby trajectories with the same realization of the noise. In this way it is possible to compute the maximum Lyapunov exponent λ_σ associated with the separation rate of two nearby trajectories with the same realization of the stochastic term.

Although the Lyapunov exponent λ_σ is a well defined quantity, it is neither unique nor the most useful characterization of complexity. In addition, a moment of reflection shows that it is practically impossible to extract λ_σ from experimental data.

In this paper we discuss how a more natural indicator of complexity, for noisy and random systems, can be obtained by computing the separation rate of nearby trajectories evolving in two different realizations of the noise, instead of only one. Let us stress that, such a procedure exactly corresponds to what happens when experimental data are analyzed with the Wolf *et al.* algorithm.[13] Basically, our measure of complexity, defined in Refs. 14 and 15, obviously inspired by the pioneering contribution of Shannon,[16] is related to the average number per unit time of the bits necessary to specify the sequence generated by a random evolution law.

2. The Naive Approach: The Noise Treated as a Standard Function of Time

The simplest approach to the "complexity" is to treat the random term as a usual time-dependent external force, and, therefore, to consider the separation of two nearby trajectories with the same realization of the noise. Such a characterization can be misleading, as illustrated in the following example.

Let us consider a one-dimensional Langevin equation

$$\frac{dx}{dt} = -\frac{\partial V(x)}{\partial x} + \sqrt{2\sigma}\,\eta\,, \tag{1}$$

where $V(x)$ diverges for $|x| \to \infty$, e.g. the usual double well $V = -x^2/2 + x^4/4$, and $\eta(t)$ is a white noise.

The Lyapunov exponent λ_σ associated with the separation rate of two nearby trajectories with the same realization of the stochastic term $\eta(t)$ is

$$\lambda_\sigma = \lim_{t\to\infty} \frac{1}{t} \ln|z(t)|\,, \tag{2}$$

where the evolution of the tangent vector (that should be regarded as an infinitesimal perturbation of the trajectory $x(t)$) is

$$\frac{dz}{dt} = -\frac{\partial^2 V(x(t))}{\partial x^2} z(t)\,. \tag{3}$$

Since the system is ergodic with invariant probability distribution $P(x) = Ce^{-V(x)/\sigma}$, one has

$$\begin{aligned}\lambda_\sigma = \lim_{t\to\infty} \frac{1}{t} \ln|z(t)| &= -\lim_{t\to\infty} \frac{1}{t} \int_0^t \partial^2_{xx} V(x(t'))dt' \\ &= -C \int \partial^2_{xx} V(x) e^{-V(x)/\sigma} dx \\ &= -\frac{C}{\sigma} \int (\partial_x V(x))^2 e^{-V(x)/\sigma} dx < 0\,.\end{aligned} \tag{4}$$

This result is rather intuitive: the trajectory $x(t)$ spends most of the time in one of the "valleys" where $-\partial^2_{xx}V(x) < 0$ and only for short periods on the "hills" where $-\partial^2_{xx}V(x) > 0$, so that the distance between two trajectories evolving in the same noise realization decreases in average. Notice that in Ref. 18, supported by a wrong argument, an opposite result is claimed.

As a matter of fact, $\lambda_\sigma < 0$ implies a fully predictable process ONLY IF the realization of the noise is known. In the more sensible case of two initially close trajectories evolving under two different noise realizations, after a certain time T_σ, the two trajectories can be very distant, because they can be in two different "valleys". For $\sigma \to 0$, due to the Kramers formula, one has $T_\sigma \sim \exp \Delta V/\sigma$, where ΔV is the difference between the values of V on the top of the hill and at the bottom of the valley.

The result obtained for the one dimensional Langevin equation can easily be generalized to any dimension for gradient systems if the noise is small enough.[15]

Another example of the limitations of the above mentioned naive approach is provided by the case of stochastic resonance. In this case, in fact, one can find the same qualitative behavior both for a positive and a negative Lyapunov exponent. We refer to Ref. 15 for more details.

3. Dynamical Systems with Noise

The main difficulties in defining the notion of complexity of an evolution law with a random perturbation already appear in 1D maps. In fact, the generalization to N-dimensional maps or to coupled ordinary differential equations is straightforward.

Let us, therefore, consider the model

$$x(t+1) = f[x(t), t] + \sigma w(t)\,, \tag{5}$$

where t is an integer and $w(t)$ is an uncorrelated random process, e.g. w are independent random variables uniformly distributed in $[-1, 1]$. The maximum Lyapunov exponent λ_σ defined in (2) is given by the map for the tangent vector:

$$z(t+1) = f'[x(t), t]\, z(t)\,, \tag{6}$$

where $f' = df/dx$. For $\sigma = 0$, λ_0 is the Lyapunov exponent of the unperturbed map.

In order to introduce a more natural indicator of complexity of noisy dynamics it is convenient to follow a quite different approach, where two realizations of the noise, instead of only one, are used.[14]

Before discussing our alternative definition of chaos in noisy systems, we must briefly recall what characterizes the intermittency in deterministic dynamical systems. An effective Lyapunov exponent[19,20] has been introduced to measure the fluctuations of chaoticity

$$\gamma_t(\tau) = \frac{1}{\tau} \ln \frac{|z(t+\tau)|}{|z(t)|}\,. \tag{7}$$

It gives the local expansion rate in the interval $[t, t+\tau]$. The maximum Lyapunov exponent is thus given by a time average along the trajectory $x(t)$: $\lambda_0 = \langle \gamma_t \rangle$ for $\tau \to \infty$.

As first let us discuss the case of deterministic maps. Let $x(t)$ be the trajectory starting from $x(0)$ and $x'(t)$ be the trajectory starting from $x'(0) = x(0) + \delta x(0)$. Let $\delta_0 \equiv |\delta x(0)|$ and indicate by τ_1 the minimum time t_1 such that $|x'(t_1) - x(t_1)| \geq \Delta$. Then, we put $x'(\tau_1 + 1) = x(\tau_1 + 1) + \delta x(0)$ and define τ_2 as the minimum t_2 such that $|x'(\tau_1 + t_2) - x(\tau_1 + t_2)| > \Delta$ and so on. In our context, we can define the effective Lyapunov exponent as

$$\gamma_i = \frac{1}{\tau_i} \ln \frac{\Delta}{\delta_0}\,. \tag{8}$$

However, we sample the expansion rate in a non-uniform way, at time intervals τ_i. As a consequence the probability of picking γ_i is $p_i = \tau_i / \sum_i \tau_i$ so that

$$\lambda_0 = \langle \gamma_i \rangle = \frac{\sum_i \tau_i \gamma_i}{\sum \tau_i} = \frac{1}{\bar{\tau}} \ln \left(\frac{\Delta}{\delta_0} \right) \qquad \bar{\tau} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \tau_i \tag{9}$$

This definition without any modification can be extended to noisy systems by introducing the rate

$$K_\sigma = \frac{1}{\bar{\tau}} \ln \left(\frac{\Delta}{\delta_0} \right) \tag{10}$$

which coincides with λ_0 for a deterministic system ($\sigma = 0$). When $\sigma = 0$ there is no reason to determine the Lyapunov exponent in this apparently odd way, of course. However, the introduction of K_σ is rather natural in the framework of information theory.[21] Considering again the noiseless situation, if one wants to transmit the sequence $x(t)$ ($t = 1, 2, \ldots, T_{\max}$) accepting only errors smaller than a tolerance threshold Δ, one can use the following strategy:

(1) Transmit the rules which specify the dynamical system (1), using a finite number of bits which does not depend on the length $T_{\max}$.
(2) Specify the initial condition with precision δ_0 using a finite number of bits which does not depend on the length $T_{\max}$.
(3) Let the system evolve till the time τ_1 where the error equals Δ and then specify again the new initial condition $x(\tau_1 + 1)$ with a precision δ_0. The number of bits which are necessary for this reduction of the error is $n = \ln_2(\Delta/\delta_0)$. Then let the system evolve and repeat the procedure N times until $T_{\max}$ is reached. Since $T_{\max} = \sum_{i=1}^{N} \tau_i \simeq N\bar{\tau}$, the mean information for unit of time is $\simeq Nn/T_{\max} = K_\sigma / \ln 2$ bits.

In presence of noise, the strategy of the transmission is unchanged but since it is not possible to transmit the realization of the noise $w(t)$, one has to estimate the growth of the error $\delta x(t) = x'(t) - x(t)$, where $x(t)$ and $x'(t)$ evolve in two different noise realizations $w(t)$ and $w'(t)$, and $|\delta x(0)| = \delta_0$.

The resulting equation for the evolution of $\delta x(t)$ is:

$$\delta x(t+1) \simeq f'[x(t), t]\, \delta x(t) + \sigma \tilde{w}(t) \quad \tilde{w}(t) = w'(t) - w(t) \tag{11}$$

For the sake of simplicity we discuss the case $|f'[x(i), i]| = \text{const} = \exp \lambda_0$, where (11) gives a bound on the error:

$$|\delta x(t)| < e^{\lambda_0 t}(\delta_0 + \tilde{\sigma}) \qquad \text{with} \qquad \tilde{\sigma} = \frac{2\sigma}{e^{\lambda_0} - 1} \tag{12}$$

This formula shows that δ_0 and $\bar{\tau}$ are not independent variables but they are linked by the relation

$$e^{\lambda_0 \bar{\tau}}(\delta_0 + \tilde{\sigma}) \simeq \Delta \tag{13}$$

As a consequence, we have only one free parameter, say $\bar{\tau}$, to optimize the information entropy K_σ in (10), so that the complexity of the noisy system can be estimated by

$$G_\sigma = \min_{\bar{\tau}} K_\sigma = \lambda_0 + O(\sigma/\Delta) \tag{14}$$

where the minimum is reached at an optimal time $\bar{\tau} = \tau_{\text{opt}}$ from the transmitter point of view.

It should be noticed that in the case of a deterministic system K_0 does not depend on the value of $\bar{\tau}$ (i.e. it is equivalent to using a long $\bar{\tau}$ and to transmit many bits a few times or a short $\bar{\tau}$ and to transmit a few bits many times). On the contrary, in noisy systems there exists an optimal time τ_{opt} which minimizes K_σ: using relation (12) one sees that $\Delta = \exp(\lambda_0 \bar{\tau})(\delta_0 + \tilde{\sigma})$ and K_σ has a minimum for $\tau_{\text{opt}} \simeq 1/\lambda_\sigma$. Let us stress that this corresponds to transmitting the initial condition with a precision $\delta_0 \simeq (\Delta - \tilde{\sigma})/e$. For small noises $\delta_0 \gg \sigma$ and it is convenient to transmit rather often the initial condition with a small accuracy.

The interesting situation happens for strong intermittency when there are alternations of positive and negative γ during a long time interval. In this case the existence of an optimal time for the transmission induces a dramatic change for the value of G_σ. This becomes particularly clear when we consider the limiting case of positive γ_1 in an interval $T_1 \gg 1/\gamma_1$ followed by a negative γ_2 in an interval $T_2 \gg 1/|\gamma_2|$, and again a positive effective Lyapunov exponent and so on. During the intervals with positive effective Lyapunov exponent the transmission has to be repeated rather often with $\simeq T_1/(\gamma_1 \ln 2)$ bits at each time, while during the ones with negative effective Lyapunov no information has to be sent. Nevertheless, at the end of the contracting intervals one has $|\delta x| = O(\sigma)$, so that, at variance with the noiseless case, it is impossible to use them to compensate the expanding ones. This implies that in the limit of very large T_i only the expanding intervals contribute to the evolution of the error $\delta x(t)$ and the information entropy is given by an average of the positive effective Lyapunov exponents:

$$G_\sigma \simeq \langle \gamma \theta(\gamma) \rangle . \tag{15}$$

For the approximation considered above, $G_\sigma \geq \lambda_\sigma = \langle \gamma \rangle$. Note that by definition $G_\sigma \geq 0$ while λ_σ can be negative. The estimate (15) stems from the fact that δ_0 cannot be smaller than σ so the typical value of τ_i is $O(1/\gamma_i)$ if γ_i is positive. We stress again that (15) holds only for strong intermittency, while for uniformly expanding systems or rapid alternations of contracting and expanding behaviors $G_\sigma \simeq \lambda_\sigma$.

Let us stress that K_σ is strongly related to the ϵ-entropy introduced by Kolmogorov[22] and recently used for the analysis of experimental data in turbulence.[23] Indeed, the complexity we consider is defined for δ_0 not to small ($\delta_0 \gg \sigma$), nevertheless, if δ_0 and Δ are small enough K_σ is essentially independent by their values. Notice that, at variance with the ϵ-entropy, the computation of K_σ is very simple and rather similar to the usual algorithm for the computation the first Lyapunov exponent. Basically K_σ relates to the ϵ-entropy as the first Lyapunov exponent relates to the Kolmogorov–Sinai entropy.

We report the results of some numerical simulations in two different systems which are shown in Figs. 1 and 2. Let us stress that we have directly computed K_σ, and since $\tau_i = O(1/\gamma_i)$, we are automatically very close to the optimal strategy so that $K_\sigma \simeq G_\sigma$ without performing the minimization. The random perturbation $w(t)$ is an independent variable uniformly distributed in the interval $[-1/2, 1/2]$.

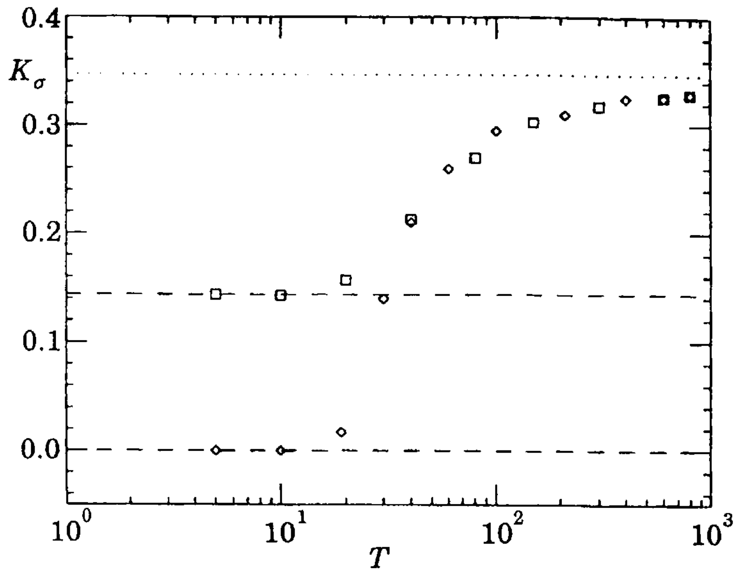

Fig. 1. K_σ versus T with $\sigma = 10^{-7}$ for the map (16). The parameters of map (16) are $a = 2$ and $b = 2/3$ (squares) or $b = 1/4$ diamonds. The dotted line indicates the Pesin-like relation (15) while the dashed lines are the noiseless limit of K_σ. Note that for $b = 1/4$ the Lyapunov exponent λ_σ is negative.

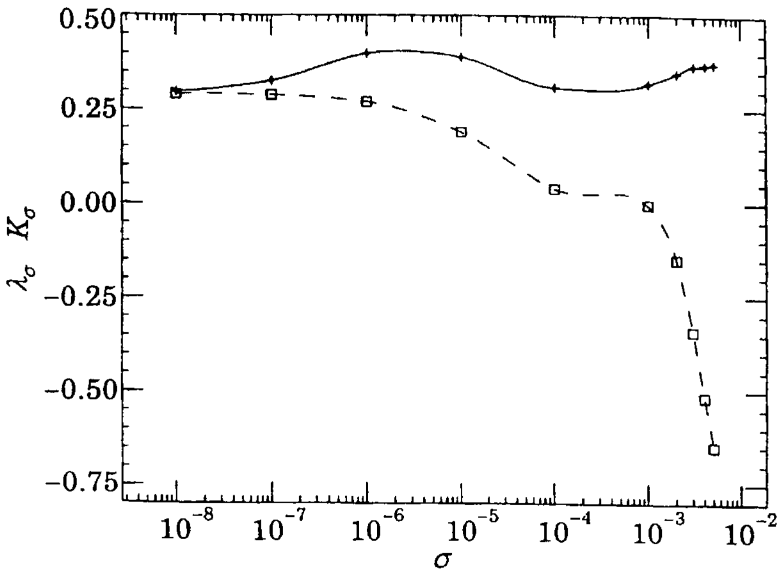

Fig. 2. λ_σ (squares) and G_σ (crosses) versus σ for map (17).

The first system is given by the periodic alternation of two piecewise linear maps of the interval $[0, 1]$ into itself:

$$f[x,t] = \begin{cases} ax \quad (\text{mod } 1) & \text{if } (2n-1)T \le t < 2nT\,; \\ bx & \text{if } 2nT \le t < (2n+1)T \end{cases} \tag{16}$$

where $a > 1$ and $b < 1$. Note that in the limit of small T, $G_\sigma \to \max[\lambda_\sigma, 0]$, because it is a non-negative quantity as shown in Fig. 1 where $b = 1/4$ and λ_σ is negative.

The second system is strongly intermittent without an external forcing. It is the Beluzov–Zhabotinsky map[4,7] related to a famous chemical reaction:

$$f(x) = \begin{cases} [(1/8 - x)^{1/3} + a]e^{-x} + b & \text{if } 0 \leq x < 1/8\,; \\ [(x - 1/8)^{1/3} + a]e^{-x} + b & \text{if } 1/8 \leq x < 3/10\,; \\ c(10xe^{-10x/3})^{19} + b & \text{if } 3/10 \leq x \end{cases} \tag{17}$$

with $a = 0.50607357$, $b = 0.0232885279$, $c = 0.121205692$. The map exhibits a chaotic alternation of expanding and very contracting time intervals. In Fig 1, one sees that while λ passes from negative to positive values at decreasing σ, G_σ is not sensitive to this transition to "order". Another important remark is that in the usual treatment of the experimental data, if some noise is present, one practically computes G_σ and the result can be completely different from λ_σ. In Ref. 6, the author studies a one-dimensional nonlinear time-dependent Langevin equation. A numerical computation shows that λ_σ is negative, but the author claims to find, using the Wolf method, a positive "Lyapunov exponent".

Our results show that the same system can be regarded either as regular (i.e. $\lambda_\sigma < 0$) when the same noise realization is considered for two nearby trajectories or as chaotic (i.e. $G_\sigma > 0$) when two different noise realizations are considered. The situation is similar to what observed in fluids with Lagrangian chaos.[24] There, a pair of particles passively advected by a chaotic velocity field might remain close, following together a "complex" trajectory. The Lagrangian Lyapunov exponent is thus zero. However, a data analysis gives a positive Lyapunov exponent because of the "Eulerian" chaos. We can say that λ_σ and G_σ correspond to the Lagrangian Lyapunov exponent and to the exponential rate of separation of a particle pair in two slightly different velocity fields, respectively.

The relation $G_\sigma \simeq \langle \gamma\theta(\gamma) \rangle$ is, in some sense, the time analogue of the Pesin relation $h \simeq \sum_i \lambda_i \theta(\lambda_i)$ between the Kolmogorov–Sinai entropy h and the Lyapunov spectrum[25] where the negative Lyapunov exponents do not decrease the value of h, because the contraction along the corresponding directions cannot be observed for any finite space partition. In the same way the contracting time intervals, if long enough, do not decrease G_σ.

4. Random Dynamical Systems

In this section we discuss dynamical systems where the randomness is not simply given by an additive noise, as in Sec. 3. This kind of systems has been the subject of much interest in the last few years in relation to the problems involving disorder,[26,27] such as the characterization of the so-called *on-off intermittency*[28] and the modeling of transport problems in turbulent flows.[29] In these systems, in general, the random part represents an ensemble of hidden variables, that are unknown observables, believed to be implicated in the dynamics: the turbulent convection in the solar cycle or several economic factors for the stock market prices are just two examples of this situation. The random part can also mimics the effect of a set of variables which vary in a chaotic way or that vary on a time scale very small relative to the time scale of the phenomenon under investigation. Random maps exhibit very interesting features ranging from stable or quasistable behaviors, to chaotic behaviors and intermittency. In particular *on-off intermittency* is an aperiodic switching between static, or laminar, behavior and chaotic bursts of oscillation. It can be generated by systems having an unstable invariant manifold, within which it is possible to find a suitable attractor (i.e. a fixed point). The intermittency is linked, in the simplest case, to the loss of stability of the fixed point. For further details we refer to Ref. 28.

A random map can be defined in the following way. Denoting with $\mathbf{x}(t)$ the state of the system at discrete time t, the evolution law is given by

$$\mathbf{x}(t+1) = f_{I(t)}(\mathbf{x}(t)) , \tag{18}$$

where $I(t)$ is a random variable. If $I(t)$ is discrete with an entropy h_s, according to the general ideas discussed in Sec. 3, a measure of the complexity of the evolution can be defined in terms of the mean number of bits that must be specified, at each iteration, in order to have a certain tolerance Δ on the knowledge of the state $\mathbf{x}$.

Let us briefly summarize some basic concepts needed to characterize random maps.

(1) The average probability distribution

After a transient, a deterministic map evolves on the invariant set of the dynamics (usually an attractor), where it is possible to define an invariant probability measure. We should now consider the probability density obtained as a limit of the histogram of points given by the iterations of (18), on the "average" invariant set. It is natural to expect that such a distribution might be obtained by means of an average over the randomness realizations. Let us recall that for a deterministic map f, the invariant probability measure ρ is the eigenfunction of the Perron–Frobenius operator L related to the maximum (in modulus) eigenvalue $\gamma_1 = 1$, i.e.

$$\rho(x) = L\rho(x) . \tag{19}$$

The operator is defined as follows

$$L\,\phi(x) = \int dy\,\delta(y - f(x))\,\phi(y) = \sum_{z=f^{-1}(x)} \frac{\phi(z)}{|f'(z)|} . \tag{20}$$

A moment of reflection shows that for a random map it is still possible to find the average probability density by the straightforward generalization of Eq. (19):

$$\rho_{\rm av}(x) = \bar{L}\rho_{\rm av}(x) , \tag{21}$$

where we have introduced the (annealed) average operator

$$\bar{L} = \sum_{j=1}^{k} p_j L_j \tag{22}$$

and L_j is the Perron–Frobenius operator associated to the deterministic map f_j.

(2) The snapshot attractor

In defining the average probability density, we have considered the long-time evolution of a single trajectory under a "typical" realization of the randomness. One can also consider the probability density obtained by starting with a cloud of initial conditions $x_0^{(1)}, \ldots, x_0^{(M)}$, with M very large, that evolve in time under *the same* randomness realization of the dynamics. There exists therefore an instantaneous probability density $\rho_t(x)$ obtained by the histogram of the $x_t^{(j)}$ in the limit $M \to \infty$. Such a probability measure is usually indicated as the snapshot attractor.[29] It is generally different from the average probability density, and asymptotically there are the following situations:

(I) $\rho_t(x) = \delta(x - x^*) = \rho_{\mathrm{av}}(x)$,
where x^* is a stable fixed point;

(II) $\rho_t(x) = \delta(x - x_t^*)$,
where x_t^* changes in time and $\rho_{\mathrm{av}}(x)$ is not a delta function;

(III) $\rho_t(x) = g(x, t)$,
where $g(x, t)$ is a nontrivial function of the space-time.

The regimes (II) and (III) can be characterized by studying other quantities such as Lyapunov exponents, time correlations and complexity measures for random systems, see e.g. Ref. 14.

(3) The Lyapunov exponent for random maps

The Lyapunov exponent λ_{I} provides the simplest information about chaoticity and can be computed considering the separation of two nearby trajectories evolving in the same realization of the random process $I(t) = i_1, i_2, \ldots, i_t$. The Lyapunov exponent λ_{I} generalizes the λ_σ in Sec. 3. It is possible to introduce the Lyapunov exponent for random maps by considering the tangent vector evolution:

$$z_{n+1} = \left.\frac{df_{i_n}}{dx}\right|_{x_n} z_n\,, \tag{23}$$

so that for almost all initial conditions

$$\lambda_{\mathrm{I}} = \lim_{N \to \infty} \frac{1}{N} \ln|z_N|\,. \tag{24}$$

It is worth stressing once more that this characterization of the chaoticity of a random dynamical system can be misleading.[14,15] A negative value of the Lyapunov exponent computed in such a way implies predictability ONLY IF the realization of the randomness is known. For trajectories initially very close but evolving under different realization of the randomness, it can happen that after a certain time the two trajectories will be very distant even with a negative λ_I.

Both in regimes (I) and (II), λ_I is negative, and in the phase (II) it also corresponds to the typical contraction rate of a cloud of points toward the "jumping" snapshot attracting point x_t^*, i.e. one has

$$\sigma_t^2 = \lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \left(x_t^{(j)} - \frac{1}{M} \sum_{i=1}^{M} x_t^{(i)} \right)^2 \sim e^{-2|\lambda|t} . \tag{25}$$

On the other hand, we expect $\lambda > 0$ in the phase (III).

(4) Measures of complexity

We have pointed out how the characterization of the chaoticity of a random dynamical system by the Lyapunov exponent can be misleading. On the other hand, as well as in the case of additive noise, it is possible to introduce a measure of complexity K. The quantity which better accounts for their chaotic properties,[14,15] as

$$K \simeq h_\mathrm{s} + \lambda_\mathrm{I}\theta(\lambda_\mathrm{I}) , \tag{26}$$

where h_s is the Shannon entropy[16] of the random sequence $I(t)$, λ_I is the Lyapunov exponent defined above and θ is the Heaveside step function. The meaning of the complexity K is rather clear: $K/\log 2$ is the mean number of bits, for each iteration, necessary to specify the sequence $x_1, \ldots, x_t$ with a certain tolerance Δ.

Therefore, there are two different contributions to the complexity:

(a) one has to specify the sequence $I(1), I(2), \ldots, I(t)$ which implies $h_\mathrm{s}/\ln 2$ bits per iteration;
(b) if λ_I is positive, one has to specify the initial condition $x(0)$ with a precision $\Delta \exp^{(-\lambda_\mathrm{I} T)}$ where T is the time length of the evolution; it is necessary to give $\lambda_\mathrm{I}/\ln 2$ bits per iteration; if λ_I is negative the initial condition can be specified using a number of bits which does not depend on T.

We know that a negative value of λ_I does not imply predictability. Nevertheless the quantity defined in (26) can overestimate the complexity of the system. To illustrate this point, let us calculate K for a system described by a random map which exhibits the so-called *on-off* intermittency[28] (see also Sec. 4). In this case one has laminar phases, i.e. $x_t \simeq 0$, of average length l_L and intermittent phases of average length l_I. It is easy to realize that for $l_\mathrm{I} \ll l_\mathrm{L}$

$$K \simeq \frac{l_\mathrm{I}}{l_\mathrm{L}} h_\mathrm{s} \tag{27}$$

since one has just to compute the contributions of the intermittent bursts whose relative weight is $l_{\rm I}/(l_{\rm I}+l_{\rm L}) \simeq l_{\rm I}/l_{\rm L}$.

Notice that the quantity in (27) coincides with that one in (15) obtained for strong intermittent noisy systems.

4.1. *A toy model: one dimensional random maps*

Let us discuss a random map which, in spite of its simplicity, captures some basic features of this kind of systems:

$$x(t+1) = a_t x(t)(1-x(t)) \tag{28}$$

where a_t is a random dichotomous variable given by

$$a_t = \begin{cases} 4 & \text{with probability } p \\ 1/2 & \text{with probability } 1-p \end{cases} \tag{29}$$

It is easy to understand the behavior for $x(t)$ close to zero. The solution of (28), keeping the linear part is

$$x(t) = \prod_{j=0}^{t-1} a_j x(0)\,. \tag{30}$$

The long-time behavior of $x(t)$ is given by the product $\prod_{j=0}^{t-1} a_j$. Using the law of large numbers one has that the typical behavior is

$$x(t) \sim x(0) e^{\langle \ln a \rangle t}\,. \tag{31}$$

Since $\langle \ln a \rangle = p \ln 4 + (1-p) \ln 1/2 = (3p-1)\ln 2$ one has that, for $p < p_{\rm c} = 1/3$, $x(t) \to 0$ for $t \to \infty$. On the contrary for $p > p_{\rm c}$ after a certain time $x(t)$ is far from the fixed point zero and the nonlinear terms are relevant. Figure 3 shows a typical *on-off intermittency* behavior for p slightly larger than $p_{\rm c}$.

Let us note that, in spite of this irregular behavior, numerical computations show that the Lyapunov exponent $\lambda_{\rm I}$ is negative for $p < \tilde{p}_{\rm c} \simeq 0.5$: this is essentially due to the nonlinear terms.

For such a system with *on-off intermittency* it is possible, in practice, to define a complexity of the sequence which turns out to be much smaller than the value given by the general formula (26).

Let us denote with $l_{\rm L}$ and $l_{\rm I}$ the average life times respectively of the laminar and of the intermittent phases for p close to $p_{\rm c}$ ($l_{\rm I} \ll l_{\rm L}$). It is easy to realize that the mean number of bits, per iteration, one has to specify in order to transmit the sequence is:

$$\frac{K}{\ln 2} \simeq \frac{l_{\rm I} h_{\rm s}}{(l_{\rm I}+l_{\rm L})\ln 2} \simeq \frac{l_{\rm I}}{l_{\rm L}} \frac{h_{\rm s}}{\ln 2}\,. \tag{32}$$

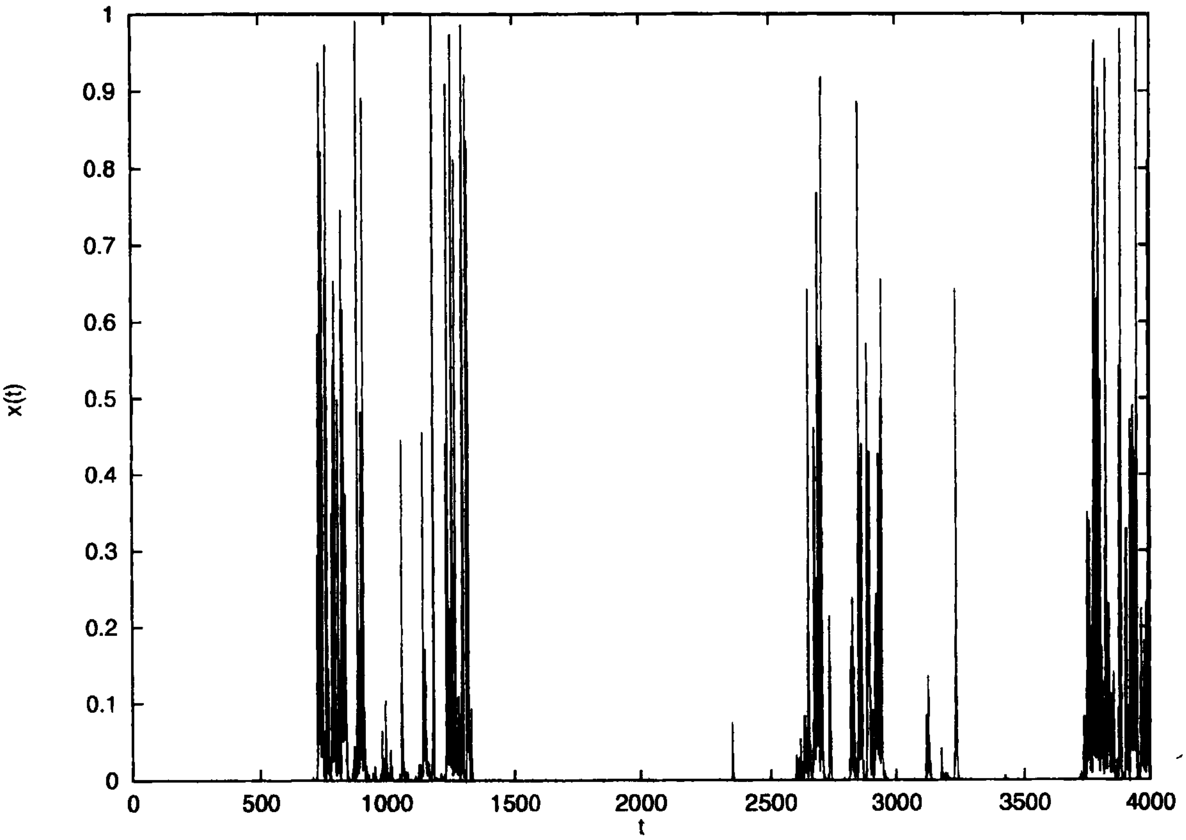

Fig. 3. $x(t)$ versus t for the random map (28, 29) with $p = 0.35$.

To obtain (32) first notice that on an interval T one has $\simeq T/(l_{\rm I} + l_{\rm L})$ intermittent bursts and the same number of laminar phases. Then notice that during an intermittent phase there is not an exponential growth of the distance between two trajectories initially close and computed with the same sequence of a_t. Since during a laminar phase one has to send a number of bits which does not depend on its duration, one can send all the necessary information just giving the sequence of a_t during the intermittent bursts. Equation (32) has an intuitive interpretation: in systems with a sort of "catastrophic" events, the most important feature is the mean time between two subsequent events. For a further discussion we refer to Ref. 30.

4.2. *Sandpile models as random maps*

Another example of a system which can treated in the framework of random maps is represented by the so-called Sandpile models.[31] These models represent an interesting example of Self-Organized Criticality (SOC).[32] This term refers to the tendency of some large dynamical systems to evolve *spontaneously* toward a critical state characterized by spatial and temporal self-similarity. The original Sandpile Models are cellular automata inspired to the dynamics of avalanches in a pile of sand. Dropping sand slowly, grain by grain on a limited base, one reaches a situation where the pile is critical, i.e. it has a critical slope. That means that a further addition of sand will produce sliding of sand (avalanches) that can be small or cover the entire size of the system. In this case the critical state is characterized by scale-invariant distributions for the size and the lifetime and it is reached without the fine tuning of any critical parameter.

We will refer in particular to the so-called Zhang model,[33] a continuous version of the original sandpile model (the BTW model),[31] defined on a d-dimensional lattice. The variable on each site x_i (interpretable as energy, sand, heat, mechanical stress etc.) can vary continuously in the range $[0, 1]$ with the threshold fixed to $x_c = 1$. The dynamics is the following:

(a) we choose a site in random way and we add to this site an energy δ (rational or irrational);

(b) if at a certain time t a site, say i, exceeds the threshold x_c a relaxation process is triggered defined as:

$$\begin{cases} x_{i+nn} \to x_{i+nn} + \dfrac{x_i}{2d} \\ x_i \to 0 \end{cases} \tag{33}$$

where nn indicates the $2d$ nearest neighbors of the site i;

(c) we repeat point (b) until all the sites are relaxed;

(d) we go back to point (a).

The dynamics of this model can be seen as described by a piecewise linear map.[34] Let us indicate with $x \equiv \{x_i\}_{i \in D}$ the configuration of the system at a certain time, where $D \subset Z^d$ is the bounded domain whose cardinality is $|D| = N^d$ with N being the linear dimension of the lattice. The operator Δ_i, corresponding to a toppling at site i, is given by

$$(\Delta_i \cdot x)_j = x_j - \delta_{i,j} x_i + \frac{1}{2d} \sum_{i'}^{*} \delta_{i',j} x_i \tag{34}$$

where $\sum^*$ means the sum over the nearest neighbors site of i.

Equation (34) shows that the single toppling is a linear operator and acts as a local laplacian. The evolution of a configuration up to the time t can be written as[34]:

$$x(t) = T^t x(0) = L_{x,t} x(0) + \delta \sum_{s=1}^{t} L_{x,t-s+1} 1_{k(s)} , \tag{35}$$

where $L_{x,t}$ is a linear operator defined as a suitable product of linear operator Δ, $x(0)$ is the initial configuration and 1_i is a vector in R^D whose component i is 1 and all the others are 0. $k(s)$ defines the sequence of site over which there will be the random addition of energy at the time s. Equation (35) shows as the evolution of the Zhang model can be seen as the sequential application of maps, chosen, time by time, in a random way. Sandpile models, thus, belong to the wide class of the random maps.

Let us now discuss the problem of predictability in Sandpile models. We recall that for chaotic systems the predictability time is given by

$$T_p \simeq \frac{1}{\lambda} \cdot \ln\left(\frac{\Delta_{\max}}{\delta_0}\right) . \tag{36}$$

where δ_0 is the error on the determination of the initial conditions and $\Delta_{\max}$ is the maximum tolerance between the real evolution and the simulation. In this case an improvement in δ_0 would increase T_p just in a logarithmic way.

In this context we would like to discuss this problem on the basis of some recent rigorous results[34] in order to clarify the role of the Lyapunov exponents for these class of systems and to address the problem of the predictability.

The Lyapunov exponent corresponding to a given trajectory $x(t) = T^t x$ can be defined, linearizing the dynamics in the neighborhood of $x(t)$, as:

$$\lambda \equiv \lim_{t\to\infty} \frac{1}{t} \ln \frac{|z(t)|}{|z(0)|} ; \tag{37}$$

where $z(t)$ represents the distance between two different configurations x and y at the time t. The distance is, for example, the L^1 norm $z(t) = \sum_i |y_i(t) - x_i(t)|$ with $i = 1, N^d$. If the two trajectories $x(t)$ and $y(t)$ make the same sequence of toppling Eq. (37) holds with the substitution $y - x \to z$. In fact, in this case, it holds $T^t y - T^t x = L_t z(0) = z(t)$. Therefore the definition (37) for the Lyapunov exponent fail when the two configuration begin to follow different sequences of toppling. It is easy to see that such a situation occurs when, for one configuration, it holds $x_i(t) = 1$ for some i and t. In this case a little difference in the second configuration $y_i(t) = x_i(t) + \epsilon$ will produce a toppling just in the y configuration. From this point onwards the two configurations will follow different sequences and the definition (37) fails definitely.

It is easy to see that the Lyapunov exponent is not positive. In fact, the dynamics in the tangent space, the dynamics of a little difference between two configurations, follows the same rules of the usual dynamics and the "error" is redistributed to the nearest neighbors site.

It is then clear that the distance between two configurations, being conserved in the toppling far from the boundaries, can just decrease when a site of the boundary topples. We can conclude that $\lambda \leq 0$.

In Ref. 34 it has been obtained rigorously that, for the maximum Lyapunov exponent λ, as defined in (37), it holds:

$$\lambda \leq -\frac{1}{N^d (R(D)+1)^2 (1/\delta + 1)(\ln N^d + 1)} \tag{38}$$

where we indicated with $R(D)$ the diameter of the domain D, that is that the Lyapunov exponent is strictly lower than 0.

An immediate consequence of the above result is that the dynamics, up to the time t (for t sufficiently large) is given by a *Piecewise Linear Contractive Map.* At first, one could think that the existence of a negative Lyapunov exponent should assure a perfect predictability. That is not true. What makes the situation complex is the existence of a splitting mechanism in the configuration space which affect the so-called snapshot attractor. A snapshot attractor is obtained by considering a cloud of initial conditions and letting it evolve forward in time under a given realization of the noisy dynamics. We can identify two different mechanisms which concur to the formation of the snapshot attractor:

(a) a volume contraction mechanism due to the effect of the negative Lyapunov exponent;
(b) a splitting mechanism which tends, by virtue of the piecewise structure of the map, to map single sets of configurations in two or more distinct sets also far apart in the phase space.

The splitting mechanism (b) tends to create a partition of the configuration space in regions which follow the same sequence of toppling, whereas mechanism (a) tends to contract the volumes of the elements of the partition.

It is worth to stress how, in same cases, it happens that the evolution of all the possible configurations shrink to the evolution of a single configuration (a point in the configuration space) whose evolution corresponds, at each time, to a snapshot attractor given by just one point.

This happens, in example, in the case of a one-dimensional (linear) chain of N sites driven with an arbitrary δ. In Ref. 34 it has been studied the case in which $\delta = 1/2$ and it has been shown that in this case the partition is time-independent. Let us discuss, for sake of simplicity and without loss of generality, this last case. A certain cloud of configurations, i.e. belonging to a same element of the partition, will evolve in a cloud of configurations, in principle smaller than the initial one due to the contractivity of the map, belonging entirely to another element of the partition; at its turn this cloud will evolve in a smaller cloud of configurations belonging to another element of the partition and so on. This process continues until all the configurations shrank to just one that continues to evolve jumping between different elements of the partition and evolving according to the map corresponding to each element of the partition. The Lyapunov exponent, in this case, gives informations about the rate typical exponential contracting rate of the radius of the snapshot attractor.

This discussion puts the problem of the definition of a predictability in a wider perspective in which the Lyapunov exponent is not the only relevant quantity. Since the Lyapunov exponent gives informations only at very large time and for infinitesimal perturbations the dynamical balance of the two effects (a) and (b) represents a basis for the definition of a predictability for such a systems.

In order to better explain this point it is possible to estimate complexity K for such kind of systems. We can give, in fact, an upper bound using (26) which we write again, for sake of clarity:

$$K = h_s + \lambda_I \theta(\lambda_I), \tag{39}$$

where h_s defines the complexity relative to the choose of the random sequence of addition of energy. In Sandpile models, for example, since each site has the same probability to be selected, one has $h_s = \ln N$, where N is the number of sites of the system and the second term does not exist in that the Lyapunov exponent is negative. We then obtain the result that the complexity for sandpile models is just determined by the randomness of the sequence of addition of energy.

5. Conclusions

A first naive approach to the "complexity" of random dynamical system is to treat the randomness as a standard time-dependent term. In this way, the Lyapunov exponent λ_σ is given by the rate of divergence of two initially close trajectories evolving under the same realization of the randomness.

Although well defined from a mathematical point of view, such an approach leads to paradoxical situations. For instance, in a system driven by the one-dimensional Langevin equation, the existence of a negative Lyapunov exponent does not imply the possibility to forecast the future state of the system unless one exactly knows the realization of the noise.

Furthermore it is practically impossible to extract λ_σ from an analysis of experimental data.

The main result of the paper is the definition of a measure of complexity K in terms of the mean number of bits per time unit necessary to specify the sequence generated by the random evolution law. We have also shown that from a practical point of view, this definition correspond to consider the divergence of nearby trajectories evolving in different noise realizations. The great advantage is that K can be extracted from experimental data.[13] The two indicators K and λ_σ have a close values and are practically equivalent in systems with weak dynamical intermittency. However, in presence of strong intermittency (say irregular alternations of long regular periods with sudden chaotic bursts) K and λ_σ become very different and in extreme situations it may happen that K is positive while λ_σ negative. It is thus questionable whether such a system is chaotic or regular and to speak of noise induced order.

A special class of systems well described by our characterization are random maps, where at each time step, different possible evolution laws are chosen according to a given probabilistic rule.

It seems to us that the study of the complexity and of the predictability is completely understood only in the case of deterministic dynamical systems with few degrees of freedom. We have just mentioned[17] the case of systems with many characteristic times where the predictability time is not trivially related to the Lyapunov exponent. Our work wants to be a first step towards a deeper comprehension of these issues in systems with many degrees of freedom or in interactions with many degrees of freedom represented by a noise, problems that are still open and sometimes controversial.

Finally we mention some recent works[35] where, following the line of thought here presented, a finite size Lyapunov exponent has been introduced. This quantity allows for the characterization of the predictability of systems with many degrees of freedom and a wide spectrum of temporal scales as in the case of three-dimensional turbulence.

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