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OPTIMAL STRATEGIES FOR PRUDENT INVESTORS

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We consider a stochastic model of investment on an asset in a stock market for a prudent investor. She decides to buy permanent goods with a fraction α of the maximum amount of money owned in her life in order that her economic level never decreases. The optimal strategy is obtained by maximizing the exponential growth rate for a fixed α. We derive analytical expressions for the typical exponential growth rate of the capital and its fluctuations by solving an one-dimensional random walk with drift.

1. Introduction

A large number of studies on finance have the main purpose of finding the optimal strategy for a given kind of investment [1-4]. These problems can be tackled by looking for simplified (but non-trivial) models which are able to describe the observed phenomenology and which can be, eventually, solved analytically. For an introduction to financial problems discussed from the point of view of the theoretical physics see [5-10].

The optimal strategy is usually defined as the one which maximizes a given utility function taking into account the risk. The use of utility functions introduce a high degree of arbitrariness since the particular choice depends on the subjective aversion to risk of the investor. This psychological arbitrariness can be removed if one considers that the rate of growth of the capital is an almost sure quantity in the long run. Therefore, the best strategy can only be the one which maximizes this rate, i.e. the one which maximizes the expected logarithm of the capital. Any other strategy almost surely ends with an exponentially, in time, smaller capital [7, 9-11].

The deep understanding of the reasons for the use of logarithmic optimization strategy comes from the Kelly's pioneering work [11]. In his paper he proposes and solves a model where an investor uses a fraction l of her capital to buy shares of a given asset at discrete time steps. It is assumed that the price of the shares can, at each time, double or vanish, so that the investor doubles or loses the fraction she has invested in it. It is assumed that the probability p of doubling is larger than $1/2$; this is a reasonable assumption since the contrary situation implies that is better to keep the money in a risk-free investment (the interest rate is supposed to be vanishing).

If the investment is absolutely sure ($p = 1$), of course she will invest all the capital ($l = 1$) at each step. In this way after t steps her capital will be 2^t times the original one. However, if the evolution of the share price is uncertain and she wants to maximize the expected value of her capital, she chooses the same strategy by investing a fraction $l = 1$. Obviously this is not the best approach: she may lose everything.

Using arguments from the theory of probability, Kelly has shown that the correct quantity to maximize is the expected value of the growth rate $\chi(t)$ of the capital: this quantity corresponds to the rate of transmission over a channel in information theory or to the Lyapunov exponent in dynamical systems and statistical mechanics of disordered systems. The value of $\chi(t)$ for a particular sequence of investments fluctuates around the expected value $\langle\chi(t)\rangle$ and the fluctuations approach zero in the limit $t \to \infty$.

If one introduces a random interest factor r_t and/or a random gain factor v_t (in the original Kelly's work $r_t = 1$ and $v_t = 2$) the model is still trivial from a mathematical point of view and it is easy to find out the optimal strategy. This is due to the fact that the model can be written in terms of a multiplicative random process. For a discussion on optimal investment strategy of a multi-asset portfolio following Kelly's approach see [7].

Recently Galluccio and Zhang [8] have considered a generalization of the previous model, where at each time step several kinds of investment are possible: a sure one (i.e. the bank) and the other risky ones (i.e. the stock market). They assume, as Kelly does, a simple behaviour for the market and find the values of the parameters which optimize the Lyapunov exponent. This model can be written as product of independent random matrices and it can be treated with standard perturbative methods [12] or by *constrained annealing* [13, 14].

In this paper we consider the case of an investment where there is a diversification between the stock market and permanent goods. A permanent good, such as a house, is characterized (at least as a first approximation) by a value that does not change in time, and it is not easy to convert into cash. The prudent investor wants, at least, to assure herself the same economic level for all her life (i.e. the capital invested in permanent goods does not decrease). In order to reach the goal she decides the permanent goods must equal a fraction α of the maximum total capital owned in the past. Therefore the capital cannot be less than this threshold and only the remaining part can be invested in the market. As a consequence the model describes the capital as a stochastic variable with memory.

Let us briefly sum up the contents of the paper. In Sec. 2, we introduce our model as a modification of Kelly's and we give an interpretation of the parameters introduced. Section 3 is devoted to the analytical solution of the model. In Sec. 4, we compare our results with Kelly's. In Sec. 5, we discuss some aspects of our model and its main features, in particular we consider the possibility of looking for time changing strategies of investment.

2. The Model

A given asset in a stock market can be always modelled in absence of memory by

$$W_{t+1} = F_t(W_t) \ , \tag{2.1}$$

where W_t is the capital at discrete time t and F_t is a random function, i.e. at each time t one chooses among different functions according to given probabilities [15, 16]. The simplest case is $F_t = u_t W_t$, where the u_t are independent stochastic variables (e.g. they can assume only two values as in the case of Kelly).

In the Kelly's model the investor keeps untouched a fraction $1-l$ (with $0 \leq l \leq 1$) of its capital, while the rest is used to buy shares with two different results: either she doubles her investment, or loses it. Therefore the capital at time $t+1$ is given by

$$W_{t+1} = (1-l)W_t + l(1+\sigma_t)W_t = (1+l\sigma_t)W_t \ . \tag{2.2}$$

It follows that u_t can be written as

$$u_t = 1 + l\sigma_t,$$

where the dichotomic random variable σ_t

$$\sigma_{\rm t} = \begin{cases} +1 & \text{with probability} \quad p \ , \\ -1 & \text{with probability } 1-p \ , \end{cases}$$

describes the change of the share price.

The growth rate of the capital at time t is $\chi(t) = \frac{1}{t}\ln W_t$. This quantity is random. Nevertheless for large t, because of the law of large numbers, it converges almost surely to the Lyapunov exponent

$$\lambda \equiv \lim_{t\to\infty} \chi(t) \ . \tag{2.3}$$

Let us notice that the optimal strategy proposed by Kelly consists in the maximization of λ (i.e. $\langle \ln W_t \rangle$) and not of $\langle W_t \rangle$. Following the naive idea to maximize $\langle W_t \rangle$ one has $l = 1$ and $\langle W_t \rangle = (2p)^t$ which is much larger (at large t) than the corresponding value obtained with the maximization of λ. Nevertheless, the naive strategy is clearly wrong since at large t one has a probability close to 1 to lose everything. On the contrary the maximization of λ has a well established theoretical motivation in the law of large numbers. Basically one has that for almost all the realizations the quantity $\chi(t)$ at large t is close to its mean value λ, i.e. the quantity $\chi(t)$ is self-averaging. Then, since we are dealing with a multiplicative process, one has that the probability distribution of W_t is close to the log-normal one [12, 17]:

$$P(W_t) \simeq \frac{1}{\sqrt{2\pi\Delta^2 t}\; W_t} \exp \frac{-(\ln W_t - \lambda t)^2}{2\Delta^2 \; t} \;, \tag{2.4}$$

where

$$\Delta^2 = \lim_{t\to\infty} t\langle(\frac{1}{t}\ln W_t - \lambda)^2\rangle \;. \tag{2.5}$$

Equation (2.4) holds for small values of $(\ln W_t - \lambda t)\,/\Delta\sqrt{t}$, on the contrary the tails depend on the details of the multiplicative process [12, 17]. Let us remark that the investor can have a small but finite probability to have a very low capital at time t.

The optimization problem consists in choosing the fraction l of the capital that maximizes λ given the probability $p > 0.5$; the result is $l_{\max} = 2p - 1$.

Our aim is to modify Kelly's model so that the capital cannot become too small in any realization of the random sequence $\{\sigma_t\}$. This time the investor decides to buy shares with only a part of the capital and uses the other part to buy permanent goods. The value of the goods equals a fraction α of the maximum total capital she has owned in the past.

The model can be written in the form:

$$W_{t+1} = \alpha\widetilde{W}_t + (1 + l\sigma_t)(W_t - \alpha\widetilde{W}_t) \;, \tag{2.6}$$

where

$$\widetilde{W}_t = \max_{\{i\le t\}} W_i \;, \tag{2.7}$$

Let us stress that $\alpha\widetilde{W}_t$ is the part of the capital kept untouched (W_{t+1} is always larger than $\alpha\widetilde{W}_t$), and l is the fraction of the available part $W_t - \alpha\widetilde{W}_t$, risked at time t. In the following, α will be considered a fixed parameter depending on the greediness (or prudence) of the investor. The Kelly model is recovered for $\alpha = 0$.

Let us remark that the optimal strategy of the model (2.6) and (2.7) is, from a conceptual point of view, equivalent to the optimal strategy of the original Kelly's problem with a suitable utility function which takes into account the prudence of the investor. The model is then a non-markovian process for the single variable W_t. A typical realization of W_t and $\widetilde{W}_t$ is shown in Fig. 1.

Moreover it is interesting that our model can be considered a Markovian process of two variables (W_t and $\widetilde{W}_t$) if we express the maximum capital owned in the investor's life (2.7) as:

$$\widetilde{W}_{t+1} = \max\{\widetilde{W}_t, W_{t+1}\} = \max\{\widetilde{W}_t, (1 + l\sigma_t)(W_t - \alpha\widetilde{W}_t) + \alpha\widetilde{W}_t\} \;. \tag{2.8}$$

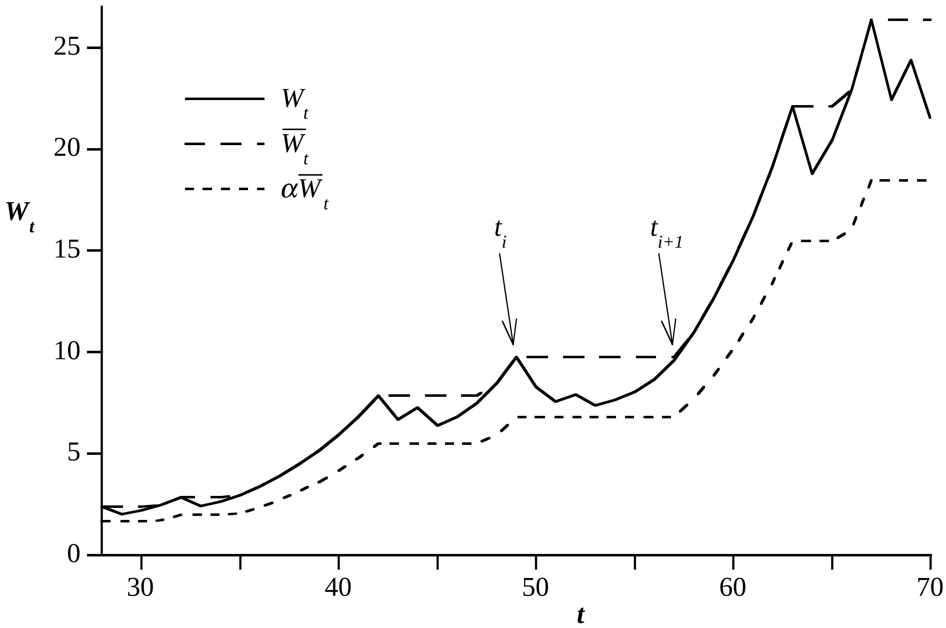

Fig. 1. A typical realization of the capital W_t (full line), of its maximum $\widetilde{W_t}$ (dashed line) and of the capital invested in permanent goods $\alpha\widetilde{W_t}$ (dotted line), for $\alpha = 0.7$ and $p = 0.75$. The two arrows shows a process of type (3.12), where $\widetilde{W_t}$ is constant.

Equations (2.6) and (2.8) are in fact a random map of two variables of the form (2.1):

$$\begin{pmatrix} W_{t+1} \\ \widetilde{W}_{t+1} \end{pmatrix} = F_t \begin{pmatrix} W_t \\ \widetilde{W_t} \end{pmatrix} \; . \tag{2.9}$$

Equation (2.9) can be considered a product of random matrices of infinite order. This is clear if one consider the process (2.6) with where now $\widetilde{W_t}$ is

$$\widetilde{W_t} = V_t^{(1)} = \max_{\{i=t-1,t\}} W_i \; , \tag{2.10}$$

or

$$\widetilde{W_t} = V_t^{(j)} = \max_{\{t-j\leq i\leq t\}} W_i \; . \tag{2.11}$$

It is easy to realize model (2.6) with $\widetilde{W_t}$ given by (2.10) can be represented in term of a Markov process of order 1 (i.e. the state at time t depends from the states at t and $t-1$). Similarly using (2.11) one has a Markov process of order j.

As far as we know for a general problem like (2.9) there isn't an Oseledec theorem [12] for the self-averaging of the quantity $\chi(t)$. Nevertheless for our specific case we shall show in the next section that $\chi(t)$ is self-averaging.

3. The Solution

The process (2.6) and (2.7) can considered a sequence of independent processes. Each of them starts at time t_i and ends at time t_{i+1} when $\widetilde{W_t}$ changes its value (i.e. W_t reaches a new maximum) so that, during this time, $\widetilde{W_t}$ is constant and equals the starting capital W_{t_i} (see Fig. 1).

Between t_i and t_{i+1} the variable $W_t - \alpha W_{t_i}$ is multiplicative as in the Kelly's model, in fact the Eq. (2.6) reduces to

$$W_{t+1} - \alpha W_{t_i} = (1 + l\sigma_t)(W_t - \alpha W_{t_i}) \ , \tag{3.12}$$

where W_{t_i} plays the role of a constant.

We notice that the final value of the i^{th} process $W_{t_{i+1}}$ turns out to be proportional to its initial value W_{t_i}:

$$W_{t_{i+1}} = e^{\gamma^{(i)}} W_{t_i} \ , \tag{3.13}$$

where $\gamma^{(i)}$ depends on all the $\{\sigma_t\}$ extracted during the time interval (t_i, t_{i+1}).

This process ends when $W_{t_{i+1}}$ becomes larger than W_{t_i} ($\gamma^{(i)} > 0$) for the first time. Of course the time interval $N^{(i)} = t_{i+1} - t_i$ is a random quantity, and it depends on the $\{\sigma_t\}$ sequence.

In this contest the global process W_t (2.6) can be expressed in terms of M independent Markovian processes $\gamma^{(1)}$,...,$\gamma^{(M)}$ as:

$$W_t = W_0 \prod_{i=1}^{M} e^{\gamma^{(i)}} \ , \tag{3.14}$$

where t equals the sum of the time duration of all the M Markovian processes $t = \sum_{i=1}^{M} N^{(i)}$. Let us remark that (3.14) is a product of independent random factors; this implies that $\chi(t)$ reaches the value λ at large t for almost all realizations.

Let us stress that the ith process described by (3.12) is a one-dimensional random walk with positive drift ($p > 1/2$), in terms of the variable $\sum_{t'=t_i}^{t} \sigma_{t'}$. The process ends as soon as the random walk reaches an *escape point* that runs away with velocity $\frac{\rho-1}{\rho+1}$, where ρ is a monotonic function of l defined by

$$\rho = -\frac{\ln(1-l)}{\ln(1+l)} \ . \tag{3.15}$$

with $\rho \geq 1$.

Supposing that it happens with $n^{(i)}$ defeats (or negative steps of the random walk), we find that

$$\gamma^{(i)} = \ln\left[\alpha + (1-\alpha)(1-l)^{n^{(i)}}(1+l)^{N^{(i)} - n^{(i)}}\right] \ . \tag{3.16}$$

Let us notice that $N^{(i)}$ and $n^{(i)}$ are not independent but must satisfy

$$N^{(i)} = 1 + n^{(i)} + \left[n^{(i)}\rho\right] \equiv N_{n^{(i)}} \ , \tag{3.17}$$

where the square bracket indicates the integer part.

Let $p(n)$ be the probability that the process γ ends with n defeats; it can be written down as

$$p(n) = C_n (1-p)^n p^{N_n - n} \ , \tag{3.18}$$

where C_n is the number of different ways to exit from the process with n defeats. The following recursive formula for C_n holds (see Appendix) for $n \geq 3$:

$$C_n = \binom{N_{n-1}-1}{n-1} - \sum_{r=1}^{n-2} \binom{N_{n-1}-N_r}{n-r} C_r \ , \tag{3.19}$$

with initial values

$$\begin{cases} C_0 = C_1 = 1 \ , \\ C_2 = N_1 - 1 \ . \end{cases} \tag{3.20}$$

Equations (3.19) and (3.20) represent a practical tool to numerically compute $p(n)$.

Notice that for a fixed p, there exists an interval of ρ (i.e. of l), such that the ith process has a non-vanishing probability to have an infinite time duration (it happens when the mean velocity $2p-1$ of the random walk is lower than the velocity $\frac{\rho-1}{\rho+1}$ of the escape point). This implies that the growth rate λ of the capital is almost surely zero, since its evolution remains confined in an ith process, with finite i. Of course it is a non-optimal situation, and the following considerations are restricted to the more interesting cases with $\lambda > 0$.

In the limit $M \to \infty$, because of the law of large numbers, the Lyapunov exponent λ can be written as

$$\lambda = \lim_{M\to\infty} \frac{\sum_{i=1}^{M} \gamma^{(i)}}{\sum_{i=1}^{M} N^{(i)}} = \frac{\overline{\gamma_n}}{\overline{N_n}} \ , \tag{3.21}$$

where the bar indicates the average according to the distribution $p(n)$.

Let us recall that the distribution of W_t is approximated by a log-normal (2.4), and furthermore the variance (2.5) can be written as

$$\Delta^2 = \lim_{M\to\infty} \overline{\left[\frac{(\sum_{i=1}^{M} \gamma^{(i)} - \lambda \sum_{i=1}^{M} N^{(i)})^2}{\sum_{i=1}^{M} N^{(i)}} \right]} = \frac{\overline{(\gamma_n - \lambda N_n)^2}}{\overline{N_n}} \ , \tag{3.22}$$

where the last result is obtained simply noticing that $\sum_{i=1}^{M} N^{(i)}$ is equal, for the law of large numbers, to $M\ \overline{N} + O(M^{\frac{1}{2}})$.

4. Discussion of the Results

As in Kelly's model we maximize the Lyapunov exponent with respect to l, to find the long time optimal strategy. We have computed λ and its variance using Eqs. (3.21) and (3.22). The probability $p(n)$ is found out for any n smaller than an appropriated $\widetilde{n}$ so that $\sum_{n=0}^{\widetilde{n}} p(n) \geq 1 - 10^{-8}$, $\widetilde{n}$ is typically $O(10^2)$.

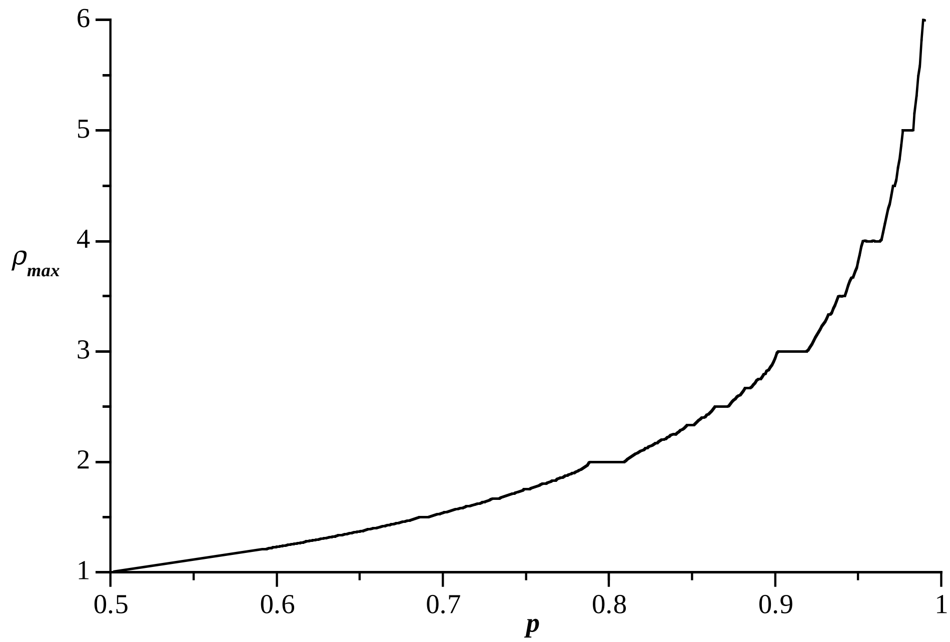

Fig. 2. $\rho_{\max} = \rho(l_{\max})$ (3.15) as function of p for $\alpha = 0.6$. The plateaux correspond to rational values of $\rho_{\max}$.

In Fig. 2 we report $\rho_{\max} \equiv \rho(l_{\max})$ as a function of p for $\alpha = 0.6$. We observe that $\rho_{\max}$ is constant for some intervals of p. These plateaux correspond to rational values of $\rho_{\max}$. This feature of the model can be explained noting that the probability $p(n)$ is discontinuous around any rational ρ. As a consequence, the Lyapunov exponent λ as function of ρ for a fixed value of p, has a cusp at any rational value of ρ. One of these cusps is a maximum of λ corresponding to the plateau of p (see, for instance, Fig. 3a).

In this context, varying p one has only a rotation of the cusp, so that at different p corresponds the same value of $\rho_{\max}$ that maximizes the Lyapunov exponent λ, while for p out of the plateau one has a decreasing or an increasing cusp, like in Fig. 3b. The width of the plateau depends on α and it becomes larger when α increases and when ρ is integer.

Let us restrict to the case of integer ρ. In order to simplify the notation, we use $\hat{\cdot}$ to indicate a quantity computed at integer values of ρ. If we study the probability distribution of the defeats (3.18) for fixed values of α and p as a function of ρ we notice that, when ρ crosses an integer value $\hat{\rho}$, the time durations $\{N_n\}$ (3.17) and the coefficients $\{C_n\}$ (3.19) change for every n. N_n changes since it depends on the integer part of $n\rho$, and C_n since the number of paths of the random walk ending with n losses depends on the cases with less defeats. For the same reason both of them remain constant immediately on the right and on the left of $\hat{\rho}$, at least until n is big enough to have negligible effects on the probability distribution $p(n)$.

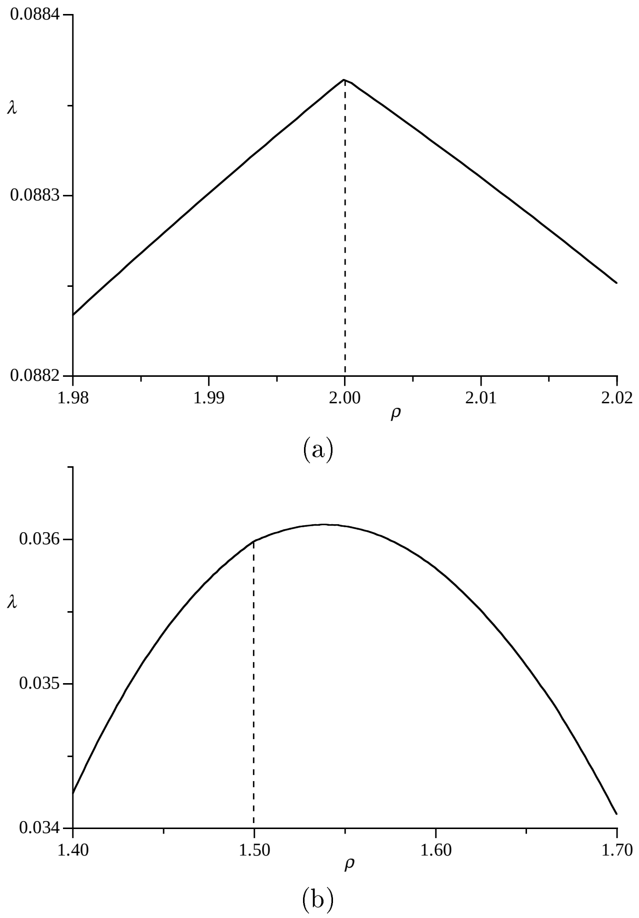

Fig. 3. Lyapunov exponent λ as function of ρ for: (a) $p = 0.8$ and $\alpha = 0.6$, where the maximum is a cusp ($\rho = 2$); (b) $p = 0.6$ and $\alpha = 0.6$, where an increasing cusp ($\rho = 1.5$) is present but not in correspondence to the maximum.

Following this idea we perform a linear expansion in l of λ around its value on the cusp. Then we write

$$\lambda^{\pm} = \lambda_0^{\pm} + \lambda_1^{\pm} \delta l,$$

where λ^+ and λ^- are respectively the right and the left limit for $l \to \hat{l} \pm 0$ of λ. After some algebra one obtains

$$\lambda_0^+ = \lambda_0^- = \hat{\lambda} = (p - \hat{\rho}(1-p)) \ln(\alpha + (1-\alpha)(1+\hat{l})) \; . \tag{4.23}$$

and

$$\lambda_1^- = \frac{1-\alpha}{1-\hat{l}^2} \left(p[2 + \hat{b}(1+\hat{\rho})] - [1 + \hat{l} + \hat{\rho}\hat{b}] \right) \; , \tag{4.24}$$

$$\lambda_1^+ = \frac{1-\alpha}{\alpha\hat{l}(1-\hat{l})^2}\hat{b}[2p-(1+\hat{l})] \ , \tag{4.25}$$

where

$$\hat{b} = \frac{\alpha\hat{l}(1-\hat{l})}{\alpha+(1-\alpha)(1+\hat{l})} \ . \tag{4.26}$$

The signs of $\lambda_1^{\pm}$ tell us when the cusp is a maximum of λ as a function of l (i.e. when $\lambda_1^- \geq 0$ and $\lambda_1^+ \leq 0$) . From (4.24) and (4.25) it is easy to see that this happens when p is between $p_{\min}$ and $p_{\max}$ where

$$p_{\min} = \frac{1+\hat{l}+\hat{\rho}\hat{b}}{2+\hat{b}(1+\hat{\rho})} \ , \tag{4.27}$$

$$p_{\max} = \frac{1+\hat{l}}{2} \ . \tag{4.28}$$

Some of the widths of these plateaux (i.e. $p_{\max} - p_{\min}$) are plotted in Fig. 4.

It is interesting to observe that the plateaux disappear ($p_{\max} \to p_{\min}$) in Kelly's limit ($\alpha \to 0$) and $p_{\max}$ is the value for which Kelly's model reaches its maximum when $l = \hat{l}$. Then we notice from (4.23) that for this value of p the maximum Lyapunov exponent rescaled with the function

$$\eta(\alpha, p) = \ln(\alpha + 2(1-\alpha)p) \ . \tag{4.29}$$

is the same for every α.

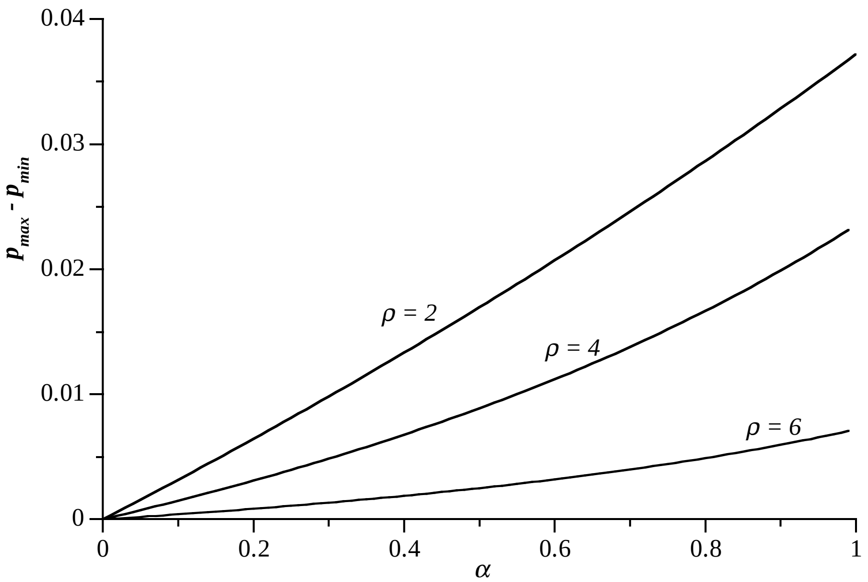

Fig. 4. Width of the plateaux $p_{\max} - p_{\min}$ ((4.27)-(4.28)) as function of α at different $\rho = 2, 4, 6$.

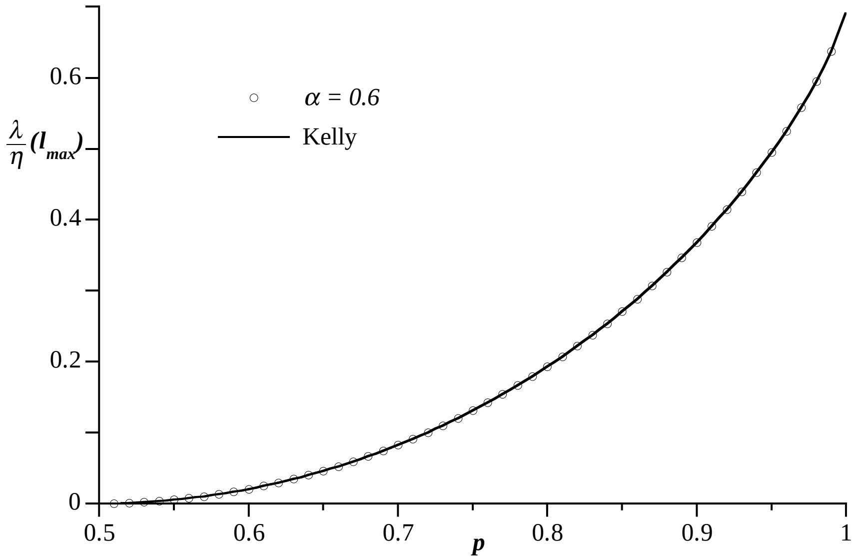

Fig. 5. Rescaled maximum Lyapunov exponent $\frac{\lambda}{\eta}(l_{\max})$ (full line) as function of p for $\alpha = 0.6$, compared with the Kelly's case (circles).

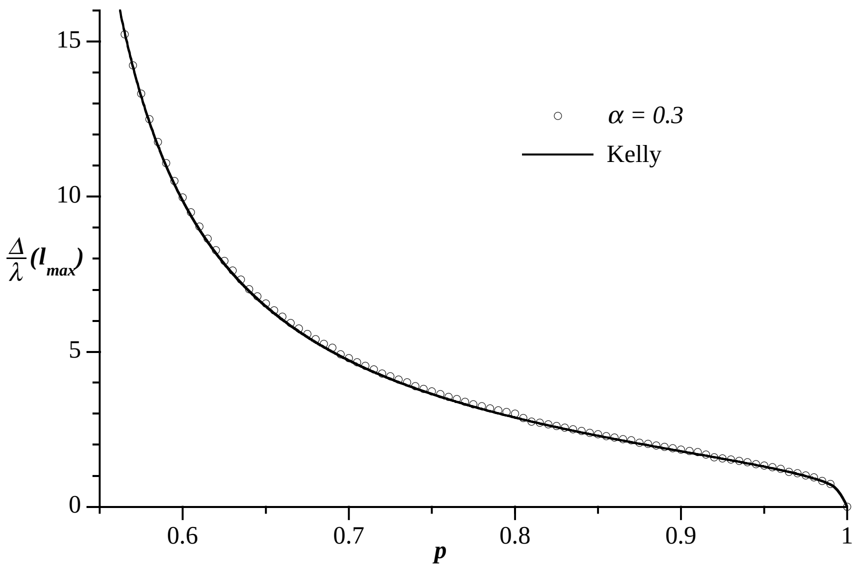

Fig. 6. Relative deviations $\frac{\Delta}{\lambda}(l_{\max})$ as function of p for $\alpha = 0.3$.

Finally we notice that the Lyapunov exponent computed at the optimal value $l = l_{\max}$, rescaled with (4.29), is quantitatively independent of α, see Fig. 5.

In addition the standard deviation Δ is proportional to the Lyapunov exponent when $l = l_{\max}$. This is an exact result, at any α, when $p = p_{\max}$ and it is qualitatively true for generic values of p, see Fig. 6.

Remember that a good parameter for quantifying the strength of the fluctuation is the ratio $R = \Delta^2/\lambda$. In fact, in the log-normal distribution (2.4) both Δ^2 and λ have the dimension of the inverse of time, and therefore $\frac{\Delta^2}{\lambda}$ is time independent. Since the Δ/λ is basically only function of p and not of α, one has

$$R = \frac{\Delta^2}{\lambda} \simeq \left(\frac{\Delta}{\lambda}\right)^2_{\text{Kelly}} \lambda \simeq \left(\frac{\Delta}{\lambda}\right)^2_{\text{Kelly}} \frac{\eta}{\eta_{\text{Kelly}}} \lambda_{\text{Kelly}} = \frac{\eta}{\eta_{\text{Kelly}}} R_{\text{Kelly}} \leq R_{\text{Kelly}} \ , \quad (4.30)$$

i.e. a reduction of the relative fluctuations.

5. Conclusions

In this paper we have considered a diversification of the portfolio between permanent goods and investments in a market. The model is a non trivial modification of Kelly's where only a part of the capital is allowed to be invested in the market. The investor keeps the remaining part as a security amount of money, equal to a fraction α of the maximum capital owned in the past. In this way the investor avoids the possibility of losing almost all her capital due to a large fluctuation as can happen in Kelly's case: the parameter α can be considered a measure of the investor's prudence.

The small fluctuations of the capital around the typical value $W_0\ e^{\lambda t}$ follow, at large t, a log-normal distribution and therefore they are well described by the typical exponential growth (or Lyapunov exponent) λ and the deviation Δ from this quantity. We give explicit analytical expressions for both these quantities. In particular we obtain a decreasing of the relative strength of the small fluctuations (i.e. Δ^2/λ).

An interesting feature of the model, from a mathematical point of view, is that the Lyapunov exponent is a continuous but not differentiable function of the parameters. This fact is particularly relevant when we look for the fraction l of the allowed capital (i.e. the capital that can be invested in the market each time), which maximizes λ. We observe a devil's stairs like behaviour [18] for $\rho_{\max}$ as a function of the probability p. The existences of plateaux can be understood if one considers in more detail the λ itself as a function of l at fixed p and α. The sizes of these plateaux can be computed; as an example we derive the width of the largest ones.

We have found that the Lyapunov exponent, when rescaled by a proper function of the parameters, and the ratio between Δ and λ, show a behaviour similar to the Kelly's case: the prudent constraint we have introduced has basically the effect of rescaling the exponential growth and the relative strength of the small fluctuations of the capital invested on the stock market according to Eqs. (4.29) and (4.30).

The study of non-commutative multiplicative models of the stock market has the great advantage that they often can be analytically treated. In generic cases one can use powerful systematic methods to obtain good approximations [12-14]. We belive they can be useful to understand problems close to the reality, such as the ones where the investor does not decide only once the best strategy to follow, but can change her mind at each step depending on the behaviour of the market. It can be shown that a fixed strategy is not the best one when we introduce correlations between successive times or different *hedgings* for prudent investors.

Appendix

In this appendix we derive the recursive formula (3.19) for C_n. The random walk (3.12) has a positive drift ($p \geq \frac{1}{2}$) and it ends as soon as the capital exceeds its initial value. The total number of steps necessary for that is N_n, where n is the number of negative steps (i.e. defeats). The stop is when n/N_n is smaller than $1/(\rho+1)$ (with $\rho \geq 1$), so that $N_n = 1 + n + [n\rho]$, where the square brackets mean integer part (see Eq. (3.17)).

The number of different ways C_n can be computed noticing that the n negative steps have to appear before the (N_{n-1})th step, in order to avoid a premature interruption of the process with only $n-1$ negative steps. This yields to $\binom{N_{n-1}}{n}$ different combinations, but in this number are also included the cases of premature arrest with r negative steps in the first N_r time steps, with $0 \leq r \leq n-2$. Each of these cases leaves out an amount of combinations equal to $\binom{N_{n-1}-N_r}{n-r} C_r$, where the combinatorial factor comes from the different ways that the remaining $(n-r)$ negative steps have to appear in the interval $[1+N_r, N_{n-1}]$.

It immediately follows the recursive formula (3.19)

$$\begin{aligned} C_n = & \binom{N_{n-1}}{n} - \sum_{r=0}^{n-2} \binom{N_{n-1}-N_r}{n-r} C_r = \\ = & \binom{N_{n-1}-1}{n-1} - \sum_{r=1}^{n-2} \binom{N_{n-1}-N_r}{n-r} C_r \end{aligned}$$

which holds for $n \geq 3$, while for $n = 2$ one has

$$C_2 = N_1 - 1.$$

The cases $n \leq 1$ can be trivially derived:

$$C_0 = C_1 = 1.$$

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