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Multiscaling and clustering of volatility

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Abstract

The dynamics of prices in stock markets has been studied intensively both experimentally (data analysis) and theoretically (models). Nevertheless, while the distribution of returns of the most important indices is known to be a truncated Lévy, the behaviour of volatility correlations is still poorly understood. What is well known is that absolute returns have memory on a long time range, this phenomenon is known in financial literature as clustering of volatility. In this paper we show that volatility correlations are power laws with a non-unique scaling exponent. This kind of multiscale phenomenology is known to be relevant in fully developed turbulence and in disordered systems and it is pointed out here for the first time for a financial series. In our study we consider the New York Stock Exchange (NYSE) daily index, from January 1966 to June 1998, for a total of 8180 working days.

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1. Introduction

One of the most challenging problems in finance is the stochastic characterization of stock market returns. This topic not only has an academic relevance but it also has an obvious technical interest. Consider, for example, the option pricing models where distribution and correlations of volatility play a central role.

It is now well established that returns of the most important indices are distributed according to a truncated Lévy and that they are uncorrelated on lags larger than a single day, in agreement with the hypothesis of efficient market. On the other hand, the distribution of volatility and its correlations are still poorly understood. What is known

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is that absolute returns (which are a measure of volatility) have memory on a long time range, this phenomenon is known in financial literature as clustering of volatility. Recent studies provide a strong evidence for power-law correlations for absolute returns [1–6]. Notice that in ARCH-GARCH approach [7–9] volatility memory is longer than a single time step but it decays exponentially, which implies that ARCH-GARCH modeling is inappropriate.

In this paper we analyse the daily returns of the NYSE composite index, and we not only find that volatility correlations are power laws on long time scales up to a year but, more importantly, they exhibit a non-unique exponent (multiscaling). This kind of multiscale phenomenology is known to be relevant in fully developed turbulence and in disordered systems and it is pointed out here for a financial series for the first time [10]. Our result is based on the fluctuation analysis of a new class of variables that we call *generalized cumulative absolute returns*. We also argue that the distribution of volatility is log-normal.

The paper is organized as follows: in Section 2 we show that the probability distribution of returns, as expected, is leptokurtic, while the Gaussian shape is recovered on a time scale of about six months. In Section 3 we perform a scaling analysis on the standard deviation of a new class of observables, the generalized cumulative absolute returns. This analysis implies power-law correlations with non-unique exponent. In Section 4 the attention is focused on the volatility whose probability distribution, indirectly deduced from the data, turns out to be log-normal on time scales of about one month. Section 5 contains some final remarks.

2. Distribution of returns

We consider the New York Stock Exchange (NYSE) daily index, from January 1966 to June 1998, for a total of $N = 8180$ working days. The quantity we consider is the (de-meaned) daily return, defined as

$$r_t = \log \frac{S_{t+1}}{S_t} - \left\langle \log \frac{S_{t+1}}{S_t} \right\rangle , \tag{1}$$

where S_t is the index value at time t ranging from 1 to N and $\langle \cdot \rangle$ is the average over the whole sequence.

As pointed out by several authors [11–13], the distribution of returns is leptokurtic. A symmetric Lévy stable distribution was proposed in [12] for the first time and more recently a strong evidence for this fact has been provided in [13]. More precisely, in [13] it is shown that the distribution is Lévy-stable for high-frequency returns except for tails, which are approximately exponential. The estimation is that the shape of a Gaussian is recovered only on longer scales, typically one month.

We start by verifying that the probability distribution $p(r)$ of the daily returns (1) is leptokurtic, i.e. that the Gaussian shape is not recovered on a time lag of one day. This fact can be appreciated in Fig. 1, where the distribution (which is obtained from data by a smoothing procedure) is compared with a Gaussian of the same variance.

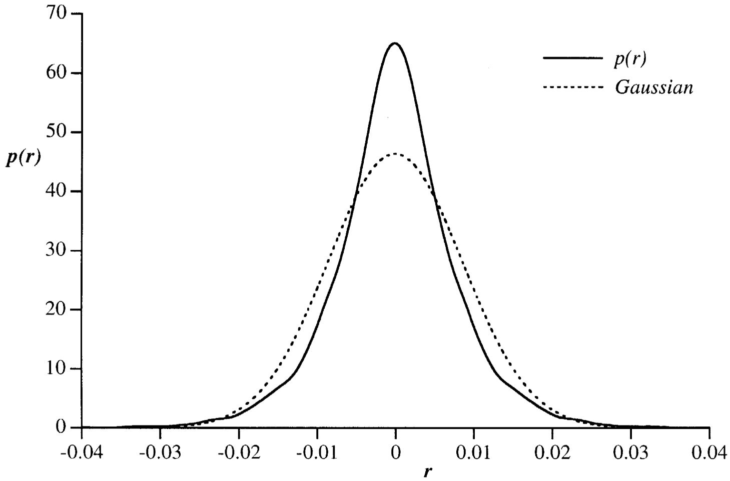

Fig. 1. Probability distribution of the daily returns (1), compared with a Gaussian distribution with same variance.

The second step is a check of the typical time necessary to fully recover the Gaussian shape. In order to obtain this information we consider the cumulative returns $\phi_t(L)$, defined as the sum of L successive returns $r_t,\dots,r_{t+L-1}$, divided by L:

$$\phi_t(L) = \frac{1}{L}\log\frac{S_{t+L}}{S_t} - \left\langle \log\frac{S_{t+1}}{S_t}\right\rangle . \tag{2}$$

Using NYSE data, one can define N/L not overlapping variables of this type. Since $N = 8180$, the statistics is not sufficient to consistently find out the distribution for this variables. Nevertheless, we can compute the kurtosis $K(L)$ of their distribution

$$K(L) \equiv \frac{\langle \phi_t^4(L)\rangle}{\langle \phi_t^2(L)\rangle^2} . \tag{3}$$

Since for a Gaussian distribution the kurtosis equals 3, it is sufficient to find the value of L for which $K(L)$ reaches this value. In Fig. 2 $K(L)$ is plotted, showing that the probability distribution of the $\phi(L)$ recovers a Gaussian after a time lag of about six months. This result is consistent with the prevision in [13]. Notice that this slow convergence of the kurtosis is due to rare large fluctuations, while the central part of the distribution converges more rapidly to a Gaussian.

3. Scaling analysis

Consistent with the efficient market hypothesis, daily returns have no autocorrelations on lags larger than a single day. In order to check this fact, we use a method which we will also apply for volatility. We consider $\phi_t(L)$ and we compute the standard

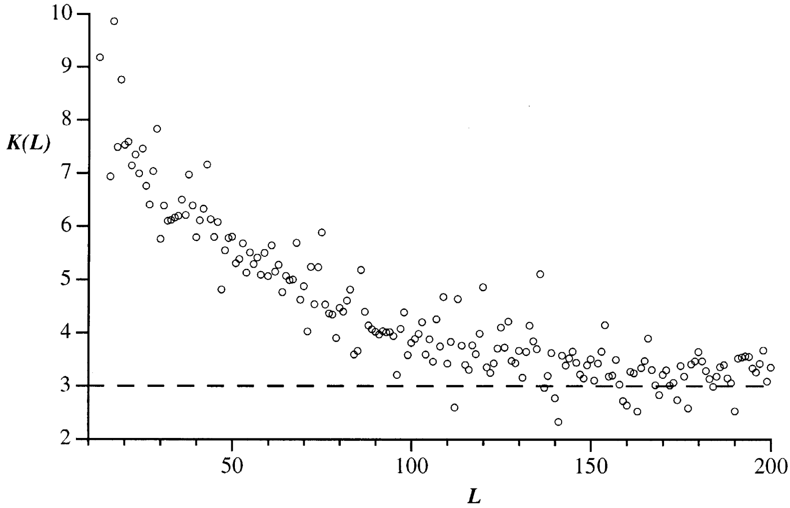

Fig. 2. Kurtosis $K(L)$ (3) of cumulative returns (2), as function of L. The value $K(L)=3$ (dashed line) of a Gaussian distribution is reached for $L \simeq 120$–150 (about six months).

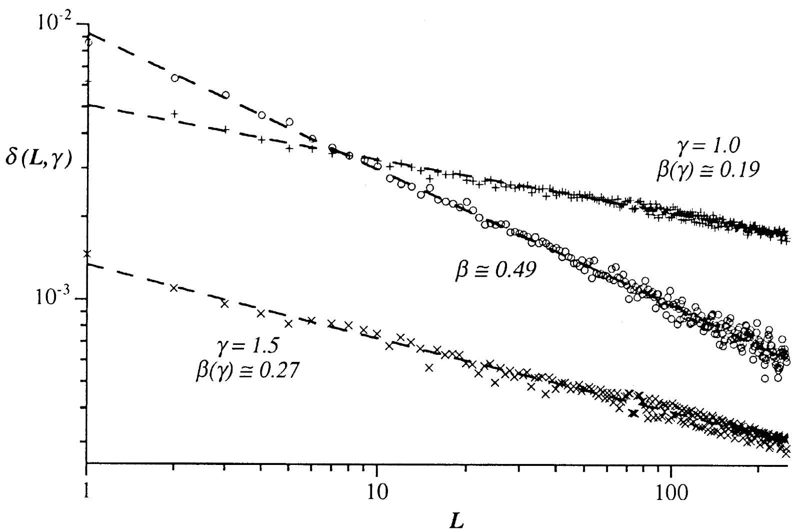

Fig. 3. Standard deviation $\delta(L,\gamma)$ of the generalized cumulative absolute returns (2) as a function of L on log–log scales for $\gamma=1$ (crosses) and $\gamma=1.5$ (slanting crosses), compared with the standard deviation $\sigma(L)$ of the cumulative returns (circles). The exponents of the best-fit straight lines (dashed lines) are, respectively, $\beta(1) \simeq 0.19$, $\beta(1.5) \simeq 0.27$ and $\beta \simeq 0.49$.

deviation $\sigma(L) = \langle \phi_t^2(L) \rangle^{1/2}$ for L ranging from 1 to 250 (one year). Larger value of L would imply insufficient statistics. Assuming that r_t are uncorrelated (or short range correlated), it follows that $\sigma(L)$ has a power-law behaviour with exponent 0.5 for large L, i.e. $\sigma(L) \sim L^{-\beta}$ with $\beta = 0.5$. The exponent for the NYSE index turns out to be

0.49 ± 0.01 (see Fig. 3 and also see [14]), according to the hypothesis of uncorrelated returns. This value of the exponent also ensures that Lévy scaling is not effective in this range of time.

On the other hand, the daily volatility is not directly observable. The observables directly related to the volatilities are the absolute returns $|r_t|$.

In order to perform the appropriate scaling analysis, let us introduce the generalized cumulative absolute returns defined as

$$\chi_t(L,\gamma) = \frac{1}{L} \sum_{i=0}^{L-1} |r_{t+i}|^{\gamma} , \tag{4}$$

where γ is a real exponent and, again, these quantities are not overlapping. If $|r_t|^{\gamma}$ were uncorrelated, one should find that their standard deviation $\delta(L,\gamma)$ has a power-law behaviour with exponent 0.5. On the contrary, a power-law autocorrelation function $C(L,\gamma)$ with exponent $\alpha(\gamma) \leqslant 1$, i.e.

$$C(L,\gamma) = \langle |r_t|^{\gamma} |r_{t+L}|^{\gamma} \rangle - \langle |r_t|^{\gamma} \rangle \langle |r_{t+L}|^{\gamma} \rangle \sim L^{-\alpha(\gamma)} \tag{5}$$

would imply that $\delta(L,\gamma)$ is a power law with exponent $\beta(\gamma) = \alpha(\gamma)/2$ ($\delta(L,\gamma) \sim L^{-\beta(\gamma)}$). For autocorrelations with exponent $\alpha(\gamma) \geqslant 1$ we would not detect anomalous scaling for the standard deviation ($\beta(\gamma) = 0.5$).

Our numerical analysis on the NYSE index shows very sharply that $\delta(L,\gamma)$ has an anomalous power-law behaviour in the range from one day to one year ($L = 250$). For example, for $\gamma = 1$ we find $\beta(1) \simeq 0.19$, while for $\gamma = 1.5$ we find $\beta(1.5) \simeq 0.27$ (see Fig. 3). For larger L the statistics becomes insufficient.

In Fig. 4, $\beta(\gamma)$ is plotted as a function of γ with error bars. The crucial result is that $\beta(\gamma)$ is not a constant function of γ in the range $-0.5 < \gamma < +4$, showing the presence of different scales. The interpretation is that different values of γ select different typical fluctuation sizes, each of them being power-law correlated with a different exponent.

The longest correlation is for $\gamma = 0.15$ ($\beta(0.15) \simeq 0.15$). The case $\gamma = 0$ corresponds to a cumulative logarithm of absolute returns. Approximately, in the region $\gamma \geqslant 4$ the averages are dominated by only few events, corresponding to a very large returns and, therefore, the statistics becomes insufficient.

The anomalous power-law scaling can be directly tested against the plot of autocorrelations. For instance, the autocorrelations of r_t and of $|r_t|$ are plotted in Fig. 5 as a function of the correlation length L. Notice that the full line, which is in good agreement with the data, is not a best fit but it is a power law whose exponent $2\beta(1) \simeq 0.38$ is obtained by the previous scaling analysis of the standard deviation. The autocorrelations for returns, as expected, vanish except for the first step ($L = 1$).

It should also be noted that a direct analysis of the autocorrelations (as in Fig. 5) would not have provided an analogous clear evidence for multiscale power-law behaviour, since the data show a wide spread compatible with different scaling hypothesis.

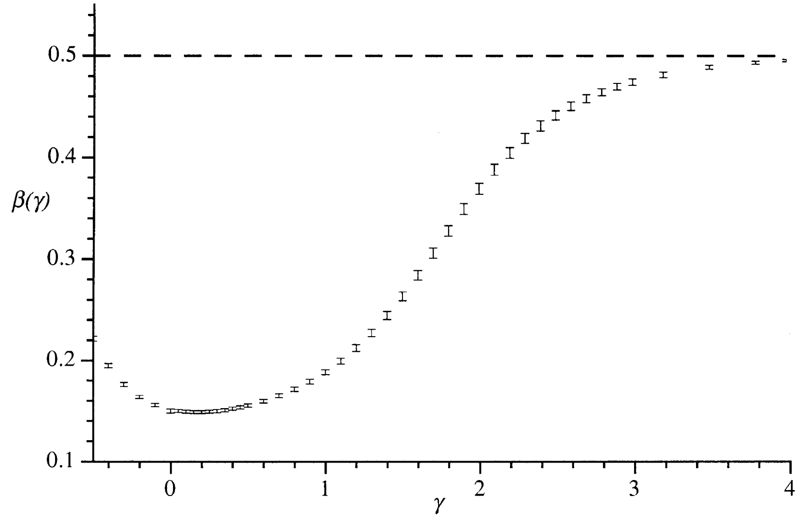

Fig. 4. Scaling exponent $\beta(\gamma)$ of the standard deviation $\delta(L,\gamma)$ as a function of L, where the bars represent the errors over the best fits. An anomalous scaling ($\beta < 0.5$) is shown in the range $-0.5 < \gamma < +4$.

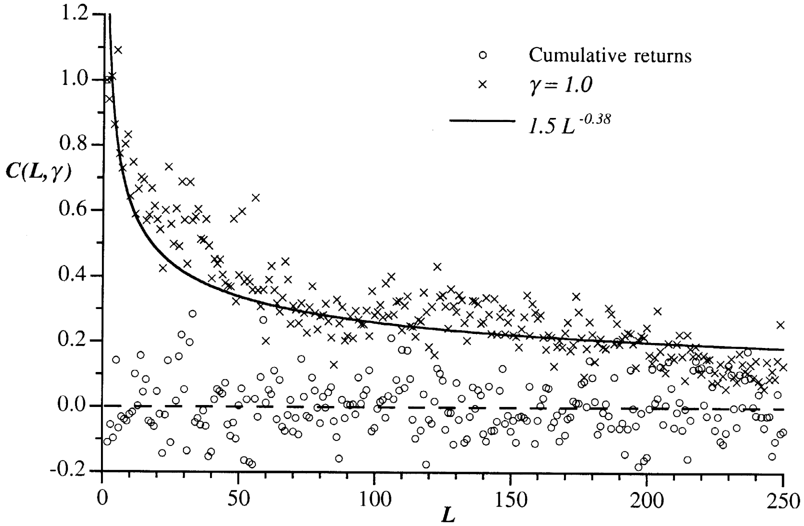

Fig. 5. Autocorrelation function $C(L,\gamma)$ of $|r_t|$ (crosses) as a function of the correlation length L, compared with the autocorrelation function of r_t (circles). The data are in a good agreement with a power law with exponent $2\beta(1) \simeq 0.38$ in the first case, and absence of correlations in the second. In both cases the scale is fixed by autocorrelations equal to 1 at $L = 1$.

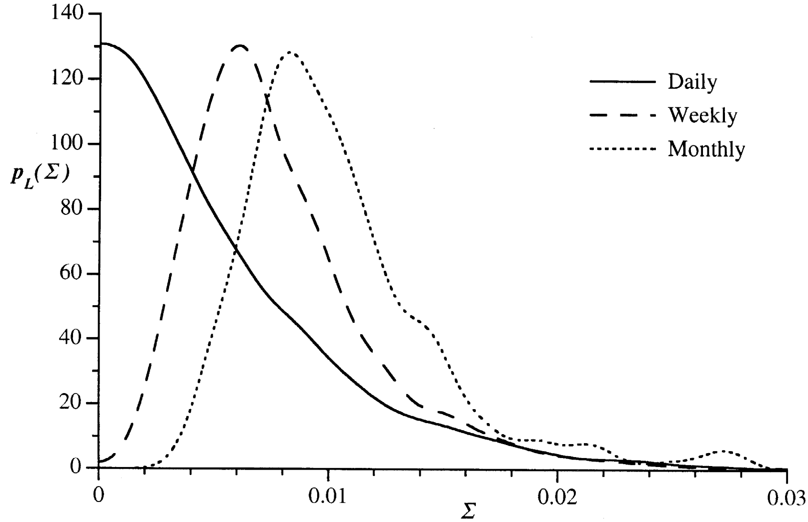

Fig. 6. Probability distribution of $\Sigma_t(L)$ (6) for $L = 1$ (daily), $L = 5$ (weekly) and $L = 20$ (monthly). The shape of the two last cases is compatible with a log-normal distribution.

4. Distribution of volatility

The discussion in previous section concerns absolute returns. An obvious question is: 'what is the relation with volatility?'. The answer is not completely trivial, since from an operative point of view, the volatility is often assumed to coincide with the intra-day absolute cumulative return or, alternatively, with the implied volatility which can be extracted from option prices.

Our point of view is that the exact definition of volatility cannot be independent from the theoretical framework. It is usually assumed that the volatility σ_t is defined by $r_t = \sigma_t \eta_t$ where η_t are identically distributed random variables with vanishing average and unitary variance. The usual choice for the distribution of η_t is the normal Gaussian. This picture is completed by assuming the probabilistic independence between σ_t and η_t. According to the above definition, all the scaling properties we have found on absolute returns directly apply to volatility.

If the time evolution of σ_t is slower than that of η_t, the L day volatility can be qualitatively observed considering sums of absolute returns:

$$\Sigma_t(L) = \frac{1}{L^{1-\beta(1)}} \sum_{i=0}^{L-1} |r_{t+i}| \, . \tag{6}$$

Notice that the anomalous exponent of L is chosen in order to have a quantity whose fluctuations are independent of L ($\Sigma_t(L) = L^{\beta(1)} \chi_t(L,1)$).

The probability distribution of $\Sigma_t(L)$ is plotted in Fig. 6, for $L = 1$ (daily), $L = 5$ (weekly) and $L = 20$ (monthly). The astonishing fact is that the last two distributions are well fitted by a log-normal distribution with the same variance. This log-normal behaviour of the distribution has been already observed in [15], where high-frequency data are considered and the time scale of one month is replaced by the time scale of one hour. This unexpected shape for the distribution of volatility suggests the existence of some underlying multiplicative process. If this prevision would be confirmed by more complete analysis, it would imply that not only indices prices are multiplicative processes, but also the associated returns.

5. Conclusions

The main result we have found is that the scaling of standard deviation of the generalized cumulative absolute returns is a power law with non-unique exponent. This fact implies power-law correlations whose exponent depends on the variable which is considered. The main theoretical consequence is that models with exponential correlations, like ARCH-GARCH, fail in describing the dynamics of financial markets, and that new models should account for the coexistence of long memory with different scales. Furthermore, we have found some evidence for log-normal distribution for the volatility. This fact should be considered as a suggestion for modelling, since it indicates that volatility dynamics is probably described by some kind of multiplicative process.

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