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# OPTIMAL LAG IN DYNAMICAL INVESTMENTS

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A portfolio of different stocks and a risk-less security whose composition is dynamically maintained stable by trading shares at any time step leads to a growth of the capital with a nonrandom rate. This is the key for the theory of optimal-growth investment formulated by Kelly. In presence of transaction costs, the optimal composition changes and, more important, it turns out that the frequency of transactions must be reduced. This simple observation leads to the definition of an optimal lag between two rearrangement of the portfolio. This idea is tested against an investment in a risky asset and a risk-less one. The price of the first is proportional to NYSE composite index while the price of the second grows according to the American Discount Rate.

## 1. Introduction

The definition of an optimal portfolio is a challenging problem in theoretical finance [1–5] and it has an obvious relevance in technical studies. Suggestions and indications for investments can be found in any economic newspaper where, usually, reference is to static strategies. The problem, in this case, consists in finding the best initial composition according to the risk attitudes of the investor and later trading is not expected. Nevertheless, if transaction costs are negligible and the investment is for a long time, it is rentable to maintain stable the composition of the portfolio by selling or buying shares. According to this dynamical point of view, the fraction of the capital invested in any stocks or security must be kept constant in time. The most important consequence is that the investor wealth grows with a nonrandom rate on a long time horizon. This fact, which is a trivial consequence of the law of large numbers, implies that the optimal-grow strategy is the only possible, while subjective risk averseness or other psychological considerations play no role.

This point is still often misunderstood in the current literature. For example, Samuelson and Merton [6, 7] demonstrated that the growth-optimal strategy does not maximize the expected value of a generic utility function. Nevertheless, an investor which would decide to optimize his strategy with respect to a generic utility function would, *almost surely*, end up with an exponentially smaller capital. The reason is that the dominant contribution to the expected value comes from events whose probability exponentially vanishes in time. This is a general probabilistic fact, widely studied in the context of large deviations theory. We should stress once again that the above considerations applies whenever one deals with long time repetition of the same investment. On the contrary, they do not apply to strategies concerning static investments, as for example the composition of a portfolio of securities which remains unchanged until they expire or they are sold out.

One could argue that in the real world an investment is never maintained for an infinite time and in any case that the underlying stochastic process is, in general, non-stationary. These facts imply deviations from the theory of large numbers which should be accounted. Nevertheless, also in this case, the most rational approach, if correlations are neglected, turns out to be the optimal-growth. We will later discuss this point.

In this paper we extend Kelly's theory showing that it still holds when transaction costs are considered. Nevertheless, in this case, it is better to reduce the frequency of trading. This simple observation leads to the definition of an optimal lag between transactions.

The paper is organized as follows. In Sec. 2 we summarize Kelly's theory assuming that interest rate may vary in time. We also consider the possibility of non-stationary process and a finite duration for the investment. In Sec. 3 we analyze the effects due to transaction costs which are assumed to be proportional to the amount of shares traded and we show how the notion of an optimal lag naturally emerges. In Sec. 4 we test our result against a portfolio with a risky asset and a risk-less one. We first consider a realistic situation where the price of the first is proportional to NYSE composite index while the price of the second grows according to the American Discount Rate. Then, for the sake of comparison, we reconsider the classical coin toss game originally proposed by Kelly. Finally, in Sec. 5, we shortly discuss the relevance of our results with respect to the notion of continuous time in finance.

## 2. The Kelly Theory of Optimal Gambling

The theory of optimal-growth investment was formulated by Kelly [8] in a contest not directly related to finance and stock market. His original purpose was mainly to find an interpretation of the Shannon [10] entropy in terms of optimal gambling strategies. This theory was later reconsidered in a more finance related contest by Breiman [11, 12], more recently it has been rediscovered and extended by various authors [13–17] and it has been also applied to the problem of pricing derivatives [18, 19] in the general case of incomplete markets.

Consider a stock, or some other security, whose price is described by

$$S_{t+1} = u_t S_t \,, \tag{2.1}$$

where time is discrete, $S_t$ is the price at time $t$ of a share and the $u_t$ are independent, identically distributed random variables. Also assume that the risk-less interest rate $r_t$ may vary in time.

Consider now an investor who starts at time 0 with a wealth $W_0$, and who decides to invest in this stock many times. Suppose that he chooses to invest at each time a fixed fraction $l$ of his capital in stock, and the remaining part in a risk-less security, i.e. a bank account with rate $r_t$. In absence of transaction costs, his wealth evolves as a multiplicative random process:

$$W_{t+1} = (1-l)r_t W_t + l u_t W_t \,. \tag{2.2}$$

It is useful to introduce the discounted prices $\tilde{u}_t \equiv u_t/r_t$, so that (2.2) rewrites as

$$W_{t+1} = r_t(1 + l(\tilde{u}_t - 1))W_t \,. \tag{2.3}$$

In the large time limit we have, by the law of large numbers, that the exponential growth rate of the wealth is, with probability one, a constant. That is,

$$\lambda(l) \equiv \lim_{T\to\infty} \frac{1}{T} \log \frac{W_T}{W_0} \,. \tag{2.4}$$

It is clear from (2.2) that the interest $r_t$ contributes to the above limit with an additive term which is independent from the strategy, and corresponds to the rate which the investor would obtain by investing all his capital in the risk-less security (bank). Therefore, the general problem with a time dependent $r_t$, can be always mapped into the $r_t = 1$ problem by properly discounting the security prices. In the second part of this section we drop the tildes, assuming that all prices are already discounted. The net growth rate (the growth rate minus the growth rate of a capital entirely invested in the risk-less security) is then, for almost all realizations of the random variables $u_t$,

$$\lambda(l) = E[\log(1 + l(u-1))] \,, \tag{2.5}$$

where $E[\,\cdot\,]$ represent the average with respect to the distribution of the $u$.

The optimal gambling strategy of Kelly consists of maximizing $\lambda(l)$ with respect to $l$. The solution is unique because the logarithm is a convex function of its argument:

$$\lambda^* = \max_l \lambda(l) = \lambda(l^*) \,. \tag{2.6}$$

The above result holds with the following assumptions: (1) there are no correlations for increments (we have stated that the $u_t$ are i.i.d. variables), (2) the process is stationary, (3) the investment is maintained for a very long time. With these assumptions the optimal-growth strategy is the best policy. In fact, it can be demonstrated that all other policies with a different $l$ or a non-constant $l$ lead to a capital which is exponentially smaller in time (see for example [20]).

Nevertheless, it happens in reality that the stochastic process which describes the evolution of prices is non-stationary and that the investment is maintained for a time which may be not so long to ensure that the large number law applies. The question then is the following, what can the agent do? Reasonably, he will apply for the future, the strategy which would have given the best performance in the past.

If the process is stationary he will look at all the past; if not, he only will look at the most recent past in which (he believes) the process has not changed.

His strategy, for the future, until the next change of the process, will be investing the fraction $l$ of the capital which would have maximized his capital at present time $t$. In other words, he will chose among the strategies of type (2.2) the one which would have maximized $W_t/W_{t_0}$. Notice that the maximization is not with respect to an expected quantity but with respect the single past realization of the process. There is no choice of a particular utility function. Now, maximizing $W_t/W_{t_0}$ is the same as maximizing

$$\log\left(\frac{W_t}{W_{t_0}}\right) = \sum_{i=t_0}^{t-1} \log\left(\frac{W_{i+1}}{W_i}\right) . \tag{2.7}$$

This expression differs from (2.5) only because the average is obtained by using a finite number of realizations of the variable $u$. Nevertheless, the true distribution of the $u$ is unknown in reality, and it can only be inferred from past realizations. Therefore, there is no real difference between (2.5) and (2.7). In conclusion, the Kelly strategy, based on the use of a logarithmic utility function, has the very deep motivation of being the only one corresponding to the optimization of the strategy with respect to the only available source of informations: the unique data history.

Before ending this section, we notice that an investor can never have a negative capital, i.e. $1 + l(u-1)$ must be always positive. This is same to say that the argument of the logarithm must be positive. Therefore, one must have that $l < 1/(1-u_{\min}) \equiv l_{\max}$ where $u_{\min}$ is the minimum value that the stochastic variable $u$ can assume. Also notice that, at variance with the original formulation of Kelly, the investor is allowed to borrow money, so that $l$ can also take values larger than the unity (but lower than $l_{\max}$). Only when $u_{\min} = 0$ (bankrupt is possible) the investor is not allowed to borrow money.

## 3. Transaction Costs and Optimal Lag

In this section we consider the effects due to transaction costs. This problem, which is a classical topic in mathematical finance (see for example [21–24]), is here reconsidered with the aim of defining an optimal lag for transactions.

Suppose that at time $t$ the agent invest a part $lW_t$ of his capital in the stock, after a time step, the capital in the stock has become $lu_tW_t$. Then he wants to restore the previous proportion, so that the capital invested in the stock is $lW_{t+1}$. In this case, he has to sell or buy the exceeding or missing shares. The entire process, assuming a trade cost proportional to the value of the traded shares, is described by the implicit equation

$$W_{t+1} = (lu_t + 1 - l)W_t - \gamma|lu_tW_t - lW_{t+1}|\,, \tag{3.1}$$

where $\gamma$ is the proportionality constant (see also [17]). This equation can be made explicit and one obtains

$$W_{t+1} = A(u_t, l, \gamma)W_t\,, \tag{3.2}$$

where

$$A(u, l, \gamma) = \frac{1 + l(u-1) + \alpha\gamma l u}{1 + \alpha\gamma l} \tag{3.3}$$

and

$$\alpha \equiv \text{sign}(u-1)\,\text{sign}(l-1)\,. \tag{3.4}$$

Notice that according to this simple rule, the behaviour of an investor is qualitatively different when $l$ is larger or smaller than 1. In the first case, in fact, one has a speculative behaviour: some of the shares are sold out after their price has decreased. In the second case, one has a prudent behaviour: some of the shares are sold out when their prise has increased.

The resulting rate (for a given $\gamma$ and for a given probability for the $u$) will be a function of $l$ and will depend on $\gamma$

$$\lambda_\gamma(l) = E[\log A(u, l, \gamma)]\,. \tag{3.5}$$

The optimal rate will be chosen by finding the fraction $l$ which maximizes the above expression.

For increasing value of $\gamma$ the amount of transactions must become smaller and the optimal $l$ has to approach one of the two limits which corresponds to a fixed portfolio: $l = 1$ or $l = 0$. The choice between the two depends on the distribution of the $u$, if $E[\log(u)] > 0$, then all the capital will be in the stock ($l = 1, \lambda_\gamma = E[\log(u)]$), otherwise, all the capital will be in the risk less security ($l = 0, \lambda_\gamma = 0$).

It is clear, at this point, that in presence of trading costs, it would be convenient to rearrange the capital less frequently. In other words, between the two limiting strategies, the static and the extremely dynamical one, it is possible to find a compromise. One can decide to rearrange the composition of the portfolio only every $\tau$ time steps. This strategy only leads to a redefinition of the reference stochastic variable. In fact, once defined

$$U_{t,\tau} \equiv \prod_{i=t+1}^{t+\tau} u_i\,, \tag{3.6}$$

one ends up with the evolution law

$$W_{t+\tau} = A(U_{t,\tau}, l, \gamma) W_t\,, \tag{3.7}$$

where $A$ has the same form as before. The associated rate of growth of the capital is

$$\lambda_\gamma(\tau, l) = \frac{1}{\tau} E[\log A(U_\tau, l, \gamma)]\,, \tag{3.8}$$

where $t$ has been eliminated from the notation because of the time translation invariance. The rate has to be maximized both with respect to $\tau$ and $l$.

$$\lambda_\gamma^* = \max_{l,\tau} \lambda_\gamma(l, \tau) = \max_\tau \lambda_\gamma(l(\tau), \tau) = \lambda_\gamma(l^*, \tau^*)\,. \tag{3.9}$$

Notice that, in absence of transaction costs, the optimal lag $\tau^*$ is always the minimal one ($\tau^* = 1$). On the contrary, it may happen that, for large transaction costs, $\tau^*$ becomes infinite, i.e. the static strategy turns out to be the best.

## 4. Real Example from NYSE Index

In order to show how this idea works in practice we consider a security whose price is proportional to the NYSE composite index. We will look to its price movement for exactly one decade, from the 1st of September 1988 to the 31st of August 1998. First of all we have to give an estimation of the the risk-less interest rate $r_t$. The simplest thing to do is to look at the American Discount Rate $R_t$ during the same period which is plotted in Fig. 1 (in %). Far from being a constant, $R_t$ ranges from 3 to 7. Then a good estimation of $r_t$ is

$$r_t = \left(1 + \frac{R_t}{100}\right)^{\frac{1}{253}} . \tag{4.1}$$

It is then easy to obtain from the NYSE index $S_t$ its discounted counterpart

$$\tilde{S}_t = \frac{S_t}{\prod_{i=0}^{t-1} r_i} . \tag{4.2}$$

In Fig. 2 we plot the discounted NYSE index whose initial value has been put equal the unity. Notice that a capital entirely invested in the stock would double in ten years with respect to the same capital invested in the risk-less security.

Using this data, we disregard all correlations, and we assume that all increments are independent. Then it is easy to compute the final value of the capital for different choices of $l$ assuming that its initial value is 1 and that $\gamma = 0$. In Fig. 3 we plot the final capital $\exp\{\lambda(l)T\}$ as a function of $l$ for vanishing transaction costs and for three different values of $\tau$ corresponding to one day, one week (5 working days) and one month (21 working days). Obviously, the best time lag will be the minimal

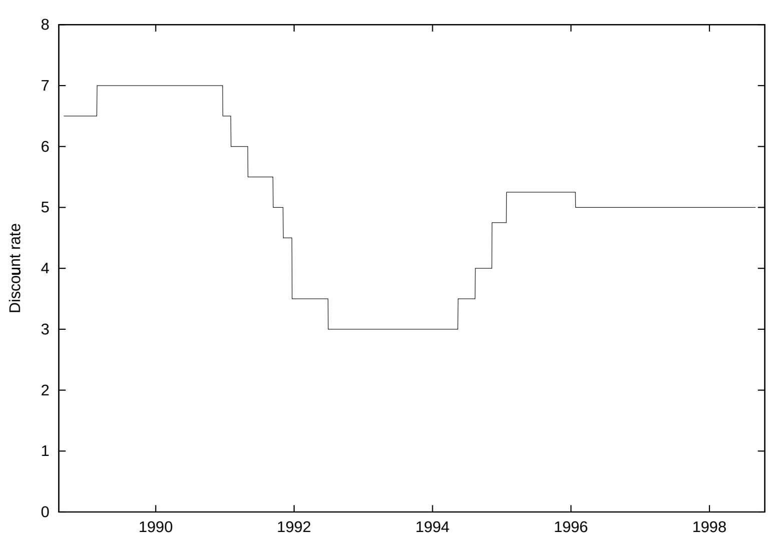


Fig. 1. American Discount Rate in % from the 1st of September 1988 to the 31st of August 1998.

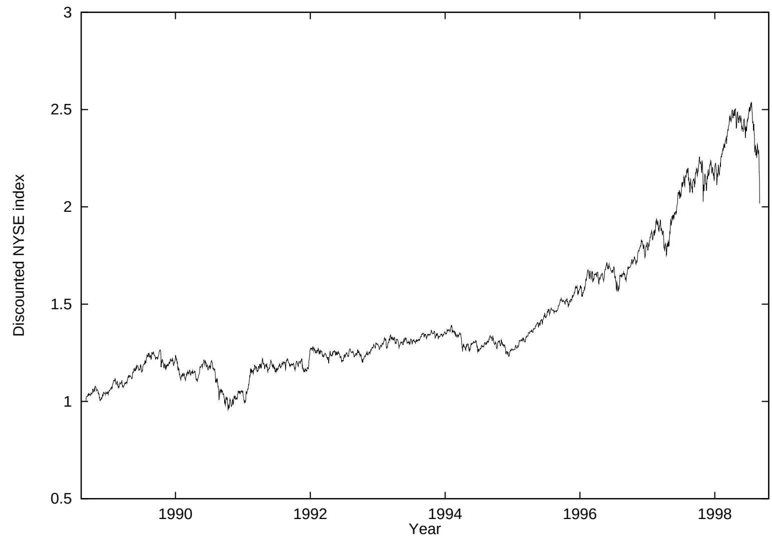


Fig. 2. NYSE discounted index from the 1st of September 1988 to the 31st of August 1998.

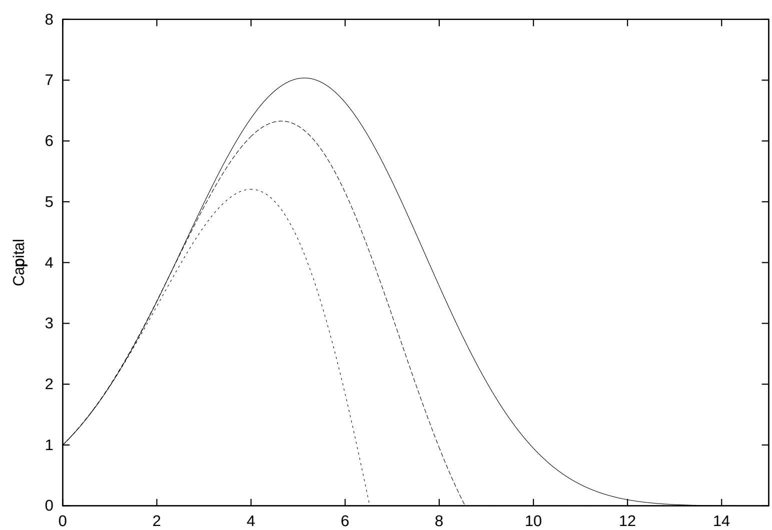


Fig. 3. The final capital $\exp(\lambda^*(l)T)$ versus $l$ for $\gamma = 0$ and for three different values of $\tau$ (the full line is one day, the dashed line is one month, the dotted line is one year).

one ($\tau = 1$, full line), in this case the maximum is reached for $l \simeq 5$, implying that the optimal investment in stock should be five time larger than the owned capital. The maximum corresponds to a final capital which is about about seven times the initial capital, much larger than the static result ($l = 1$) which gives a final capital

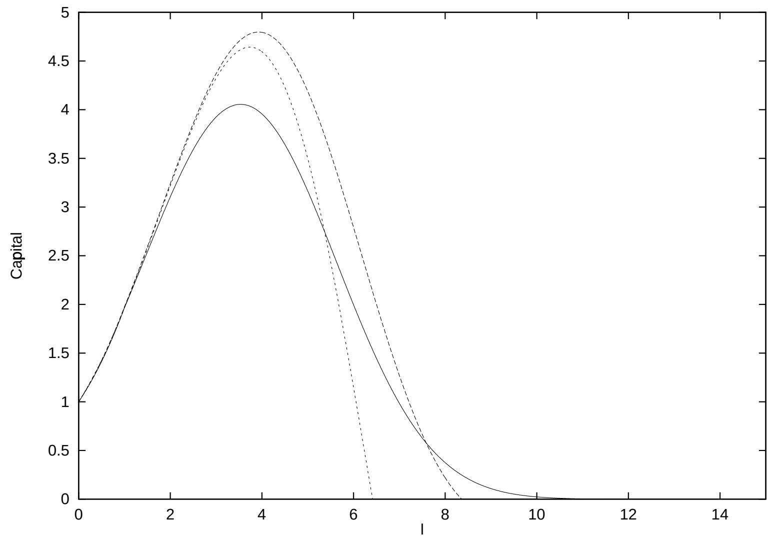


Fig. 4. The final capital $\exp(\lambda^*(l)T)$ versus $l$ for $\gamma = 0.003$ and for three different values of $\tau$ (the full line is one day, the dashed line is one month, the dotted line is one year).

only twice larger than the initial one. For $l$ larger than 14 the capital vanishes, which implies that $u_{\min} = 0.93$. The dashed and the dotted lines correspond to the more static strategies of rearranging the portfolio every week and every month. We clearly see that, as expected, these more static strategies are less efficient than the dynamical one.

In Fig. 4 we consider exactly the same situation for $\gamma = 0.003$. In this case the best result corresponds to arrangements every week (dashed line). The optimal $l$ is about 4, smaller than the cost-free result, and the final capital is now only five times larger than the initial one. The daily strategy (full line) is much less efficient for this value of $\gamma$, while there is a very small difference with the more static strategy corresponding to monthly transactions.

It may be useful to test the general strategy against the classical Kelly coin toss. In this game one has that $u = 2$ with probability $p$ and $u = 0$ with probability $1 - p$. If a lag $\tau$ and a fraction $l$ are chosen, then the corresponding growth rate is

$$\lambda_\gamma(\tau, l) = \frac{p^\tau}{\tau} \log\left(\frac{1 + l(2^\tau - 1) - 2^\tau \gamma l}{1 - \gamma l}\right) + \frac{1 - p^\tau}{\tau} \log\left(\frac{1 - l}{1 + \gamma l}\right) . \qquad (4.3)$$

A consequence of the above formula is that the minimum probability $p$ necessary to have a positive rate when $\tau = 1$ is

$$p_{\min} = \left(\frac{1 + \gamma}{2}\right) \qquad (4.4)$$

which says that for $p < p_{\min}$ it is better not to invest at all in the stock, if lags larger than 1 are not allowed.

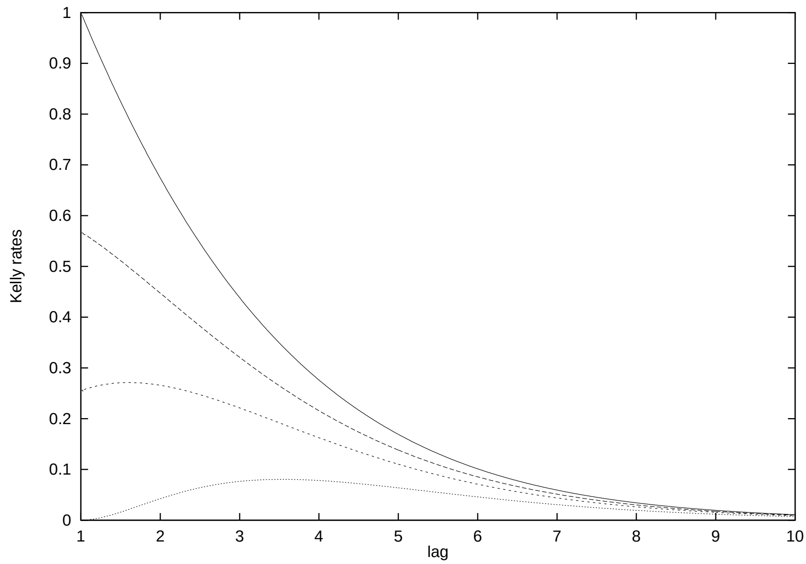


Fig. 5. Kelly dichotomic relative growth rate versus $\tau$ for $p = 0.51$ and for different values of $\gamma(0, 0.005, 0.01, 0.02)$.

If the rate (4.3) is maximized with respect to $l$ one obtains $\lambda_\gamma(l(\tau), \tau)$. It is useful to plot this quantity (for given $\gamma$ and $p$) with respect to $\tau$, in order to compare with the classical Kelly result, and in order to have a qualitative idea on the conditions for a non trivial optimal $\tau^*$. In Fig. 5 we plot the relative rate $\lambda_\gamma(l(\tau), \tau)/\lambda^*$ versus $\tau$, where $\lambda^* = \log(2) + p \log p + (1-p) \log(1-p)$ is the cost-less Kelly optimal rate. In the absence of costs (full line) we have that the relative rate equals 1 at $\tau = 1$, i.e. we recover the Kelly result. The line, as expected, monotonically decreases for larger lags, and vanishes for lags of about 10. The same qualitative behaviour is also found for $\gamma = 0.0005$ (dashed line) and $\gamma = 0.001$ (dotted line), the best lag still being the minimal one. The only difference is that now the cost-less Kelly rate cannot be entirely recovered. The qualitative behaviour changes only for $\gamma = 0.002$ and the optimal $\tau^*$ turns out to be about 4.

## 5. Final Remarks

The proposal of this paper is to introduce the notion of an optimal lag for transactions in order to bridge between static and dynamical portfolio strategies. The lag is chosen to be a deterministic quantity, nevertheless, one could choose more refined strategies in which it is allowed to be a stochastic variable. For example, one could choose to sell some of the shares when the composition of the portfolio becomes sufficiently far from the optimal one. Such a strategy implies that lags depend on the evolution of the price and their probability distribution can be found out in the context of first hitting time theory.

Nevertheless, in this case the important fact is that lags are discrete. The consequence is that the idea of a continuous trading time turns out to be only a fictitious assumption, even when an asset price is established with high frequency. The lag between transactions, in fact, is usually much larger than lag between two consecutive fixing of a price.

This simple consideration has relevance for the classical problem of derivative pricing. The most successful approach, due to Black and Scholes, works for a complete market, which means that trading time is assumed to be continuous. In the light of the present discussion, it is clear that a complete market only can be considered as an approximation and more realistic pricing, accounting for incomplete markets (i.e. discrete lags), has to be considered [18, 19].

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## References

[1] J.-P. Bouchaud and M. Potters, *Théorie des Risques Financiers*, Aléa-Saclay (1997).
[2] R. Merton, *Continuous-Time Finance*, Blackwell (1990).
[3] J. Ingersoll, *Theory of Financial Decision Making*, Rowman and Littlefield (1987).
[4] J. Lintner, *The valuation of risky assets and selection of risky investments in stock portfolios and capital budgets*, Rev. Econ. and Stat. **47** (1965) 13–37.
[5] H. Markowitz, *Portfolio selection: Efficient Diversification of Investment*, John Wiley and Sons (1959).
[6] P. Samuelson, *The "fallacy" of maximizing the geometric mean in a long sequence of investing and gambling*, Proc. Nat. Acad. Sci. USA **68** (1971) 2493–2496.
[7] R. Merton and P. Samuelson, *Fallacy of the lognormal approximation to optimal portfolio decision making over many periods*, J. of Financial Econ. **1** (1974) 67–94.
[8] J. L. Kelly, Jr., *A new interpretation of the information rate*, Bell Syst. Tech. J. **35** (1956) 917.
[9] D. Bernoulli, *Exposition of a new theory on the measurement of risk*, Econometrica **22** (1738:1954) 22–36.
[10] C. Shannon, *A mathematical theory of communication*, Bell Sys. Tech. J. **27** (1948) 379–423, 623–656.
[11] L. Breiman, *Investment policies for expanding business optimal in the long-run sense*, Naval Research Logistics Quarterly **7**(4) (1960) 647–651.
[12] L. Breiman, *Optimal gambling systems for favorable games*, Fourth Berkely Symposium on Mathematical Statistics and Probability, Prague 1961 (Univ. of California Press, Berkeley, 1961) 65–78.
[13] S. Galluccio and Y.-C. Zhang, *Product of random matrices and investment strategies*, Phys. Rev. **E54** (1996) 4516–4519.
[14] S. Maslov and Y.-C. Zhang, *Optimal investment strategy for risky assets*, Int. J. Theoretical and Appl. Finance **3** (1998) 377–387.

[15] M. Marsili, S. Maslov and Y.-C. Zhang, *Dynamical optimization theory of a diversified portfolio*, Physica A **253** (1998) 403–418.

[16] R. Baviera, M. Pasquini, M. Serva and A. Vulpiani, *Optimal strategies for prudent investors*, Int. J. Theoretical and Appl. Finance [in press], cond/mat 9804297.

[17] F. Slanina, *On the possibility of optimal investment*, in preparation.

[18] E. Aurell, R. Baviera, O. Hammarlid, M. Serva and A. Vulpiani, *Gambling and pricing of derivatives*, J. F. Qual. Anal. (submitted).

[19] E. Aurell, R. Baviera, O. Hammarlid, M. Serva and A. Vulpiani, *A general methodology to price and hedge derivatives in incomplete markets*, Int. J. Theoretical and Appl. Finance [to be submitted].

[20] S. Browne, *Can you do better than Kelly in the short run?*, Proc. of "The 10th Int. Conf. on Gambling and Risk Taking" (Montreal 1997) Ed. E. Thorpe.

[21] M. Akian, J.-L. Menaldi and A. Sulem, *On an investment-consumption model with transaction costs*, SIAM J. Control and Optim. **34** (1996) 329–364.

[22] M. J. P. Magill and G. M. Constantinides, *Portfolio selection with transaction costs*, J. Econ. Theory **13** (1976) 245–263.

[23] S. E. Shreve and H. M. Soner, *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab. **4** (1994) 909–962.

[24] T. Zariphopoulou, *Investment-consumption model with transaction fees and Markov chain parameters*, SIAM J. Control and Optim. **30** (1992) 613–636.