

# CLASSICAL LIMIT: LOCALIZATION INDUCED BY NOISE

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We introduce a model for a two configurations system, and we study the transition from quantum to classical behaviour. We first consider the effect of the interaction with the environment as an external noise and we show that it produces decoherence and suppression of tunnelling. These features are necessary ingredients for *classicality*, but we believe that classicality also implies that quantum delocalized states spontaneously evolve into localized ones. We then show that this evolution takes place only when both noise and non linearity in the equations are present.

## 1 Introduction

In this paper we focus our attention on a very old question: how and why macroscopic objects behave classically. If we disregard any answer which invokes a non-quantum or quantum-modified status for macroscopic objects we remain with the widely accepted point of view that environment induces decoherence. This fact, which has been shown to hold in a huge number of models (see for example <sup>1,2,3,4,5</sup>), can be stated as follows: the density matrix corresponding to a superposition of eigen-states of a macroscopic variable becomes very soon almost diagonal. Therefore, the superposition is *for all practical purposes* indistinguishable from the statistical mixture, whose probabilities are given by the diagonal elements <sup>6</sup>.

Our point is that this is not a complete answer, since it has only a statistical meaning, it remains to be explained why individual macroscopic objects are always found in a localized state. If we want such an explanation we are left with only a possibility: to show that any superposition quantum state spontaneously evolves in a eigen-state of the macroscopic variable.

We study the problem by considering a system with two minimal energy configuration positions. Systems of this kind are, at low temperature, classically localized in one of the two minima. On the contrary, quantum behaviour allows for delocalized states and coherent tunnelling. Real objects of this type are, for example, superconducting rings with a *weak* junction crossed by a magnetic flux or a class of tetraedrical molecules.

We start considering *ab initio* a two configurations model. The isolated system tunnels coherently and periodically. Then we consider the effect of a time dependent perturbation (interaction with the environment). Now there is decoherence and, for strong noise, suppression of tunnelling (see also <sup>7,8,9</sup>).

Our point, as explained before, is that decoherence and suppression of tunnelling which implies stability of localization are not enough for *classicality*. *Classicality* also needs spontaneous evolution into one of the two possible localized states.

Therefore, in a next step we consider the same model, but we assume a non linear evolution. In absence of perturbation, it turns out that the system behaves qualitatively as in the linear unperturbed case. But, this time, if interaction with the environment is introduced, one has that there is spontaneous evolution into one of the two localized states. The conclusion is that both non linearity and noise are fundamental ingredients for *classicality*.

## 2 The model

The density matrix can be written in a convenient form using the vector  $\mathbf{x} = (x, y, z)$  defined by

$$\rho = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \quad (1)$$

The trace of this density operator equals 1 (normalization) and the determinant vanishes for a pure state. This last requirement implies  $|\mathbf{x}| = 1$  (a pure state is represented by a point on the unitary sphere).

The third component  $z$  of this vector encodes the informations about the localization ( $z = \pm 1$  means that the system is exactly localized in one of the two states) while  $x$  and  $y$  encode the informations about coherence (when they vanish there is complete decoherence).

The above considerations show that the quantum evolution  $\dot{\rho} = -i[H, \rho]$  corresponds to the motion of a point on a unitary sphere. When the point is in one of the two poles the system is localized.

The hamiltonian which describes an isolated two configurations system allowed for tunnelling is typically

$$H = \alpha \sigma_x \quad (2)$$

where  $\sigma_x$  is the Pauli matrix. This hamiltonian produces coherent quantum tunnelling of period  $\frac{\pi}{\alpha}$  between the two *macroscopic* configurations of the system (in mathematical terms, the two eigen-states of the Pauli matrix  $\sigma_z$ ).

The corresponding equations for  $\mathbf{x}$  are

$$\begin{aligned}\dot{x} &= 0 \\ \dot{y} &= -2\alpha z \\ \dot{z} &= 2\alpha y\end{aligned}\tag{3}$$

It is clear that (2) induces a rotation with constant angular velocity around the  $x$  axis. The value of the first component of the vector remains unchanged. Therefore, (2) never allows for localization or decoherence.

### 3 Decoherence induced by noise

Assume now that the interaction with the environment is described by

$$H = \alpha\sigma_x + \epsilon(t)\sigma_z\tag{4}$$

where  $\sigma_x$  and  $\sigma_z$  are Pauli matrices and  $\epsilon(t)$  is a given realization of a stochastic process. The presence of a non vanishing  $\epsilon(t)$  means that we introduce an interaction with the environment which randomly breaks in time the energy symmetry of the two configurations.

We find that the vector  $\mathbf{x}$  satisfies

$$\begin{aligned}\dot{x} &= -2\epsilon(t)y \\ \dot{y} &= +2\epsilon(t)x - 2\alpha z \\ \dot{z} &= +2\alpha y\end{aligned}\tag{5}$$

which is the superposition of two rotations, the first around the  $x$  axis with constant angular velocity  $2\alpha$ , the second around the  $z$  axis with time dependent angular velocity  $2\epsilon(t)$ .

Assume  $\epsilon(t) = \beta\eta(t)$  where  $\eta(t)$  is a given realization of a white noise (i.e.  $w(t) \equiv \int_0^t \eta(s)ds$  is a brownian motion). In Stratanovitch notation the only thing we have to do is to substitute  $\epsilon(t)$  with  $\beta\eta(t)$  in the above equations. Nevertheless, it is much more practical to use Ito notation for which the above equations (5) rewrite as

$$\begin{aligned}dx &= -2\beta^2 x dt - 2\beta y dw \\ dy &= -2\beta^2 y dt - 2\alpha z dt + 2\beta x dw \\ dz &= +2\alpha y dt\end{aligned}\tag{6}$$

From (6), taking the expectation value, we immediately obtain

$$\begin{aligned}\dot{\bar{x}} &= -2\beta^2 \bar{x} \\ \dot{\bar{y}} &= -2\beta^2 \bar{y} - 2\alpha \bar{z} \\ \dot{\bar{z}} &= +2\alpha \bar{y}\end{aligned}\tag{7}$$

Since the density matrix depends linearly on  $\mathbf{x}$ , the above equation describes the evolution of the averaged density matrix  $\bar{\rho}$ . This matrix describes the mixture of all the states associated to the ensemble of all the realizations of the noise in the hamiltonian (4). In other words, if one can not know which is the noise realization, all the statistical previsions are those that can be derived from the averaged density matrix  $\bar{\rho}$ .

Equation (7) can be easily solved. When  $\beta^2 < 2|\alpha|$  the solution is:

$$\begin{aligned}\overline{x(t)} &= e^{-2\beta^2 t} x(0) \\ \overline{y(t)} &= e^{-\beta^2 t} [y(0) \cos(\omega t) + c_1 \sin(\omega t)] \\ \overline{z(t)} &= e^{-\beta^2 t} [z(0) \cos(\omega t) + c_2 \sin(\omega t)]\end{aligned}\tag{8}$$

where  $\omega = \sqrt{|\beta^4 - 4\alpha^2|}$ ,  $c_1 = \frac{-\beta^2 y(0) - 2\alpha z(0)}{\omega}$  and  $c_2 = \frac{\beta^2 z(0) + 2\alpha y(0)}{\omega}$ . Looking at the third component  $z(t)$  one realizes that in this region a quantum coherent behavior partially survives to the noise for a certain time.

When  $\beta^2 > 2|\alpha|$  the solution is purely exponential and is given by

$$\begin{aligned}\overline{x(t)} &= e^{-2\beta^2 t} x(0) \\ \overline{y(t)} &= e^{-\beta^2 t} [y(0) \cosh(\omega t) + c_1 \sinh(\omega t)] \\ \overline{z(t)} &= e^{-\beta^2 t} [z(0) \cosh(\omega t) + c_2 \sinh(\omega t)]\end{aligned}\tag{9}$$

In this strong noise region the coherent behaviour is completely destroyed since the localization probability relaxes exponentially. The interesting fact is that for large  $\beta$ ,  $\overline{z(t)}$  relaxes with an exponent which vanishes as  $\frac{2\alpha^2}{\beta^2}$ . This fact means that the time necessary for delocalization becomes infinite in the limit of extremely strong noise.

Summarizing, we have damped oscillations for a weak interaction with the environment ( $\beta^2 < 2|\alpha|$ ) and incoherent relaxation for strong interaction ( $\beta^2 > 2|\alpha|$ ). Furthermore, in the limit of very strong noise a localized configuration turns out to be almost stable.

Since in both cases  $\overline{\mathbf{x}} \rightarrow 0$  exponentially, we have that

$$\bar{\rho} \rightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\tag{10}$$

which says that in any case we have almost complete decoherence at large times, the off-diagonal elements being damped whereas the diagonal ones carry the probabilities.

From (10) we could conclude that an individual system is in one of the two configuration positions. This is not true, in fact, from the stochastic

equations (6) one can derive the set of linear equations

$$\begin{aligned}\dot{\overline{z^2}} &= 4\alpha\overline{zy} \\ \dot{\overline{y^2}} &= -4\alpha\overline{zy} - 4\beta^2\overline{y^2} + 4\beta^2\overline{x^2} \\ \dot{\overline{yz}} &= -2\beta^2\overline{zy} + 2\alpha\overline{y^2} - 2\alpha\overline{z^2}\end{aligned}\tag{11}$$

Using the condition  $x^2 + y^2 + z^2 = 1$ , one can replace  $\overline{x^2}$  with  $1 - \overline{y^2} - \overline{z^2}$  in the second of the above equations. For any  $\beta \neq 0$  one finds that the solution converges to the stationary solution  $\overline{x^2} = \overline{y^2} = \overline{z^2} = \frac{1}{3}$ ,  $\overline{zy} = 0$ . This means that the typical state is not localized (localization for all states would imply  $\overline{z^2} = 1$ ). A new ingredient is thus necessary to produce localization.

#### 4 Non-linearity

The effect of the interaction with the environment can be taken into account also considering a non linear term in the differential equations that describe the evolution of the density matrix. In our case, we can assume that  $\alpha$  depends on the state ( $\alpha = \alpha(x, y, z)$ ). For the sake of simplicity assume, as usual, that this dependence reduces to a dependence on the modulus square of the wave function. In our language it means that  $\alpha$  depends only on  $z$ . There are many reasons for this choice, that we will discuss in a forthcoming paper <sup>10</sup>.

Also assume that:

a)  $\alpha(z)$  is an even function of  $z$ :  $\alpha(z) = \alpha(-z)$ ;

b)  $\alpha(z)$  is zero at the poles:  $\alpha(+1) = \alpha(-1) = 0$ , positive otherwise and sufficiently smooth around the poles.

The requirement b) implies that a localized state remains, in fact, localized. Nevertheless, *in absence of noise*, the system is not able to spontaneously localize. In this case, in fact, equation (3) still holds, with the only difference that  $\alpha$  depends on  $z$ . Notice that the first component  $x$  of the vector  $\mathbf{x}$  remains constant during the motion. This means that there is no decoherence, and, furthermore, that the system can never reach one of the poles. Indeed, the motion remains periodic, since we are on a two dimensional variety. The only exception is when the system is initially on the meridian  $x = 0$ , in which case it moves along this meridian toward one of the poles, producing localization, but this ensemble of initial conditions has probability zero.

In conclusion, non linear differential equations, like the equation modeling the interaction with the environment, are not able to reproduce a classical behaviour in the more restricted sense of induced localization. Nevertheless, this goal can be achieved if one takes into account the simultaneous presence of these two different mechanisms: noise and non linearity.

## 5 Localization induced by noise and non-linearity

Let us consider again the hamiltonian (4) with  $\alpha = \alpha(z)$ ; the previous requirements on  $\alpha(z)$  are also assumed. The differential equations (6) which remain formally unchanged (the only difference is that  $\alpha$  is no longer constant, but it depends on  $z$ ), have two fixed points corresponding to the poles. What we can show is that, independently on the initial conditions, one has  $z(t) \rightarrow \pm 1$ . This convergence is not due to the attractiveness of the poles but to the fact that the motion along the meridians becomes more and more slow approaching the poles. In other words, the system stays for long time intervals around them, the distribution of this time intervals having sufficiently long tails to guarantee that, at the end the system will spend almost all its time around a pole (it will be almost surely in a pole).

We can sketch how to show that the system localizes for large times; a rigorous proof being beyond the scope of this paper. The requirements on  $\alpha(z)$  imply that it can be approximated around the poles by

$$\alpha(z) = \alpha_0(1 - z^2) + \dots \quad (12)$$

We now prove that the limiting (steady) distribution is concentrated on the poles. Using equations (6) and taking the average, we have

$$\begin{aligned} \frac{d}{dt} \left[ \overline{\theta(z - z_c)y} \right] = & -2\beta^2 \overline{\theta(z - z_c)y} + \\ & -2\overline{\alpha(z)\theta(z - z_c)z} + 2\overline{\alpha(z)\delta(z - z_c)y^2} \end{aligned} \quad (13)$$

where  $0 \leq z_c \leq 1$ , and  $\theta(\cdot)$  is the step function.

We can safely assume that any initial distribution on the surface of the unitary sphere will evolve, after a transient time, toward a stable distribution. For this stable distribution, the above time derivative  $\frac{d}{dt} \overline{\theta(z - z_c)y}$  vanishes.

Furthermore, from equations (6) it also follows that

$$\frac{d}{dt} \left[ \overline{\theta(z - z_c)} \right] = 2\overline{\alpha(z)\delta(z - z_c)y} \quad (14)$$

The above equation implies for the steady distribution  $\overline{\delta(z - z_c)y} = 0$ , which, in turn, implies  $\overline{\theta(z - z_c)y} = 0$ .

This last equality, inserted in (13) gives for the stationary distribution

$$\overline{\alpha(z)\theta(z - z_c)z} = \alpha(z_c)\overline{\delta(z - z_c)y^2} \quad (15)$$

which leads to

$$\lim_{z_c \rightarrow 1} \frac{\overline{\alpha(z)\theta(z-z_c)z}}{\alpha(z_c)\overline{\delta(z-z_c)y^2}} = 1 \quad (16)$$

It is intuitive, and it can be also easily shown (again from (6)) that the relative difference between  $\overline{\delta(z-z_c)y^2}$  and  $\overline{\delta(z-z_c)x^2}$  vanishes in the limit  $z \rightarrow z_c$ , in fact the random rotation around the  $z$  axis becomes very fast compared with the rotation around the  $x$  axis. Therefore, since  $x^2 + y^2 + z^2 = 1$ , we can replace  $\overline{\delta(z-z_c)y^2}$  with  $\overline{\delta(z-z_c)\frac{1-z^2}{2}}$  in equation (16), which we rewrite as

$$\lim_{z_c \rightarrow 1} \frac{\int_{z_c}^1 (1-z^2)z\rho(z)dz}{\frac{(1-z_c^2)^2}{2}\rho(z_c)} = 1 \quad (17)$$

where we have used the expansion of  $\alpha(z) \simeq \alpha_0(1-z^2)$  around the north pole and we have introduced explicitly the steady state probability  $\rho(z)$  that the system is at a quote  $z$ .

Normalization implies that the probability distribution  $\rho(z)$  cannot diverge around the north pole faster than  $\rho(z) \simeq \frac{1}{(1-z_c)^\gamma}$  with  $\gamma < 1$ . With this assumption, we find that the above equation gives

$$\frac{1}{2-\gamma} = 1 \quad (18)$$

which cannot be satisfied for any normalizable distribution.

Finally, since (15) is satisfied for a distribution which is concentrated (Dirac's delta) in the north pole we find that this is the only possible stable distribution. Repeating the argument for  $-1 \leq z_c \leq 0$ , and taking into account the symmetry of the system (6) with respect to the changes  $y \rightarrow -y$  and  $z \rightarrow -z$ , one can conclude that the steady distribution is made of two Dirac's delta distributions centered on the poles, with equal weights.

Notice that this result implies again (10), i.e.  $\bar{\rho}$  is equal one half of the identity operator for  $t \rightarrow \infty$ . Nevertheless, in this case one has

$$\rho = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{with prob. } \frac{1}{2} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{with prob. } \frac{1}{2} \end{cases} \quad (19)$$

and the density matrix decomposition corresponds to the combination of the two localized states.

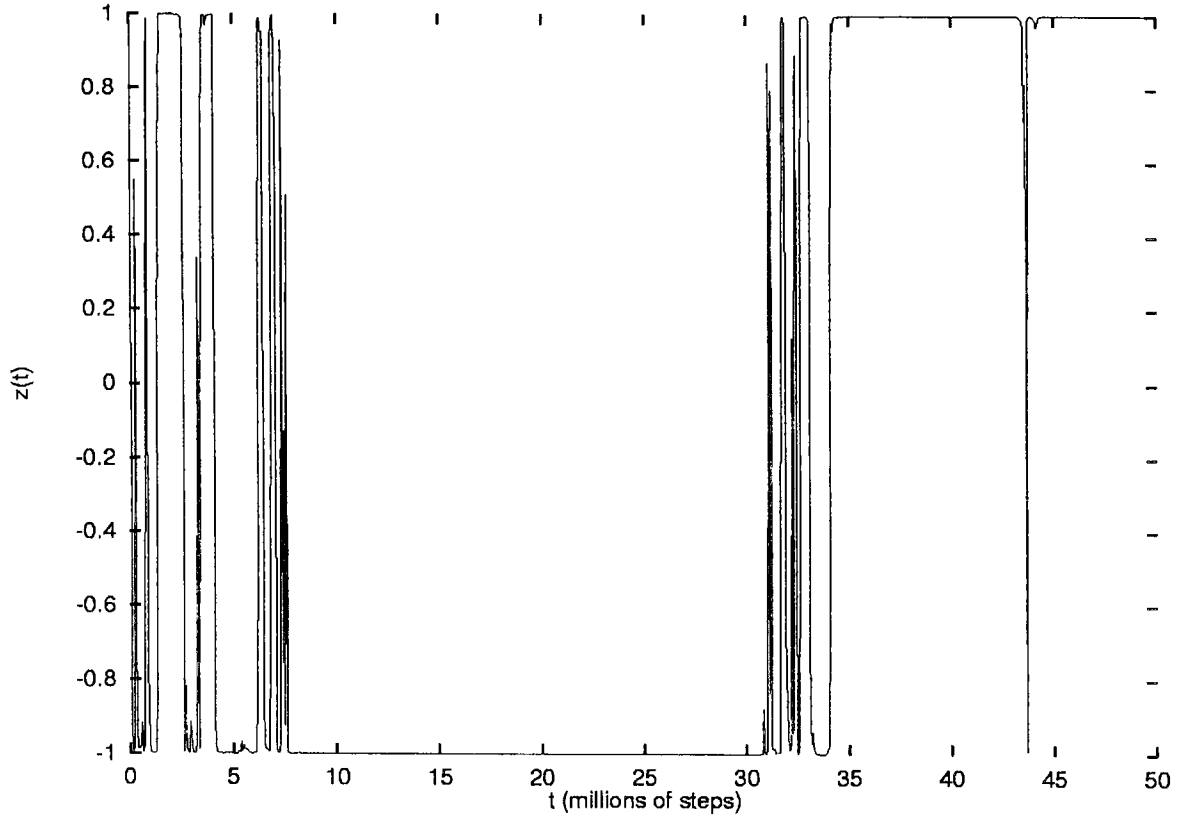


Figure 1. A typical realization of  $z(t)$  as function of  $t$  for  $\alpha_0 = 1$  and  $\beta = 7$ . The differential equation [6] is numerically solved with a time step of  $10^{-2}$ .

Therefore, for what concerns decoherence nothing is changed, the important difference being that now  $\overline{\rho^2} = \bar{\rho}$  for large times.

In fig. 1, where  $z(t)$  is plotted starting from a numerical solution of the differential equations (6) for  $\alpha_0 = 1$  and  $\beta = 7$ , one sees clearly that the system spends a large part of the time in very narrow regions around the poles.

The time necessary to make a transition between the poles turns out to be negligible with respect to the long periods in which  $z \simeq \pm 1$ , which become infinitely large with  $t$ . Therefore, if one perform an observation of the system, it is almost sure to find it in a localized configuration. A useful quantity in order to show this fact is the time average of  $z(t)^2$ , defined by:

$$L(t) = \frac{1}{t} \int_0^t [z(s)]^2 ds \quad (20)$$



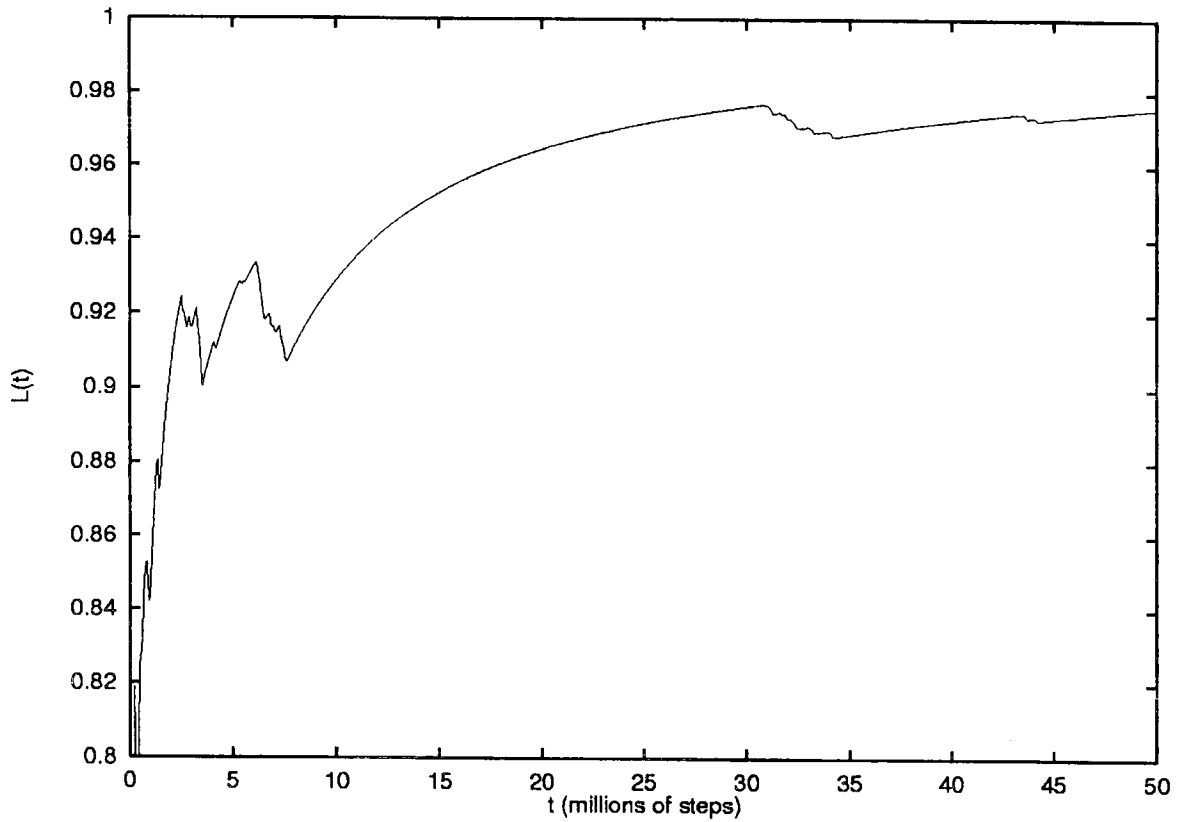


Figure 2.  $L(t)$  (19) as function of  $t$  for the same noise realization of fig. 1, with  $\alpha_0 = 1$  and  $\beta = 7$ .

This average tends to grow in time since the periods when  $z(t)^2$  is substantially different from 1 become soon negligible, as shown by fig. 2, where  $L(t)$  is plotted for the same noise realization as fig. 1.

## 6 Conclusions

In conclusion, we have presented a model where the coupling with the environment is represented both by a noisy potential and a non linear correction to the differential equations. This second effect should be a consequence of the feed-back of the system on the environment. Non linearity or noise alone cannot produce localization, while together they induce the system to localize in one of its minimal energy configuration positions. Moreover, the localization positions are chosen randomly and they are not fixed in time, in the sense that the system can always make a transition between the two different

macroscopic states. Therefore, it is not possible to predict the result of an observation.

The specific features of the model we have chosen are merely dictated by simplicity. Nevertheless, we consider a genuine quantum dynamics without more or less phenomenological dissipative terms in the hamiltonian.

In our opinion, the results we found could provide some insight for the understanding of the subtle transition associated with the emergence of the classical world.

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