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A GENERAL METHODOLOGY TO PRICE AND HEDGE DERIVATIVES IN INCOMPLETE MARKETS

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We introduce and discuss a general criterion for the derivative pricing in the general situation of incomplete markets, we refer to it as the No Almost Sure Arbitrage Principle. This approach is based on the theory of optimal strategy in repeated multiplicative games originally introduced by Kelly. As particular cases we obtain the Cox–Ross–Rubinstein and Black–Scholes in the complete markets case and the Schweizer and Bouchaud–Sornette as a quadratic approximation of our prescription. Technical and numerical aspects for the practical option pricing, as large deviation theory approximation and Monte Carlo computation are discussed in detail.

1. Introduction

The *classical* ingredients for derivative pricing and hedging are the absence of costs and the efficient and complete market. According to these hypothesis, the absence of arbitrage opportunities determines the price.

The most general formulation of the no-arbitrage argument is due to Harrison and Kreps [18]. They show that a price system (under certain restrictions such as no trading costs) admits no arbitrage opportunities if and only if all price processes are martingales. This theory is discussed in several excellent monographs [10, 14, 19, 20, 26]. Nevertheless the martingale is not unique except for a complete market. In this case it is possible, using options and shares of a single stock, to build up a portfolio with deterministic return. Then, the capital growth rate equals the bank interest rate. The most successful (and famous) applications are the Black and Scholes formula [3] for continuous time and the Cox, Ross and Rubinstein for binomial processes [9].

Unfortunately, real stock markets are not complete and a universally accepted pricing procedure is still lacking. Nevertheless, the mean-variance methods seem to have the largest consensus [5, 31]. In these theories, the optimal hedge portfolio is found by assuming that investors are risk averse. One of the main results is that the expected capital invested in this portfolio grows according to the bank interest rate. A serious conceptual and practical problem obviously rises: why should a risk averse investor, or any rational investor, put the money in such a portfolio? In fact, return is not deterministic and it is expected to be merely equal to the fixed bank return.

In the present paper we shall consider an incomplete market under the standard assumption of no transaction costs. We state an argument which is able to specify the appropriate martingale out of the many possible ones. Considering the derivative as a possible asset in the diversification of a portfolio we take the point of view of a speculator, who is a market operator interested in the "best" investment in the long run limit. We shall show in Sec. 3 in which sense this point of view is reasonable.

The criterion is based on the theory of optimal gambling due to Kelly [21] and hereafter we will refer to it as the Principle of No Almost Sure Arbitrage. Some results of this paper have been already presented in [1].

Remark that our methodology differs from the standard approach, where one takes the point of view of the investor who tries to minimize risks with an appropriate hedging procedure. In the context of a speculator, we consider in this paper, the word "hedging" indicates the portfolio strategy he chooses to reach his aim.

The paper is organized as follows: in Sec. 2 we discuss the Kelly theory in the general framework of incomplete markets. In Sec. 3 we state the Principle of No Almost Sure Arbitrage and we obtain the derivative price. In Sec. 4 we show the *almost sure* consequences of an incorrect pricing. Section 5 is devoted to a discussion of relations and differences with other pricing procedures. In particular, we obtain the Cox–Ross–Rubinstein and Black–Scholes prices in the case of complete markets and we derive the Schweizer and Bouchaud–Sornette price as a quadratic approximation of our prescription. We discuss the large deviation approach in Sec. 6 and we develop a Monte Carlo method in Sec. 7. Finally, in Sec. 8 we summarize and discuss our results. In the Appendices we discuss in detail some technical points: the case of time correlated returns and heavy-tail distributions.

2. The Kelly Theory of Optimal Portfolio

In this section we summarize Kelly's theory of optimal gambling [21]. Kelly originally was looking for an interpretation of Shannon's Information Theory [32] outside the context of communication. Later Breiman [7, 8] reconsidered Kelly's theory as a model for optimal portfolio in a stock market. For a recent review of growth-optimal investment strategies, see [16, 25].

In the paper we shall consider, except where differently stated, a discrete time price movement of stock (or some other security) described by

$$S_{t+1} = u_t S_t \,, \tag{2.1}$$

where time is discrete, S_t is the price at time t of a share and the u_t's are independent, identically distributed (i.i.d. hereafter) random variables.

Let us consider a market operator endowed with a wealth W_0 at time zero, who decides to gamble on such a stock many times. He invests at each time a fraction l of his capital in stock, and the rest in a risk-less security, i.e. a bank account. A non-zero risk-less rate corresponds to a discount factor in the definition of the share prices, and can be accounted for by a redefinition of the u_t's. We set for simplicity the risk-less rate to zero. We will write the result for the general case at the end. At time t the market operator will hold a number lW_t/S_t of shares, and his wealth at successive instants of time follows a multiplicative random process

$$W_{t+1} = (1-l)W_t + lu_tW_t = (1 + l(u_t - 1))W_t \,. \tag{2.2}$$

As a consequence of the large numbers law the exponential growth rate of the wealth is in the large time limit a constant with probability one. That is,

$$\lambda(l) = \lim_{\tau\to\infty} \frac{1}{\tau} \log \frac{W_\tau}{W_0} = E^p[\log(1 + l(u-1))] \tag{2.3}$$

holds for almost all realizations of the random variables u_t, where

$$E^p[\,(\cdot)\,] \equiv \int (\cdot)\, p(u) du \,, \tag{2.4}$$

and $p(u)$ is the probability density for the random factors u.

The optimal gambling strategy of Kelly consists in maximizing $\lambda(l)$ in (2.3) by varying l. We call speculator a market operator who follows this rule. The solution is unique because the logarithm is a convex function of its argument. Let us discuss which values of l are reasonable in our problem. First, the optimum l must be such that $1+l(u-1)$ is positive on the support of u. Second, we must decide if borrowing of cash or short selling of stock is allowed. In the original formulation of Kelly it is not, but here it is useful to allow l to take any finite positive or negative value, and look for the maximum of $\lambda(l)$.

The desired strategy is specified by the only finite l^* which solves

$$0 = \left.\frac{d\lambda(l)}{dl}\right|_{l=l^*} = E^p\left[\frac{u-1}{1+l^*(u-1)}\right] \tag{2.5}$$

and the maximum growth rate is

$$\lambda^* = E^p[\log(1 + l^*(u - 1))]\,. \tag{2.6}$$

Let us define

$$q(u) \equiv \frac{p(u)}{1 + l^*(u - 1)}\,, \tag{2.7}$$

where l^* is the optimal fraction. It is easy to show that $q(u)$ is a probability density. Since $1+l^*(u-1)$ is positive also $q(u)$ is positive. From (2.5) and the normalization of $p(u)$ one has

$$\int q(u)du = \int q(u)(1 + l^*(u - 1))du = 1\,. \tag{2.8}$$

The $q(u)$ is a new probability with respect to the given stock. Furthermore, according to this probability the associated multiplicative process is a martingale. From the definition

$$E^q[\,(\cdot)\,] \equiv \int (\cdot)q(u)du \tag{2.9}$$

and using (2.5) it is immediate to obtain

$$E^q[u] = 1\,. \tag{2.10}$$

Maximizing any monotonic convex function gives the same formal result as (2.3) i.e. a unique solution and, in the absence of constraints, the price is a martingale. However, Kelly has shown that repetition of the investment many times gives an objective meaning to the statement that the growth-optimal strategy is the best, regardless to the subjective attitude to risk or other psychological considerations.

The generalization to a speculator who can diversify his investment on N risk assets is straightforward and the interested reader can find it in [7]. In the next section we shall consider the particular case of a two risky assets portfolio composed by a share and a derivative written on the share. We shall show that the probability (2.7) plays a privileged role for pricing derivatives.

3. Principle of No Almost Sure Arbitrage

We consider now the problem of pricing a derivative in incomplete markets. As in the previous section the speculator can invest his capital only at discrete times.

The speculator's portfolio is composed at time $t = 0$ by a risk-less security, a stock and a derivative written on the same stock with maturity time T and strike price $K = kS_0$.

At time T the derivative expires and the speculator decides to buy a new derivative with the same k and maturity $2T$. Let us focus on the investment at time t in order to fix the notation. We call C_t the value of derivative at time t and $C_{t+1}(u_t)$ the value at the following instant of time $t + 1$. Finally

$$f_t(u_t) \equiv \frac{C_{t+1}(u_t)}{C_t} \tag{3.1}$$

is the return of the derivative. The notation $f_t(u_t)$ is to stress that we are dealing with a derivative, i.e. the return of this asset depends on the return of the share u_t. The simplest example is a European call option with a strike price kS_0, one time step before the expiration date, i.e. $t = T - 1$; in this case one has $C_T(u_T - 1) = \{u_T - 1S_{T-1} - kS_0\}^+$.

At time t the speculator diversifies his portfolio by investing a fraction l_t of his wealth in shares and a fraction d_t in derivatives. If the speculator keeps some money W_S in stock and W_D in derivatives, after the market fall out the two capitals are worth respectively $W_S u_t$ and $W_D f_t(u_t)$ where the same random variable u_t appears in both expressions. The total wealth is therefore changed to

$$W_{t+1} = (1 + l_t(u_t - 1) + d_t(f_t(u_t) - 1))W_t \,. \tag{3.2}$$

We shall consider hereafter, as in Sec. 2, the return u_t an i.i.d. random variable. This assumption is relaxed in Appendix A where the Markovian case is treated.

The return of the derivative f_t is a function of $T - t$ if $0 < t < T$ and of the random variable u_t. Let us stress that if u_t is an i.i.d. random variable the returns f_t are periodic of period T. Hence the fractions $\{l_t\}_t$ and $\{d_t\}_t$ take on only T different values.

We consider the case of a speculator who plays the game every time and repeats his investment many times. In this situation his wealth will almost surely grow at an exponential rate which is

$$\begin{aligned}\lambda(\{l_t\}_t, \{d_t\}_t; \{f_t\}_t) &= \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{n=0}^{\tau} \log(1 + l_n(u_n - 1) + d_n(f_n(u_n) - 1)) \\ &= \frac{1}{T} \sum_{t=0}^{T-1} E^p[\log(1 + l_t(u_t - 1) + d_t(f_t(u_t) - 1))] \,. \end{aligned} \tag{3.3}$$

The speculator is then interested in the fractions attaining the optimal growth rate of the capital $(\{l_t^*\}_{t=0,\ldots,T-1}, \{d_t^*\}_{t=0,\ldots,T-1})$. This set of values is unique because of the convexity of the logarithm given the set of the derivative returns $\{f_t\}_{t=0,\ldots,T-1}$.

We are now ready to state the Principle of No Almost Sure Arbitrage (hereafter NASA) introduced in [1]: *prices of derivative securities must be such that a speculator cannot construct a portfolio out of combinations of the derivatives and the underlying security which grows almost surely at a faster exponential rate than a portfolio containing only the underlying security.*

In other words, the NASA Principle says that the returns $\{f_t\}_t$ must be such that the global maximum of (3.3) equals λ^*, i.e.

$$\lambda(\{l_t^*\}_t, \{d_t^*\}_t; \{f_t\}_t) = \lambda^* \,, \tag{3.4}$$

where λ^* is the Kelly optimal rate of Eq. (2.6). Since one trivially has $\lambda(\{l_t = l^*\}_t, \{d_t = 0\}_t; \{f_t\}_t) = \lambda^*$, the maximum must be in $(l_t^* = l^*, d_t^* = 0)$, because of the uniqueness of the maximum of a convex function.

Let us give a simple intuitive interpretation of this fact. If d_t^* is larger than zero all speculators would like to buy the derivative in order to achieve a larger exponential rate of their capital. As a consequence the derivative price rises and its return rate falls. The resulting fraction d_t^* decreases. On the other hand, if d_t^* is less than zero, then all speculators want to go short of derivatives causing their price to fall and the return rate to rise so that d_t^* tends to grow.

From a technical point of view, the idea of setting d_t^* to zero, is closely related to the Samuelson's methodology [30] of warrant pricing in the "incipient case", see [26, Chap. 7]. We apply such a principle here to derivatives and not only to warrants. Furthermore, the condition $d_t^* = 0$ stems from a general principle. The incipient case is intrinsically important and not only a convenient assumption to simplify the computations.

The condition for l_t^* is

$$0 = \left.\frac{\partial\lambda(\{l_t\}_t, \{d_t\}_t; \{f_t\}_t)}{\partial l_t}\right|_{\{l_t=l^*\}_t, \{d_t=0\}_t} = E^p\left[\frac{u-1}{1+l^*(u-1)}\right]. \tag{3.5}$$

This equation is identical to (2.5) of the Kelly model and we find again the probability q (2.7). The equation specifying d_t^* is

$$0 = \left.\frac{\partial\lambda(\{l_t\}_t, \{d_t\}_t; \{f_t\}_t)}{\partial d_t}\right|_{\{l_t=l^*\}_t, \{d_t=0\}_t} = E^p\left[\frac{(f_t(u_t)-1)}{1+l^*(u_t-1)}\right]. \tag{3.6}$$

Then using (2.7)

$$C_t = E^q[C_{t+1}(u)], \tag{3.7}$$

which states that the NASA price for derivatives is a martingale with respect to q: in the following we refer to it as the equilibrium price. In Appendix B we deal with some simple cases where the q can be computed analytically, and we show that even a $p(u)$ very wildly divergent for large u will lead to a $q(u)$, which has at least finite first and second moments.

We have derived the rule which gives the price at time t given the price a time $t+1$. In particular, the option price at time $T-1$ is

$$C_{T-1}(S_{T-1}) = E^q[C_T(u\,S_{T-1})], \tag{3.8}$$

since the price of the derivative is known at the expiration time T. If we iterate backwards the above equation we obtain

$$C_t(S_t) = E^Q[C_T(U_{T-t}\,S_t)], \tag{3.9}$$

where the expectation value $E^Q[\cdot]$ is taken with respect to the random variable

$$U_{T-t} \equiv \frac{S_T}{S_t} = \prod_{s=t+1}^{T} u_s \tag{3.10}$$

which is the product of $T-t$ independent identically q-distributed variables u_s. The probability distribution $Q_{T-t}(S_T|S_t)$ of the product can be obtained as the convolution of $T-t$ probability distributions of a single variable. The price depends on the actual share price S_t at time t. In the particular case

$$C_T(S_T) = \{S_T - kS_0\}^+ \tag{3.11}$$

we obtain

$$C_t(S_t) = E^Q[\{U_{T-t}S_t - kS_0\}^+] , \tag{3.12}$$

which is the NASA price of a European call option.

We have taken the risk-less rate to be zero. A non-zero risk-less rate can be reintroduced as a discount factor in the share prices. Actually, having defined $\tilde{u}$ as u/r we can reproduce all the computations for the new stochastic variable. We start from the effective probability of the $\tilde{u}$ and we find the probability $q(\tilde{u})$. Then, for $r > 1$ the European call option price is

$$C_t(S_t) = E^Q\left[\left\{\tilde{U}_{T-t}S_t - \frac{kS_0}{r^{T-t}}\right\}^+\right] , \tag{3.13}$$

where $\tilde{U}_{T-t} = \prod_{s=t+1}^{T} \tilde{u}_s$.

Before we turn to the consequences of a derivative price different from the one imposed by the NASA Principle, let us pause and comment. We are interested in the derivative value assuming that the speculator knows but cannot change the distribution of stock price returns. This is of course an idealization but it looks a reasonable starting point. We stress once again that the Principle, as it is stated above, can be applied *only* to an asset and a derivative written on it, because both of them use the same source of information: the distribution of the returns of the underlying. Surely it cannot be applied to determine the stock price. In this case two different financial objects are involved, a risk-free interest rate and a risky asset. It is not reasonable to have the same growth rate in both cases, otherwise nobody would invest in the second one. However we expect to observe in liquid markets, the tendency of the average discounted return to become closer and closer to zero. This is true for example in the case of currency markets. In fact after 1989 for the currencies of the most developed countries the purchasing power parity is well verified, i.e. the exchange rate of two countries should adjust according to relative prices.[a] A null average return, for periods of one or few years, is a simple consequence of the purchasing power parity and it implies that the optimal fraction in a stationary policy is zero.

4. Arbitrage in Non Equilibrium Case

We show here that an incorrect pricing of a derivative allows for *almost sure* arbitrage, i.e. a rate of capital grow larger than the one obtained with the optimal portfolio of shares and risk-less securities.

[a]The observed deviation from this behavior is generally due to macroeconomic reasons, such as the rapid depreciation of Italian Lira in the wake of ERM crisis in 1992 or the large appreciation of UK exchange rate in the eighties with the development of the North Sea oil fields [13].

Let us suppose that the derivative is not correctly priced (i.e. it is not at its equilibrium value C_t^*):

$$C_t = C_t^* + \Delta C_t \quad \text{with} \quad t = 0, \ldots, T-1\,. \tag{4.1}$$

Given the price of the derivative the problem is to find the optimal growth rate of the capital for a mixed portfolio composed by a risk-less security, a stock and the derivative written on it. Assuming that ΔC_t is small and expanding the growth rate up to the second order in $\delta l_t \equiv l_t - l^*$, $\delta d_t \equiv d_t$ and ΔC_t we find that the optimal portfolio corresponds to δl_t^* and δd_t^* given by

$$\delta l_t^* = \frac{\Delta C_t}{C_t^*} \frac{\Gamma_{12}^{(t)}}{\det \Gamma^{(t)}}\,, \tag{4.2}$$

$$\delta d_t^* = -\frac{\Delta C_t}{C_t^*} \frac{\Gamma_{11}^{(t)}}{\det \Gamma^{(t)}}\,, \tag{4.3}$$

where the matrix $\Gamma^{(t)}$ is

$$\Gamma^{(t)} = \begin{pmatrix} E^p\left[\dfrac{(u-1)^2}{(1+l^*(u-1))^2}\right] & E^p\left[\dfrac{(f_t(u_t)-1)(u_t-1)}{(1+l^*(u_t-1))^2}\right] \\ E^p\left[\dfrac{(f_t(u_t)-1)(u_t-1)}{(1+l^*(u_t-1))^2}\right] & E^p\left[\dfrac{(f_t(u_t)-1)^2}{(1+l^*(u_t-1))^2}\right] \end{pmatrix}. \tag{4.4}$$

The corresponding maximal rate is

$$\lambda(\{C_t\}_t) = \lambda^* + \frac{1}{2T}\sum_{t=0}^{T-1}\left(\frac{\Delta C_t}{C_t^*}\right)^2 \frac{\Gamma_{11}^{(t)}}{\det \Gamma^{(t)}}\,. \tag{4.5}$$

Remark that $\Gamma_{11}^{(t)}$ and $\det\Gamma^{(t)}$ are positive quantities. Therefore, the correction to the equilibrium optimal growth rate λ^* is always positive both for under-valued and over-valued derivatives.

5. Comparisons with Other Approaches and Limiting Cases

In this section we compare our approach with previous pricing procedures both in case of complete and incomplete markets.

5.1. *Classical pricing prescriptions for complete markets*

The above proposed pricing procedure agrees with no-arbitrage pricing in the case of complete markets. It is, therefore, instructive to carry through the calculations for the dichotomic Cox–Ross–Rubinstein case and the Black–Scholes continuous time limit.

Following Cox–Ross–Rubinstein we assume that the share price can go up by a factor u_u with probability p and down by a factor u_d with probability $(1-p)$. We can safely assume that the risk-free interest rate r equals unity. The case $r > 1$ can always be recovered by simply replacing u with $\tilde{u} = u/r$. The probability density is therefore:

$$p(u) = p\delta(u - u_u) + (1 - p)\delta(u - u_d)\,, \tag{5.1}$$

where the $\delta(\cdot)$ are Dirac delta functions.

The optimization equation is then solved as

$$l^* = \frac{p}{1 - u_d} - \frac{(1 - p)}{u_u - 1} \tag{5.2}$$

which leads to a probability density $q(u)$ having exactly the same form as (5.1), the only difference being that the probability p is replaced by q:

$$q = \frac{p}{1 + l_s^*(u_u - 1)} = \frac{1 - u_d}{u_u - u_d}\,, \tag{5.3}$$

which is the Cox–Ross–Rubinstein probability q. The new probability q is independent of p: a result which suggests that this case is somehow atypical.

Consider now the continuous limit case of Black and Scholes. The prices of shares are continuously monitored and the financial operator is allowed for a continuous time hedging. In an infinitesimal time interval dt the share changes by a factor

$$u = \exp\{\eta dt + \Delta dw\}\,, \tag{5.4}$$

where η and Δ are constants while dw is a random increment with vanishing average and variance $E^p[(dw)^2] = dt$. This infinitesimal time expansion is equivalent to say that

$$E^p[\log u] = \eta dt\,, \quad \mathrm{Var}^p[\log u] = \Delta^2 dt \tag{5.5}$$

while it is assumed that higher moments of $\log(u)$ have expected value of higher order in dt.

With the above assumptions one can easily solve the equation for l^*, in fact, up to order dt equation for l^* is

$$E^p[(u - 1)(1 - l^*(u - 1))] = 0 \tag{5.6}$$

which gives

$$l^* = \frac{\eta + \frac{\Delta^2}{2}}{\Delta^2}\,. \tag{5.7}$$

Using (5.7) it is easy to verify that

$$E^q[\log u] = -\frac{\Delta^2}{2}dt\,; \quad \mathrm{Var}^q[\log u] = \Delta^2 dt \tag{5.8}$$

and therefore for a finite time T the variable $U_T = S_T/S_0$ is distributed according to a lognormal martingale which follows the same prescription of Black and Scholes. Notice that the continuous case of Black and Scholes refers, as well the Cox–Ross–Rubinstein case, to a complete market. In both cases it is possible a perfect hedging, i.e. a complete replication of the option price process by a combination of shares and risk-less investments. This fact is not surprising since the continuous case can always be recovered from the dichotomic case, when the the time step becomes infinitesimal.

It is useful to stress that, even if the price with our principle is the same as the price with the *classical* approaches in complete markets, this does not mean that the strategy is the same. The speculator, in our case, does not worry about hedging continuously in order to obtain a risk-less portfolio but he tries to do the best with the information he has, i.e. he attempts to get the largest *almost sure* rate of exponential growth. Even in the complete market case, the speculator obtains an almost sure exponential growth rate λ generally larger than the risk-less interest rate r (obtained by the classical pricing prescriptions), explaining why one should invest in a portfolio which needs an active and frequent trade.

5.2. *Schweizer and Bouchaud–Sornette approach as quadratic approximation*

In the Schweizer [31] and Bouchaud–Sornette [5] approach the option price is obtained searching a hedging strategy which minimizes a quadratic risk function. This is a natural generalization of the Black and Scholes idea of looking for a zero risk strategy (perfect hedging) to the case in which this is not anymore possible Black–Scholes theory is recovered. The problem is that the above approach gives negative values of the option price for large S_T [17, 35] as shown in a different context by Dybvig and Ingersoll [15]. Nevertheless, negative prices appear in very unrealistic situations while, for more realistic cases, the formula gives sensible values.

We show in this section that the Schweizer and Bouchaud–Sornette price can be considered a quadratic approximation of the NASA one.

Let us first recall their result in a convenient form. It has been shown in [35] that their price can be expressed in the same way of Eq. (3.9) where the expectation is now constructed using the pseudo-martingale probability

$$q(u) = p(u)\left\{1 - \frac{\mu}{\sigma^2}[(u-1) - \mu]\right\}, \tag{5.9}$$

where $\mu \equiv E^p[u-1]$, $\sigma^2 \equiv \ \mathrm{Var}^p[u-1]$.

According to a rather general theory, one can show [12] that the martingale probability associated to an utility function $U(W)$ in a two times problem is

$$q(u) = p(u)\frac{\frac{d}{dW}U(W)|_{l=l^*}}{E^p\left[\frac{d}{dW}U(W)|_{l=l^*}\right]}, \tag{5.10}$$

where

$$W = (1 + l(u-1))W_0 \tag{5.11}$$

is the capital after the investment and W_0 before it; l^* is the optimal fraction associated to the utility U.

From Eq. (5.10) it is clear why negative prices appear: this happens only when the utility is a decreasing function of the capital.

Let us introduce a quadratic utility function of the capital W

$$U^{\mathrm{quad}}(W) = \alpha W - \frac{\beta}{2}(W - \bar{W})^2 \,, \tag{5.12}$$

where $\bar{W}$ is the expected capital and α and β two positive parameters.

Inserting this utility U^{quad} in Eq. (5.10) one gets

$$q(u) = p(u)\left\{1 - l^* W_0 \frac{\beta}{\alpha}[(u-1) - \mu]\right\} . \tag{5.13}$$

The optimal l^* can be derived simply by the imposition of the martingale property or equivalently from the equation

$$l^* W_0 \frac{\beta}{\alpha} = \frac{\mu}{\sigma^2} \,, \tag{5.14}$$

obtaining that the above pseudo-probability coincides with (5.9).

Notice that $U^{\mathrm{quad}}(W)$ is nothing but a Taylor expansion up to the second order of a generic utility function around $\bar{W}$ and then also of the logarithmic case of the speculator we have considered. Our price and the Schweizer and Bouchaud–Sornette one coincide for all practical purposes in *mild* periods but can lead to different results if strong fluctuations of the returns are present (i.e. near financial crisis).

6. Large Deviations

In Sec. 3 we have shown that the NASA price of a derivative is determined by a probability distribution $Q_T(S_T|S_0)$ which is related to the one-time step equivalent martingale probability q defined in (2.7).

The probability $Q_T(S_T|S_0)$ is constructed by compounding the one time step probability q. If u can take only a finite number of values N, then $Q_T(S_T|S_0)$ can be written as a multinomial formula, a simple generalization of the Cox–Ross–Rubinstein case with binomial coefficients.

Unfortunately, if the number of time steps is large, it is not possible to compute explicitly the multinomial average which gives the price because it involves $O(N^T)$ operations, and some other approach must be found. Therefore we need some analytical and numerical techniques for an estimation of the price. In this section we discuss the large deviation approach, next section is devoted to the Monte Carlo method.

The natural answer to the problem of a correct approximation of the price comes from large deviation theory. To state the problem we have to define the proper large deviation variable

$$z \equiv \frac{1}{T} \log S_T/S_0 \,, \tag{6.1}$$

whose probability distribution $\tilde{Q}_T(z)$ is trivially obtained from $Q_T(S_T|S_0)$ as

$$\tilde{Q}_T(z) = T \exp\{zT\} Q_T(\exp\{zT\}) . \tag{6.2}$$

In a nutshell, the mathematical essence of the large deviations theory [34] is the existence (under suitable hypothesis: essentially the finiteness of all moments of $\exp(Tz)$) of a convex function $G(z)$ (usually called Cramers function) such that

$$\tilde{Q}_T(z) = \Phi_T(z) \exp\{-G(z)T\} , \tag{6.3}$$

where $\Phi_T(z)$ is subexponential for large T.

It is often reported in literature that the distributions are heavy-tailed, and therefore it could be doubtful the existence of Cramer function because the hypothesis is not verified. However distributions obtained by historical sequences always have all finite moments, this is a consequence of the presence of a natural cut-off which can also be reflected at the level of models [23].

Because of the Oseledec theorem [29] one has that z, in the limit of very large T, is almost surely equal to its expected value λ, therefore $G(z)$ has to be strictly positive except for $z = \lambda$ where it vanishes.

In order to find out in practice the Cramer function it is sufficient to compute the scaling exponents $L(n)$ for the moments of S_T

$$L(n) \equiv \lim_{T\to\infty} \frac{1}{T} \log E^Q[(S_T/S_0)^n] , \tag{6.4}$$

where

$$E^Q[(S_T/S_0)^n] = \int \tilde{Q}_T(z) \exp\{zTn\}\, dz . \tag{6.5}$$

In fact, the Cramers function and the $L(n)$ are related via a Legendre transformation:

$$G(z) = \max_n [nz - L(n)] = zn^*(z) - L(n^*(z)) , \tag{6.6}$$

where $n^*(z)$ is the value where the maximum is realized.

The lognormal approximation can be recovered keeping only the first two terms of the Taylor expansion of $L(n)$ around zero. One obtains

$$G(z) = \frac{1}{2\Delta^2}(z - \lambda)^2 , \tag{6.7}$$

where

$$\lambda \equiv E^q[\log u] , \quad \Delta^2 \equiv \mathrm{Var}^q[\log u] \tag{6.8}$$

which is a good approximation only for relatively small fluctuation of S_T around its typical value $\exp \lambda T$. The lognormal approximation is from a mathematical point of view rather peculiar. This is basically due to the fact its moments grow too fast and therefore the Carleman criterion does not hold [28]. A simple way is to consider the correction to the parabolic shape of the lognormal approximation is to consider the Taylor expansion of order R of $L(n)$

$$L(n) \simeq \sum_{j=1}^{R} \lambda_j n^j \tag{6.9}$$

that implies

$$G(z) \simeq \sum_{j=2}^{R} g_j (z-\lambda)^j \tag{6.10}$$

where

$$\begin{aligned}
g_2 &= \frac{1}{2\lambda_2} \\
g_3 &= -\frac{\lambda_3}{3\lambda_2^3} \\
g_4 &= \frac{1}{\lambda_2^4}\left(\frac{\lambda_3^2}{2\lambda_2} - \frac{\lambda_4}{4}\right) \\
g_5 &= -\frac{1}{\lambda_2^5}\left(\frac{\lambda_3^3}{\lambda_2^2} - \frac{\lambda_4\lambda_3}{\lambda_2} + \frac{\lambda_5}{5}\right) \\
g_6 &= \frac{1}{\lambda_2^6}\left(\frac{7}{3}\frac{\lambda_3^4}{\lambda_2^3} - \frac{7}{2}\frac{\lambda_3^2\lambda_4}{\lambda_2^2} + \frac{1}{\lambda_2}\left(\frac{\lambda_4^2}{2} + \lambda_5\lambda_3\right) - \frac{\lambda_6}{6}\right). \\
&\vdots
\end{aligned} \tag{6.11}$$

The above expansion of $L(n)$ around $n = 0$ is quite reasonable because in this way one has a rather good approximation of the probability distribution around the maximum. On the other hand, at least in non trivial cases, one does not obtain good results for the option pricing, even if probability normalization factor are properly taken into account (see Fig. 1).

In order to understand this fact it is enlightening to compute the scaling exponents $L(n)$ using (6.3) and (6.4).

$$E^Q[(S_T/S_0)^n] \sim \exp\{T(nz^*(n) - G(z^*(n)))\}\,, \tag{6.12}$$

where the symbol $\sim$ indicates logarithmic equivalence and $z^*(n)$ is the maximum of $nz - G(z)$. The most important contribution to this integral comes from rare events of exponentially small probability such that $z = z^*(n) \neq \lambda$. Only when $n = 0$ the integral is dominated by the most probable events $z^*(0) = \lambda$.

Now we can easily see how the above considerations enters into the problem of pricing options.

Let us notice that the price C_0 of a European call option can be written as

$$C_0 = S_0 \int dz \tilde{Q}_T(z)\{\exp Tz - k\}^+ = S_0 I_1 - K I_2\,, \tag{6.13}$$

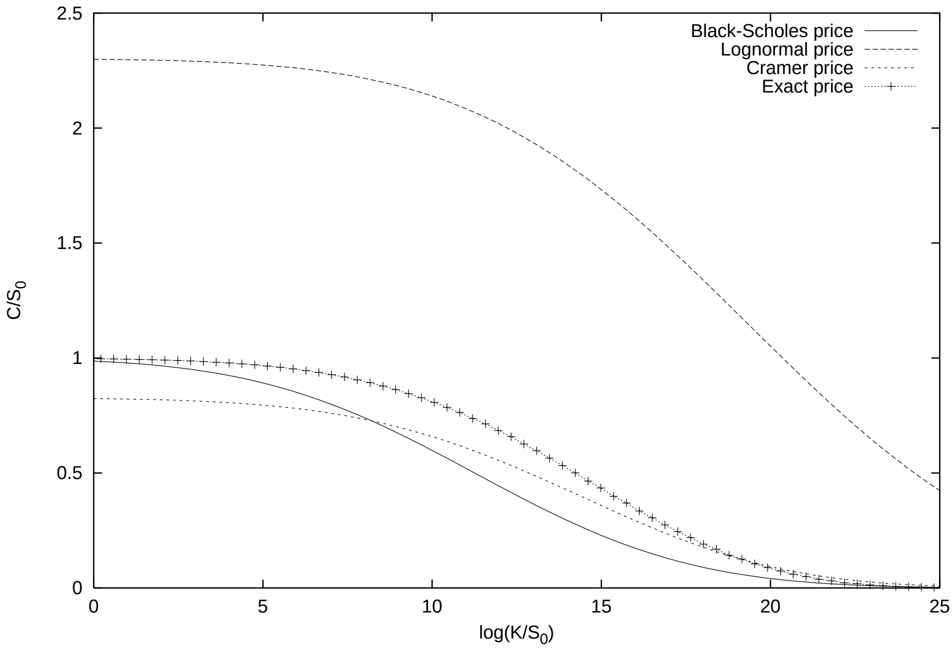

Fig. 1. Option prices C rescaled by the initial value of the stock S_0 as function of $\log(K/S_0)$ for $T = 30$. The returns of the price over one elementary time step take three discrete values 0.2, 1 and 2.5 with probability 0.15, 0.15 and 0.7, respectively. We compare the exact prices, and the Black–Scholes ones with the large deviation prescription with an expansion of $L(n)$ around 0 with $R = 2$ (lognormal approximation) and $R = 6$ (see text).

where

$$I_1 = \int_{\frac{1}{T}\log k}^{\infty} dz \tilde{Q}_T(z) \exp Tz \quad I_2 = \int_{\frac{1}{T}\log k}^{\infty} dz \tilde{Q}_T(z)\,. \tag{6.14}$$

Then the origin of the problems both for the *naive* Monte Carlo (see the next section) and large deviation becomes clear: the most important contribution to the integral I_1 is given by the exponentially rare events for which $z = z^*(1)$. It is then important to observe that because of the martingale property

$$\tilde{\Omega}_T(z) \equiv \tilde{Q}_T(z) \exp Tz \tag{6.15}$$

is a probability distribution. Since the most important contributions to I_1 come from the most probable events of the $\tilde{\Omega}_T(z)$ distribution, we can simply repeat the large deviations expansion of $L(n)$ around $n = 0$ for this new probability distribution while I_2 is computed in the previous way.

It easy to check that $L(n)$ computed with respect to $\tilde{\Omega}_T$ exactly equals $L(n+1)$ computed with respect to $\tilde{Q}_T$. Therefore the expansion with respect to $n = 0$ for the new probability distribution corresponds to an expansion around $n = 1$ for the old one.

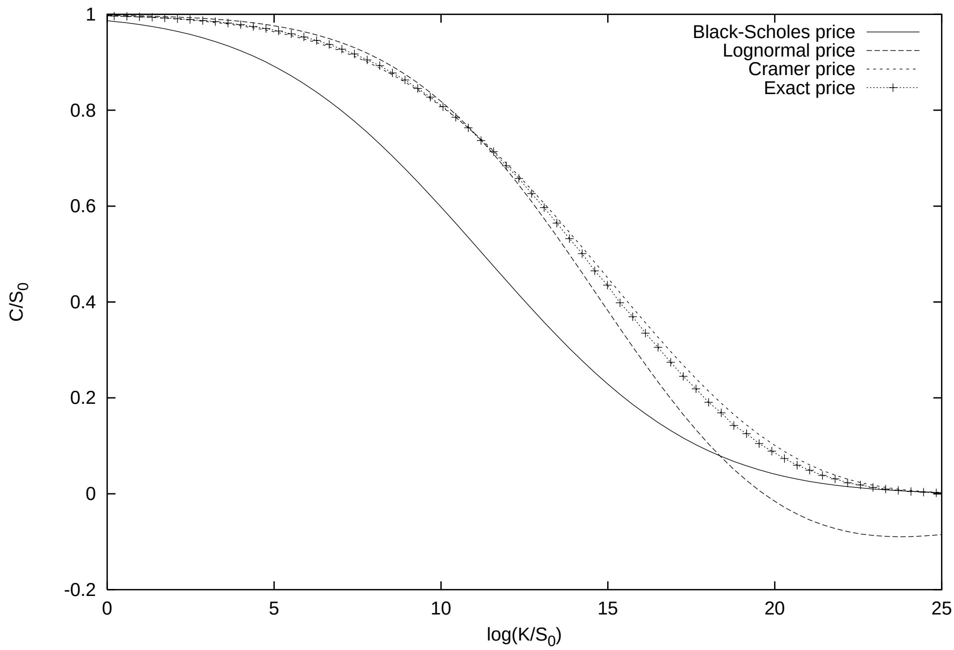

Fig. 2. Option prices C rescaled by the initial value of the stock S_0 vs. $\log(K/S_0)$ for $T = 30$. The returns of the price over one elementary time step are the same of Fig. 1. We compare the exact prices, and the Black–Scholes ones with the large deviation prescription described in the text with $R = 2, 6$.

All this integrals are not very sensitive to the choice of the sub-exponential function $\Phi_T(z)$ which can be fixed by means of a normalization constant or, if more computational precision is needed, as

$$\Phi_T(z) \simeq \sqrt{\frac{TG^{(2)}(z)}{2\pi}}\,, \tag{6.16}$$

where $G^{(2)}$ is the second derivative of the function G.

In Fig. 2 we show the results obtained in this way for the cases with $R = 2$ (lognormal approximation) and $R = 6$. We observe a quite good agreement with the exact prices.

As a final remark we want to stress that the lognormal approximation does not coincide with the Black and Scholes except in the continuous time limit. The differences can be appreciated in Figs. 1 and 2.

7. Monte Carlo

The idea to use the Monte Carlo method [27], to compute quantities which depend on Markovian processes, is very common in both the physicist and economic community. Instead of computing an average of a function $A(S_T)$ according the probability $Q_T(S_T|S_0)$:

$$E^Q[A(S_T)] = \int A(S_T)\, Q_T(S_T|S_0)\, dS_T \,, \tag{7.1}$$

one performs a large number M of trials (i.e. M different realizations of the random process S_T according to the probability Q_T) and then compute the average as

$$A_T^{(M)} = \frac{1}{M} \sum_{j=1}^{M} A(S_T^{(j)}) \,, \tag{7.2}$$

where $S_T^{(j)}$ is the jth realization and $E^Q[A(S_T)] = \lim_{M\to\infty} A_T^{(M)}$. We discuss the case in which the u can take only a finite number of values $u^{(i)}$, $i = 1, 2, \ldots, N$ with probability p_i. The jth realization is obtained in the following way. From S_0 one constructs $S_1^{(j)}$ as $S_1^{(j)} = S_0 u_1$, where u_1 is drawn with probability (2.7), one extracts in the same way u_2 and so on; with this procedure one obtains $S_2^{(j)}$, $S_3^{(j)}, \ldots$ and finally $S_T^{(j)}$. In this way the number of operations involved is $O(MT)$, while the error estimation using standard variance considerations is

$$\Delta A_T^{(M)} = \sqrt{\frac{1}{M-1} \sum_{j=1}^{M} (A(S_T^{(j)}) - A_T^{(M)})^2} \,. \tag{7.3}$$

For example in Fig. 3 we show the price of a European call option as a function of a trinomial probability distribution. The prices obtained with Monte Carlo simulations with 1 million trials are compared with the Black–Scholes ones and the exact ones given by Eq. (3.12).

The large errors involved in this approximations are mainly due to the fact that the most important contribution in (3.12) comes from the tails of the distribution $Q_T(S_T|S_0)$. We need then a good control of the tails; this is provided by the large deviations theory or a suitable Monte Carlo.

Taking into account the results of the previous section we have now all the ingredients to build a reasonable Monte Carlo algorithm. As previously shown in (6.13) the price is a linear combination of two integrals, I_1 and I_2. The most important contribution to I_1 come from the most probable events of $\tilde{\Omega}_T$. Let us write

$$I_1 = \int_{\frac{1}{T}\log k}^{\infty} dz \tilde{\Omega}_T(z) = \sum_{S_T > K} \Omega_T(S_T/S_0) \,, \tag{7.4}$$

where Ω, in the case u assumes only a finite number of values N, is the convolution of a number T the one-time probability defined by

$$\omega(u^{(i)}) = u^{(i)} q(u^{(i)}) \tag{7.5}$$

that is properly normalized because of the martingale property. The algorithm follows then the same steps of the simple Monte Carlo, using to compute the integral I_1 the ω-probability instead of the q-probability. To improve the results we have also used the control variate technique [6]. Even in the extreme situation of a very

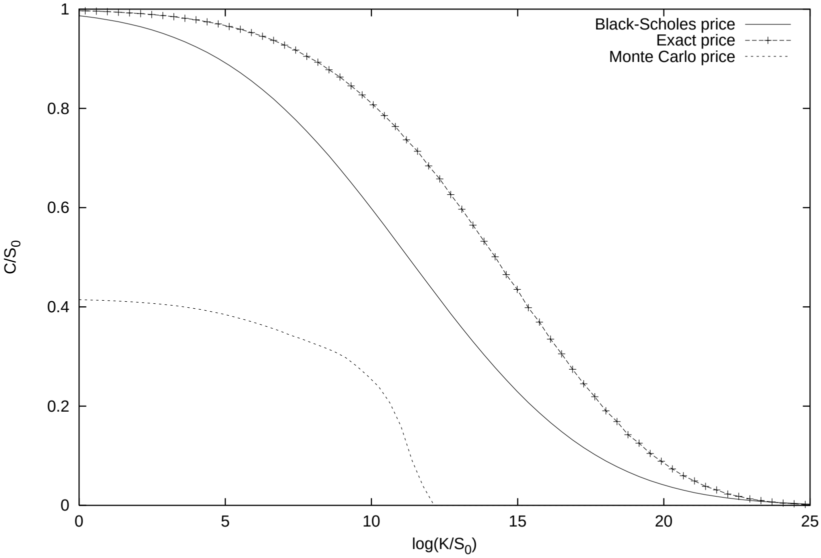

Fig. 3. Option prices C rescaled by the initial value of the stock S_0 as function of $\log(K/S_0)$ where K is the strike price, T is chosen equal to 30. The returns of the price over one elementary time step are the same of Fig. 1. We compare the exact prices, the ones obtained with the *naive* Monte Carlo algorithm with 1 million trials and the Black–Scholes prescription. The values have been intentionally chosen unrealistic in order to stress the differences between the different approaches.

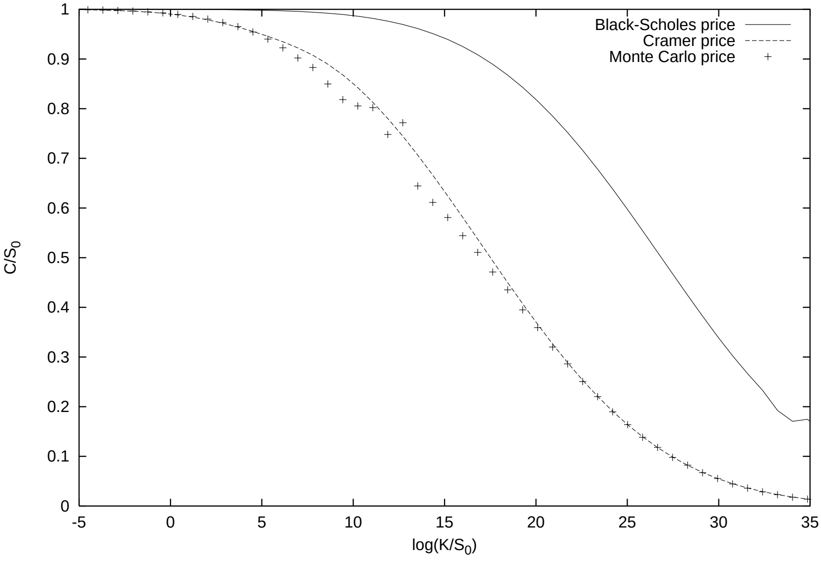

Fig. 4. Option prices C rescaled by the initial value of the stock S_0 vs. $\log(K/S_0)$ for $T = 30$. The returns can take the values $u^{(n)} = u^{(0)}\alpha^n$, with $u^{(0)}$ equal to 0.4, α equal to 1.2 and the index n ranging from zero to $N = 30$. The probability $p(u^{(n)})$ is $C_N(u^{(n)})^{-\beta}$, with $\beta = 0.5$ and C_N a normalization constant. We compare the large deviations ($R = 6$) and Monte Carlo approximations and the Black–Scholes prices. The Monte Carlo simulation has been performed with 10^7 trials.

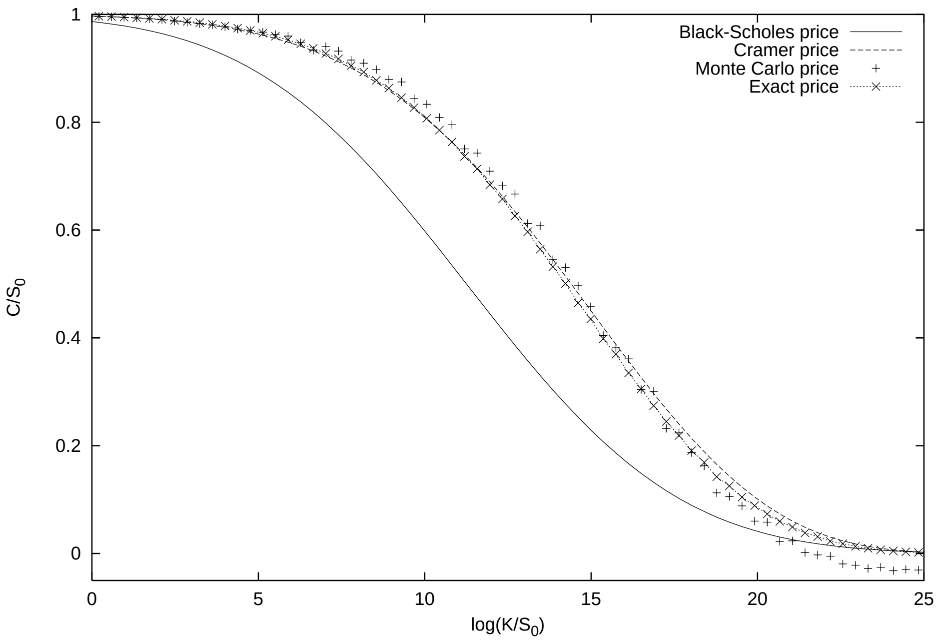

Fig. 5. We compare the same quantities of Fig. 4 with the exact prices in the case of the trinomial returns of Fig. 1. The Monte Carlo simulation has been performed with 10^5 trials.

large $\mathrm{Var}[u]$ such as in the case of a truncated Levy distribution of Fig. 4 we obtain good results both for the Monte Carlo and the large deviations approximation. In Fig. 5 we show the results obtained the trinomial distribution of Fig. 1.

8. Final Remarks

In this paper we have discussed a general criterion to price derivatives. This principle (No Almost Sure Arbitrage) basically says that it is not possible to build a portfolio including derivatives which grows almost surely with an exponential rate larger than one with only the underlying securities.

This approach holds for the general case of incomplete markets and for a generic probability distribution of the return. Let us stress that it is possible to apply the procedure also for returns correlated in time, e.g. a Markov process.

In the cases of dichotomic distribution for the return and continuous time limit, our method gives the same results of Cox–Ross–Rubinstein and Black–Scholes respectively. If $p(u)$ is not too spread, the results obtained with our approach are rather close to those ones of the Schweizer and Bouchaud–Sornette theory. This is not true in presence of strong fluctuations for the return. In this case an accurate computation of the option price can be obtained in the framework of the large deviations theory (which systematically takes into account the deviation from the lognormality) or via a properly realized Monte Carlo simulation.

Appendix A

In this appendix we consider a Markovian model for the price movement of the stock (i.e. Eq. (2.1)) in order to extend our price prescription to this case. Now the u_t are random variables with a Markovian probabilistic rule. For sake of simplicity we discuss the case of Markov chain, i.e. u_t can take only N different values $u^{(1)}, u^{(2)}, \dots, u^{(N)}$.

Let us remind that a Markov chain is completely characterized by its transition matrix

$$P_{i\to j} = \mathrm{Prob}(u_{t+1} = u^{(j)} | u_t = u^{(i)})\,. \tag{A.1}$$

In a realistic market it is natural to assume that the Markov chain is ergodic and therefore there exist an invariant probability:

$$p_j = \sum_{i=1}^{N} p_i P_{i\to j}\,. \tag{A.2}$$

For ergodic chain the convergence to the invariant probability is exponentially fast

$$(P^n)_{i\to j} = \mathrm{Prob}(u_n = u^{(j)} | u_0 = u^{(i)}) = p_j + O(e^{-\alpha n})\,, \tag{A.3}$$

where α is given by the second eigenvalue of the matrix $P_{i\to j}$.

The optimization strategy for the shares problem is rather obvious. If at time $t-1$, $u_{t-1} = u^{(i)}$ then the probability to have $u_t = u^{(j)}$ is nothing but $P_{i\to j}$. Therefore the optimal fraction of capital in stock l_i, is obtained from the maximization of

$$\sum_{j=1}^{N} P_{i\to j} \log[1 + l_i(u^{(j)} - 1)]\,. \tag{A.4}$$

Of course l_i can change at any i. In Kelly's paper the relation between the Shannon entropy of the Markov chain and the optimal exponential growth rate of the capital is discussed.

Following the same step of Kelly's strategy described in Sec. 2 it is straightforward to construct from $P_{i\to j}$ an equivalent martingale with transition probability $Q_{i\to j}$:

$$Q_{i\to j} = \frac{P_{i\to j}}{1 + l_i^*(u^{(j)} - 1)}\,, \tag{A.5}$$

where l_i^* is given by the maximization of the quantity (A.4). It is easy to realize that

$$E^Q(u|u^{(j)}) = 1\,. \tag{A.6}$$

If the Markov chain with transition matrix $P_{i\to j}$ is ergodic then also the Markov chain with transaction matrix $Q_{i\to j}$ has to be ergodic and one can introduce an invariant probability

$$q_j = \sum_{i=1}^{N} q_i Q_{i\to j}\,. \tag{A.7}$$

Also for the new Markov chain the convergence to the invariant probability is exponentially fast

$$(Q^n)_{i\to j} = q_j + O(e^{-\beta n})\,, \tag{A.8}$$

nevertheless, in general, $\beta \neq \alpha$.

Now we have all the ingredients for the construction of $Q_T(S_t|S_0)$ since we can assign the probability to each trajectory. Then, by repetition of the same steps of Sec. 3, one can construct the price of the derivatives. The only difference is that now the expectation is taken with respect to the Markovian martingale.

Let us notice that for the dichotomic case $N = 2$, the present construction gives the same probabilistic rule of Cox–Ross–Rubinstein. In this case the $Q_{i\to j}$ do not depend on $P_{i\to j}$ but only on the values $u^{(1)}$ and $u^{(2)}$. The Monte Carlo computation is rather simple. In each trial, at time t, if $u_{t-1} = u^{(i)}$, one chooses with probability $Q_{i\to j}$ the jth branch of the tree.

Also the lognormal approximation for the Markovian case is straightforward but interesting. Equation (6.7) holds, now one has

$$\lambda = \lim_{T\to\infty} \frac{1}{T} E^Q \left[\sum_{t=1}^{T} \log u_t \right] = E^Q[\log u] \tag{A.9}$$

and

$$\Delta^2 = \lim_{T\to\infty} \frac{1}{T} E^Q \left[\left(\sum_{t=1}^{T} (\log u_t - \lambda) \right)^2 \right]. \tag{A.10}$$

After having defined

$$R_n \equiv \sum_{i,j} q_i (\log u^{(i)} - \lambda)(\log u^{(j)} - \lambda)(Q^n)_{i\to j}\,, \tag{A.11}$$

it is a matter of computation to check, using the exponential convergence of R_n due to the Markovian nature of the process, that

$$\Delta^2 = E^Q[(\log u - \lambda)^2] + 2 \sum_{n=1}^{\infty} R_n\,. \tag{A.12}$$

Let us note the fact that, at variance with λ which depends only on the invariant probability, for Δ^2 are also relevant the time correlations. Equation (A.12) shows that the "effective" volatility takes into account the time correlation.

Appendix B

It is useful to have some cases in which the probability q is easy to handle analytically. In this appendix we consider three particular distributions of the return of the asset, in the case of small excess rate of return $\mu \equiv E^p[u - 1]$.

Compact Support Distribution

We first consider a probability distribution $p(u)$ in a compact support $[0, u_{\max}]$. Equation (2.5) can be rewritten as

$$\sum_{n=0}^{\infty} E^p[(-1)^n (l^*)^n (u-1)^{n+1}] = 0\,. \tag{B.1}$$

We can solve for l^* as a power series in the excess rate of return μ

$$l^* = \frac{1}{\sigma^2}\mu + \frac{\chi}{\sigma^6}\mu^2 + O(\mu^3)\,, \tag{B.2}$$

where σ^2 and χ are the second and the third cumulant of the rescaled return $u-1$ respectively. From Eqs. (2.7) and (B.2) we get the probability

$$q(u) = p(u)\left(1 - \frac{\mu}{\sigma^2}(u-1) + \mu^2\left(-\frac{\chi}{\sigma^6}(u-1) + \frac{1}{\sigma^4}(u-1)^2\right) + O(\mu^3)\right)\,, \tag{B.3}$$

hence up to order μ the result of the NASA principle is identical to the minimal risk prescription of Schweizer and Bouchaud–Sornette.

Expansion (B.1) is convergent if the optimal value l^* is small enough. The individual terms in the expansion however exist under the much weaker conditions that the respective moments are finite, but the support of $p(u)$ can be unbounded.

Levy Distribution

There are some known experimental evidences that the one-time return u is well described by a proper Levy distribution [22, 23], but it is still an open problem if the variance is finite or not. In this case expansion (B.1) is meaningless. We assume here that the returns are described by a probability distribution with a power law decay at large arguments

$$p(u) \sim \beta u^{-1-\nu} \text{ for } u \gg 1\,. \tag{B.4}$$

The exponent of the power law ν is in the interval $(1, 2]$ and β is a constant. Hence the expected value of u exist and is finite, but all the higher moments are infinite.

Let us notice that the maximizing l^* cannot be less than zero, because the support of p is unbounded. Hence, $l = 0$ is at the boundary of the domain for which $\lambda(l)$ is the well-defined function of l.

Let us now assume that l^* is small, but not necessarily of the same order as μ, in this case the probability distribution $q(u)$ is well approximated by

$$q(u) \sim \begin{cases} p(u) & \text{if} \quad u \ll \dfrac{1}{l^*} \\ \dfrac{1}{l^* u} p(u) & \text{if} \quad u \gg \dfrac{1}{l^*} \end{cases}\,, \tag{B.5}$$

and then its second moment exists. Using the approximation (B.5) for the probability q and the definition of μ, after some algebra one obtains

$$l^* \approx \left(\frac{\mu\nu(\nu-1)}{\beta}\right)^{\frac{1}{\nu-1}} . \tag{B.6}$$

In the limit when ν tends to 2, l^* scales linearly with μ, in agreement with (B.2). When ν is in the interval between 1 and 2, the optimal fraction however goes to zero faster than μ.

Log-Levy Distribution

We construct here the probability q from a p of the form

$$p(u) \sim \begin{cases} Cu^{-1}(\log u)^{-1-\nu^{(+)}} & \text{for} \quad u \gg 1\,, \\ Cu^{-1}|\log u|^{-1-\nu^{(-)}} & \text{for} \quad u \ll 1\,. \end{cases} \tag{B.7}$$

We will refer to these laws as "log-Levy" laws, since they are distributed as Levy laws with exponents $\nu^{(+)}$ and $\nu^{(-)}$ in the logarithmic variable. In this case the expected value of the logarithm of the return is finite but the variance is infinite. As the Levy laws are stable under addition, so the log-Levy laws are stable under multiplication, if the exponents of the power laws, $\nu^{(+)}$ and $\nu^{(-)}$, are in the interval [1, 2]. The limit case of an exponent equal to 2 corresponds to the lognormal distribution.

However, this does not mean that the log-Levy laws are necessarily as natural as the Levy laws. The divergence at large u in (B.7) implies that the expectation value with respect to p of the random variable u does not exist. Therefore the excess rate of return μ is undefined.

If the probability distribution p is nicely behaved for small u we have that the growth rate of the wealth (2.3) exists and is finite in a region around $l = 0$. We can therefore find the maximizing l^*, attained at some positive l^*. Using the variational equation, we obtain the probability (2.7) that can be approximated by (B.5). Just as in the Levy distribution, we have that $E^q[u^2]$ is finite, and it follows the much weaker condition that $E^q[(\log u)^2]$ is convergent at large u.

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