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Growth optimal investment and pricing of derivatives

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Abstract

We introduce a criterion how to price derivatives in incomplete markets, based on the theory of growth optimal strategy in repeated multiplicative games. We present reasons why these growth-optimal strategies should be particularly relevant to the problem of pricing derivatives. Under the assumptions of no trading costs, and no restrictions on lending, we find an appropriate equivalent martingale measure that prices the underlying and the derivative security. We compare our result with other alternative pricing procedures in the literature, and discuss the limits of validity of the lognormal approximation. We also generalize the pricing method to a market with correlated stocks. The expected estimation error of the optimal investment fraction is derived in a closed form, and its validity is checked with a small-scale empirical test. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

The option pricing formula of Black–Scholes is arguably one of the most successful modern applications of quantitative mathematical analysis. Despite its great simplicity it gives prices of the most actively traded options on leading markets which are typically not off by more than a few percent. This has paved the way for an influx of advanced ideas from the theory of stochastic processes into finance. In the most general formulation of the no-arbitrage argument of Black–Scholes [1] and Cox et al. [2] it

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was established by Harrison–Kreps [3] that a price system (under certain restrictions such as no trading costs) admits no arbitrage opportunities if and only if there exists an equivalent probability measure, with respect to which all discounted price processes of underlying and derivative securities are martingales.

This theory is presented in several excellent monographs, see Refs. [4–8]. Nevertheless, these developments do not yield a fully general solution to the derivative pricing problem, since the equivalent martingale measure is not uniquely defined, except if the market is complete. If the market is incomplete there are many possible equivalent martingale measures, and each of these specifies a price system without arbitrage opportunities.

In the lognormal model of Black–Scholes the market of one share and one bond is already complete. One can thus test its predictions by computing the volatility of the price process of the underlying security, plugging that into the Black–Scholes formula, and then comparing with observed market prices of a derivative: such tests were initiated already by Black–Scholes [9] and Black [10]. The volatility from historical data is obviously subject to measurement errors and statistical fluctuations. A more robust test is therefore if observed option prices can be fitted by a Black–Scholes formula with volatility treated as a free parameter. There are several recent reports that this is not so, see Rubinstein [11], and earlier investigations [12–14].

In addition to the lognormal model there exists a wider class of Itô processes in continuous time for which the market is also complete, and the derivative pricing problem can be solved only by no-arbitrage conditions. To distinguish which model in this class is the best approximation to observed data, and then to use it, is however already not a trivial task, see e.g. Refs. [11,15–18].

For the rest of this paper we will consider the general case of an incomplete market, but we do keep the standard assumption of no market friction. We introduce a criterion to price derivatives in this situation. Stated differently, we advance an argument as to which out of many possible equivalent martingale measures is the appropriate one. We refer to this criterion as the principle of no almost sure arbitrage [19]: it is based on the theory of optimal gambling of Kelly [20]. We will consider an investor that gambles on investing in an underlying and a derivative security, and who looks for the growth-optimal investment strategies among such combinations.

The paper is organized as follows. In Section 2 we summarize and discuss the Kelly theory in a version appropriate for our purposes. We discuss also here the papers of Samuelson [21] and Merton–Samuelson [22]. In Section 3, the technical core of the paper, we apply the Kelly theory to derivative pricing, and in Section 4 we discuss the relations and differences to other proposed alternative derivative pricing procedures.

It is straightforward to see that in complete markets our procedure agrees with the standard ones, e.g. with the models of Cox–Ross–Rubinstein and Black–Scholes. In Section 5 we discuss the lognormal limit of our procedure, the limits of its applicability, and that it is not exactly equivalent to the Black–Scholes formula. In Section 6 we study the expected scatter of the optimal investment fraction and report a small-scale empirical test of pricing options on the Swedish OMX index. In Section 7 we generalize the method to a market with correlated stocks. In Section 8 we summarize and discuss our results.

An earlier version of this paper was circulated in 1998 [23]. A longer sequel was also made available in the electronic archive [19], lacking however in the version accepted for publication, material corresponding to Sections 6 and 7 below. The presentation in the present paper is significally different from Ref. [19] and contains a fuller discussion of the implementation of the model and the economic background.

2. The Kelly problem of optimal gambling

The theory to be exposed in this section is due to Kelly [20] who was looking for an interpretation of information theory of Shannon [24] outside of the context of communication, and to the treatment of the St. Petersburg Paradox by Bernoulli [25]. For a recent review of growth-optimal investment strategies, see Hakanson–Ziemba [26].

Consider a price movement of stock or some other security which is described by

$$S_{i+1} = u_i S_i \, , \tag{1}$$

where time is discrete, S_i is the price at time i and the u_i's are independent, identically distributed random variables. The assumptions on the u_i's can be relaxed.

Consider now an investor which starts at time 0 with a wealth W_0, and who decides to gamble on this stock repeatedly. Suppose that the investor chooses to commit at each time a fraction l_s of his capital in stock, and the rest in a risk-less security. We set the risk-less rate to zero. A non-zero risk-less rate corresponds to a discount factor in the definition of the share prices, and can be accounted for by a redefinition of the u_i's. The investor will hence hold at time i a number $l_s W_i / S_i$ of shares, and his wealth at successive instants of time follows a multiplicative random process

$$W_{i+1} = (1 - l_s)W_i + l_s u_i W_i = (1 + l_s(u_i - 1))W_i \, . \tag{2}$$

In the large time limit we have by the law of large numbers that the exponential growth rate of the wealth tends with probability one to a constant. That is,

$$\lambda(l_s) = \lim_{T\to\infty} \frac{1}{T} \log \frac{W_T}{W_0} = E^p[\log(1 + l_s(u - 1))] \, . \tag{3}$$

The optimal gambling strategy in the sense of Kelly consists in maximizing $\lambda(l_s)$ in (3) by varying l_s. The solution must be unique because the logarithm is a concave function of its argument. Which values of l_s are reasonable in our problem? First, the optimum l_s must be such that $1 + l_s(u - 1)$ is positive on the support of u. Second, we must decide if the investor is allowed to borrow money. In the original formulation of Kelly he is not, but here it is useful to assume that the investor may do so at a risk-less rate. Furthermore we also assume that the investor can go short i.e., it is possible to borrow stocks. Hence we allow l_s to take any finite positive or negative value, and look for the maximum of $\lambda(l_s)$.

The desired strategy is hence the only finite l_s which solves

$$\left.\frac{d\lambda(l_s)}{dl_s}\right|_{l_s=l_s^*} = E^p\left[\frac{u-1}{1+l_s^*(u-1)}\right] = 0 \tag{4}$$

and the maximum realized growth rate is

$$\lambda^* = E^p[\log(1+l_s^*(u-1))]\,. \tag{5}$$

Let us look more closely at (4) if u can take the values v_n with probabilities p_n. Let us write

$$q_n = \frac{p_n}{1+l_s^*(v_n-1)} \tag{6}$$

such that (4) reads

$$\sum_n q_n(v_n-1) = 0\,. \tag{7}$$

From $1+l_s^*(v_n-1)$ being positive it follows that the q_n's are all positive, and from (7) it follows that the sum of the q_n's is one. The q_n's thus form a new set of probabilities, and, again by (7), with respect to these the stock price process is a martingale.

It is worthwhile to discuss this point at some length in light of the well-known papers of Samuelson [21] and Merton–Samuelson [22]. These authors demonstrated that if the risk/return preferences of an agent is described by an expected utility, then the growth-optimal strategy does not maximize that utility, except if the utility is logarithmic, and the price process is similar to (1).

It is a mathematical fact that the growth rate (3) holds with probability one. If a gambler would choose any other strategy in l_s but that which maximizes λ then he would in the large time limit almost surely end up with an exponentially smaller capital.

In the counterexamples of Samuelson–Merton the dominant contribution to non-logarithmic utility comes from events with asymptotically (in time) vanishing probability. When sets of measure zero are involved, the relation between sample and ensemble averages becomes delicate. Suppose some agents want to maximize a non-logarithmic utility, and that they choose strategies accordingly. If they do so, and we compare with them using the growth-optimal strategy, they would almost surely end up with less utility according to their own criterion. Only if there would be an ensemble of infinitely many agents, all using this same strategy, then some few would actually end up with higher utility. And, in fact, so much higher that the averaged utility over the ensemble would be increased. This is the sense in which other strategies can be 'better' than the growth-optimal strategies. Therefore, Merton–Samuelson give preference to ensemble averages over sample average. This is a problematic point, which has apparently not received much attention in the economic literature.

A simple counter-argument against the growth-optimal criterion is that time in economical problems, in the sense considered here, is perhaps often not very long. However, this is essentially a quantitive question, which has to be decided case-by-case.

For recent discussions of characteristic times in the Kelly problem with transaction costs, see Refs. [27–29].

3. Fixing the price of the derivative

Let us now consider the stock of Section 2 and one derivative security on the same stock. The argument generalizes at every step in an obvious way to an arbitrary number of derivatives, but to keep the notation simple we will write out the formulae for one derivative. To begin, let us write the present share price at the time i S_i, and let us assume that the realized value of the derivate at the next future instant of time is $C_{i+1}(S_{i+1}) = C_{i+1}(S_i u_i)$. The unknown parameter is the present price of derivative, which we will write C_i. We will further suppose that this situation is repeated over and over again. That is, we assume that the investor finds himself many times in the situation that if he keeps some money W_i in stock it will be worth $u_i W_i$, while if he keeps the same money in the derivative it will be worth $C_{i+1}(S_i u_i) W_i / C_i(S_i)$, with the same random variable u_i entering into both expressions. The simplest example is one time step before expiration time of a call option with a present share price S_{T-1} and a strike price K; in this case one has $C_T(S_T) = \max(S_T - K, 0)$.

If at the ith stage in this gamble the investor commits a fraction l_s of his wealth in shares and a fraction l_d in the derivative, this means $l_s W_i / S_i$ shares and $l_d W_i / C_i$ derivatives. After the random variable u_i has taken a value, the wealth is changed to

$$W_{i+1} = \left(1 + l_s(u_i - 1) + l_d \left(\frac{C_{i+1}}{C_i} - 1\right)\right) W_i \,. \tag{8}$$

If the investor plays this game many times, his wealth will almost surely grow at an exponential rate (over these games) which is

$$\lambda(l_s, l_d; C_i) = E^p \left[\log\left(1 + l_s(u - 1) + l_d \left(\frac{C_{i+1}}{C_i} - 1\right)\right)\right] \,. \tag{9}$$

Let us now assume that we compute (9) and that we find an optimal growth rate

$$\lambda^*(C_i) = \operatorname*{Max}_{l_s,\, l_d} E^p \left[\log\left(1 + l_s(u - 1) + l_d \left(\frac{C_{i+1}}{C_i} - 1\right)\right)\right] \tag{10}$$

and that this is realized at fractions l_s^* and l_d^*. If l_d^* is larger than zero, then all operators would like to buy the derivative. Hence its present price C_i would tend to rise, and its rate of return would fall. The fraction l_d^* would thus tend to fall. If, on the other hand, l_d^* would be less than zero, then all operators would want to go short on the derivative, its present price would fall, its rate of return would rise and l_d^* would tend to rise. The only value of C_i in which the market can be in equilibrium is therefore the one such that l_d^* is zero. This statement was referred to as the principle of no almost sure arbitrage in Ref. [19]. We remark that from the technical point of view, this procedure, where l_d^* is taken to vanish, is closely similar to the Samuelson's [30] methodology of warrant pricing in the 'incipient case', see Ref. [6, Chapter 7].

C_i and the fractions l_s and l_d thus satisfy two equations. One is

$$0 = \left.\frac{d\lambda(l_s, l_d; C_i)}{dl_s}\right|_{l_s=l_s^*,\, l_d=0,\, C_i=C_i^*} = E^p\left[\frac{u-1}{1+l_s^*(u-1)}\right], \tag{11}$$

which is identical to (4) of the Kelly model, and therefore determines l_s^*. The other is

$$0 = \left.\frac{d\lambda(l_s, l_d; C_i)}{dl_d}\right|_{l_s=l_s^*,\, l_d=0,\, C_i=C_i^*} = E^p\left[\frac{((C_{i+1})/C_i^* - 1)}{1+l_s^*(u-1)}\right]. \tag{12}$$

If we now compare with (6) and (7) we see that (12) can be written as

$$C_i^* = \sum_n q_n C_{i+1}(S_i v_n), \tag{13}$$

which states that the derivative price process is a martingale with respect to q.

We have constructed the equivalent martingale measure q over one elementary time step. By compounding we can construct, from T convolutions of q distribution, the price distributions over many time steps $Q_T(S_T|S_0)$. The price of, say, a European call option with expiration time T at time t, can thus be written in the standard manner as

$$C(i, S_i) = E^{Q_{T-i}}[C(T, S_T)] \quad C(T, S_T) = \max(S_T - K, 0), \tag{14}$$

where the expectation value is taken with respect to the many time steps distribution $Q_{T-i}(S_T|S_i)$ over share prices S_T at time T, conditioned by the share price having been S_i at time i. We have taken the risk-less rate to be zero. A non-zero risk-less rate can be put back in as a discount factor in the share prices, which leads to a different definition of the u's, the returns over elementary periods of time, and hence to different q's.

4. Comparison with other derivative pricing prescriptions

The pricing procedure proposed in Section 3 is based on a particular choice of an equivalent martingale measure. It therefore follows that this procedure is arbitrage-free. It also follows that it must agree with no-arbitrage pricing in complete markets.

It may nevertheless be instructive to carry through the calculations of Section 3 explicitly for the Cox–Ross–Rubinstein model. We hence assume that the discounted share price can go up by a fraction u/r with probability p, down by d/r with probability $(1-p)$, and that $u > r > d$, where r is the risk-free interest rate. Eq. (11) is then solved for l_s^* as

$$l_s^* = \frac{rp}{r-d} - \frac{r(1-p)}{u-r}, \qquad q = \frac{p}{1+l_s^*((u/r)-1)} = \frac{r-d}{u-d}, \tag{15}$$

and this reproduces the Cox–Ross–Rubinstein measure q, independent of p. The maximum growth rate however still depends on p, and is

$$\lambda^* = -p\log\frac{p}{q} - (1-p)\log\frac{1-p}{1-q}. \tag{16}$$

The Black–Scholes model can now be derived as the continuous-time limit of the Cox–Ross–Rubinstein model.

The growth rate (16) is the Kullback contrast of the measure q with respect to the measure p in the dichotomic case. It is easy to see that this is the general form of the solution, and that the measure q can be obtained as follows: Consider a security with price process determined by a measure p, and consider all measures q with respect to which the price process of the security is a martingale. Then the particular measure q which we compute from (11) (see also Eqs. (6) and (7)) is the one that minimizes the Kullback contrast

$$d_p(q) = E^p\left[-\log\frac{p}{q}\right] = -\sum_n p_n \log\frac{p_n}{q_n}\,. \tag{17}$$

Our proposal therefore coincides, over one time step, with one of the two proposals recently put forward by Stutzer [31]. Stutzer's formulae differ from ours in that we posit the minimization of (17) under the martingale constraint for the elementary process over each time step, while Stutzer applies (17) directly on the probability distribution over a finite time interval. A similar approach minimizing the relative entropy distance to calibrate a pricing model has been suggested by Avellaneda et al. [32].

It is also interesting to remark that if we perform the quadratic approximation of the logarithm in (3) around the expected return on capital, $E^p[W_{i+1}/W_i]$ and then look for the optimal portfolio, we obtain

$$q(u) = \left(1 - \frac{\mu}{\mathrm{Var}^p[u]}(u-1-\mu)\right)p(u), \quad \mu = E^p[u-1]\,. \tag{18}$$

Eq. (18) reproduces the least-squares option pricing procedure of Follmer–Sondermann [33]. Schäl [34] and Schweizer [35]. Another approach to option pricing based on global least-squares minimization was recently proposed by Bouchaud–Sornette [36], see also Ref. [37], but has now been proved by Wolczynska [38] and Hammarlid [39] to give the same least-squares solution, see also Schweizer [40]. The empirical tests of Refs. [36,37], which pertain mainly to the case of small μ, hence also validate our theory.

A conceptual problem of least-squares option pricing, first shown by Dybvig–Ingersoll [41], is that it can assign negative prices of some state-contingent claims. This may lead to negative prices of derivatives with strictly non-negative pay-off, and hence to arbitrage opportunities. In (18) this happens when u is sufficiently large, provided μ is positive. From our point of view this result is clear. Negative prices appear only at capital values for which the quadratic approximation is a decreasing function. It is precisely because quadratic utility is not monotonically increasing that its maximization only gives a pseudo-martingale measure, which can take both positive and negative values. Our theory is practically equivalent to the least-squares method when that one has no problems, i.e., when we look at small deviations from the most probable realizations of the capital returns and when μ is small. In the opposite limits our theory deviates from least-squares minimization, always prices derivatives by a proper equivalent martingale measure, and the problem of possible negative prices does not appear.

5. On the lognormal limit

The distribution $Q_T(S_T|S_0)$ is constructed by compounding the measure q of one time step. If u only takes a finite number of values, then $Q_T(S_T|S_0)$ can be written as a multinomial expansion. However, if the number of time steps is large, then the multinominal expansion is unwiedly, and some other approach must be found for analytic and numerical work.

The asymptotic price distribution is of course determined by the behavior of aggregated logarithmic returns, and this is intimately linked to the lognormal distribution, its applicability and its limits. Let us assume that $E^q[(\log u)^2]$ is finite. Then, for the most probable realizations, $Q_T(S_T|S_0)$ is well approximated by a lognormal distribution, i.e.,

$$Q_T(S_T|S_0) \simeq Q_T^{(\mathrm{LN})}(S_T|S_0) = \frac{1}{S_T\sqrt{2\pi\Delta^2 T}} \exp - \frac{(\log(S_T|S_0) - \lambda T)^2}{2\Delta^2 T}\,, \tag{19}$$

where

$$\lambda = E^q[\log u], \quad \Delta^2 = E^q[(\log u)^2] - (E^q[\log u])^2\,. \tag{20}$$

For large T only low-probability events are not distributed according to (19). For the treatment of such large, but very rare, events the proper mathematical setting is the theory of large deviations, see Ref. [42]. In Ref. [19] we discuss the use of large deviations theory to price derivatives related to rare events.

We note that $Q_T(S_T|S_0)$ is by construction a martingale measure, while, for the lognormal approximation,

$$E^{Q^{(\mathrm{LN})}}[S_T] = \int Q_T^{(\mathrm{LN})}(S_T|S_0) S_T\, \mathrm{d}S_T = S_0 \exp(\lambda + \tfrac{1}{2}\Delta^2)T\,. \tag{21}$$

In general $\lambda + \frac{1}{2}\Delta^2 \neq 0$. That is, in the lognormal approximation one looses the martingale property.

Our lognormal approximation does not agree with a straightforward application of the Black–Scholes formula. We recall that the Black–Scholes theory also involves lognormal distribution. In the parameterization of (19) it is characterized by drift coefficient λ_{BS} and volatility Δ^2_{BS}, related through

$$\lambda_{\mathrm{BS}} = -\tfrac{1}{2}\Delta^2_{\mathrm{BS}}, \quad \Delta^2_{\mathrm{BS}} = E^p[(\log u)^2] - (E^p[\log u])^2\,. \tag{22}$$

It is easy to understand that if u has support concentrated around one, then the difference between $Q_T^{(\mathrm{LN})}$ and $Q_T^{(\mathrm{BS})}$ is small. In the continuous limit the differences between $Q_T, Q_T^{(\mathrm{LN})}$ and $Q_T^{(\mathrm{BS})}$ disappear altogether.

On the other hand, the risk-neutral price distribution can actually be close to lognormal but still significantly different from Black–Scholes. In an extreme situation of very large $\mathrm{Var}^q[u]$, $Q_T^{(\mathrm{LN})}$ is only close to Q_T in a very limited region. As an example we show in Fig. 1 the case where $p(u)$ is a truncated Levy distribution. The distribution Q_T has been computed by a Monte Carlo procedure. Here it is obvious that one has to consider the corrections to the lognormal to get a good approximation to Q_T.

We observe that if u has support around one the predictions of the Black–Scholes theory and the lognormal approximation are both rather close to the exact formula:

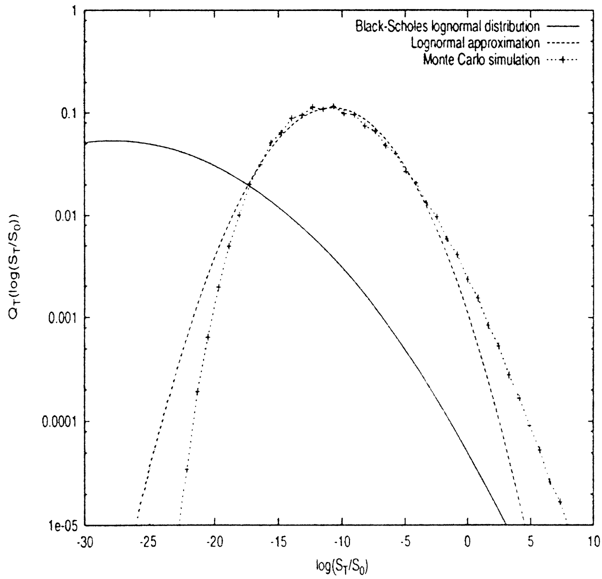

Fig. 1. The probability distribution is a discrete approximation to a Lévy distribution. The relative price increments can take the values $u_n = u_0\alpha^u$, with u_0 equal to 0.4, α equal to 1.2 and the index n ranging from zero to N, which in this example has been set to 30. The probability of an elementary event, $p(u_n)$, is $C_N u_n^{-\beta}$, with β equal to 0.5 and C_N a normalization constant. The Monte Carlo simulation was performed with 20 million trials.

this is the case in Fig. 2. In Fig. 2 we show the price of a European call option as a function of strike price where the probability distributions are: the increment u takes three discrete values $0.8, 1$ and 1.2 with probabilities $0.2, 0.3$ and 0.5.

On the other hand, in general, both Black–Scholes formula and the lognormal approximation can be far from the exact formula. Large deviations have therefore to be taken into account to compute properly the expectation value, see Ref. [19].

6. The experimental error on the optimal l^*

The realistic application of the theory we have presented depends on the possibility of estimating the distribution $p(u)$ from real data. In fact, $p(u)$ enters into the construction of the $q(u)$ both directly and with the optimal l^*, which is a functional of $p(u)$. Therefore, it is very important to estimate l^* together with the magnitude of its error.

The information which is possible to obtain from the market is a series of stock prices, $S_0,\dots,S_H$, where H is the number of data. We consider the time interval between observations to be large enough to ensure the time independence of the returns $u_i = S_i/S_{i-1}$.

In order to estimate experimentally the true distribution $p(u)$ we consider a partition of the support of the distribution in M intervals of width Δ, the requirement being that

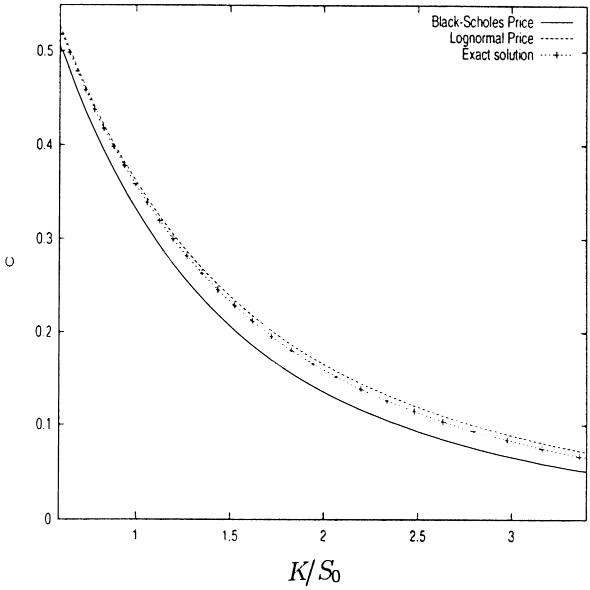

Fig. 2. Option prices C as function of the strike prices K for $T=30$. The relative price increments over one elementary step in the process are, that u takes three discrete values 0.8, 1 and 1.2 with probabilities 0.2, 0.3 and 0.5. $S_0=1$.

$M \leqslant H$. Then we estimate the probability in one of the interval by the ratio between the number of observed data in the interval and the total number of data H. In this way, one creates a histogram of the approximating probability $P_{\Delta,H}(u)$, the deviation of which from the 'true' probability $p(u)$ decreases as H becomes larger and Δ smaller. Let us call this deviation

$$\delta P_{\Delta,H}(u) = P_{\Delta,H}(u) - P(u) . \tag{23}$$

Because observed data are either in an interval or not, the number of observed data in an interval follows a binomial distribution. This observation immediately leads to

$$E[\delta P_{\Delta,H}(u)] = 0 , \tag{24}$$

$$\mathrm{Var}(\delta P_{\Delta,H}(u)) = P_{\Delta,H}(u)(1 - P_{\Delta,H}(u)\Delta) . \tag{25}$$

To estimate the true l^* with l_H^* using the approximating probability $P_{\Delta,H}(u)$ we have to solve the equation

$$\int \frac{P_{\Delta,H}(u)(u-1)}{1 + l_H^*(u-1)}\, \mathrm{d}u = 0 . \tag{26}$$

The error on the optimal fraction invested in the stock is the difference

$$\delta l = l_H^* - l^* , \tag{27}$$

where l^* is the true optimal fraction obtained from the distribution $p(u)$.

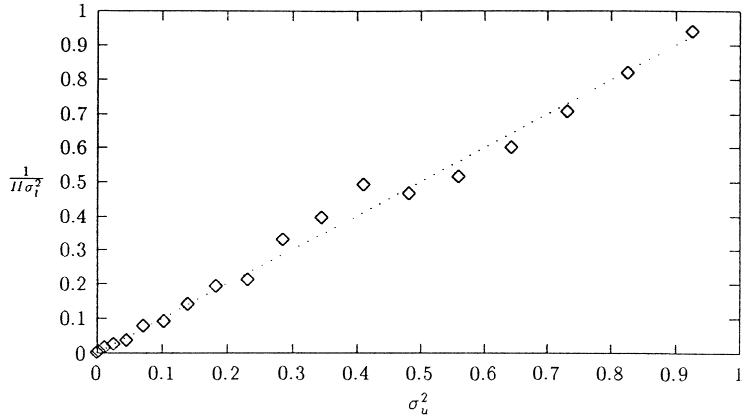

Fig. 3. Comparison between the error on l^* estimated by Eq. (28) and by the bootstrap technique. We plot $1/(H\sigma_1^2)$ vs. σ_u^2, where the σ_l are obtained by the bootstrap technique with $J = 200$ samples and the distribution of u is a Gaussian with $E[u] = 1$ and standard deviation σ_u.

A Taylor expansion up to the second-order around l^* of (26), and the above considerations about the error in estimating the probability give after some calculations

$$\sigma_l = \frac{1}{\tilde{\sigma}\sqrt{H}} \, . \tag{28}$$

where

$$\tilde{\sigma} \equiv \left(\int \frac{P_{\Delta,\,H}(u)(u-1)^2}{(1 + l^*(u-1))^2} \, \mathrm{d}u \right)^{1/2} , \tag{29}$$

where the fact that for a sensible partitioning of the sample space $P_{\Delta,\,H}(u)\Delta \ll 1$ has been used.

The error on l^* can also be estimated from the historical distribution using bootstrap technique [43], i.e., drawing with replacement a set of J new samples of the same size H of the original one from the empirical distribution, and computing a new l^* for each bootstrap sample. In Fig. 3 we compare the variance of this distribution with the quadratic estimation (28). The agreement is very good for different values of the $\sigma_u \equiv \sqrt{\mathrm{Var}[u]}$.

According to Eq. (28), it is evident that in most cases the error of l^* is large. A non-intuitive result is that it is easier to determine l^* with accuracy for distribution with larger variance because of the $\tilde{\sigma}$ in the denominator. This fact can be understood as follows. A stock with small fluctuations allows for a bigger fraction be held against the interest rate to make money, but just a small change in the distribution will make a big change in the optimal fraction. On the contrary, for a wildly fluctuating stock the fraction invested is smaller, and a little change in the distribution will not have much effect on the optimal portfolio.

6.1. Data analysis

Empirical study is needed to verify the growth optimal pricing procedure here discussed. To verify the model we looked at the European call option on the OMX-index, traded at the OM exchange in Stockholm, during the period 2 January 1995 until 22 September 1995. We consider closing prices of each day. The time period between trading is one day. The time series contained 183 observations, providing a sample of u_i of size (data points) 182. The empirical distribution of u_i of the OMX-index is built by the fraction of the price of the OMX-index today and tomorrow, $u_i = S_{i+1}/S_i$. Weekends and other holidays are treated as non-existing.

The thirty day interest rate was as used discount factor r to compute the discounted daily returns. $\{u_i/r_i\}_{i=1}^{182}$. During the period the interest rate fluctuate between 7.15 and 9.83% of yearly yield. The sample space was divided into $M = H$ cells and the optimal l^* determined using the Newton–Raphson algorithm [44]. Because of the concave of the logarithm the convergence is very fast, the optimal l^* for the OMX-index is equal to 14.46.

The distribution of l^* is obtained using bootstrap technique with a set of 200 new samples. The standard deviation of l^* is 8.6 and can be compared to 8.7 obtained estimating the standard deviation using formula (28). From the empirical distribution p the empirical q-distribution can be created and from q-distribution the empirical Q-distribution can be derived.

The empirical q-distribution is used to estimate parameters of the lognormal approximation (20) $\lambda = 1.9 \times 10^{-4}$ and $\Delta^2 = 7.3 \times 10^{-5}$. We determine, using the bootstrap technique, the distribution and standard deviation for λ and $\Delta^2, \sigma_\lambda = 3.6 \times 10^{-6}$ and $\sigma_{\Delta^2} = 3.9 \times 10^{-6}$, respectively.

These parameters correspond inside the experimental errors, to the Black–Scholes model. The consequence is that in this case lognormal approximation of the Kelly pricing scheme is practically equivalent to Black–Scholes model.

The comparison between empirical data and theoretical one is shown in Fig. 4, for the call options with maturity date less or equal then 10 working days (two weeks). We observe a coincidence inside the errors. Finally, we observe that to be able to tell a difference between classical way of pricing derivatives and growth-optimal pricing procedure either the price movement of the underlying has to have 'wilder' fluctuations, or one has to have access to more high-frequency data.

7. The general case with correlated stocks

In order to study the effect of correlated stocks, the market in this section is assumed to have L stocks $\{S^k\}_{k=1}^{L}$ and D_k derivatives $\{C^{(j,k)}(S_T^k)\}_{j=1}^{D_k}$ written on them. To get transparency of the principle we discuss the case of just two stocks $L = 2$ with one derivative each and a risk-free interest rate equal to 1. The conclusion of this section will be given in a general notation.

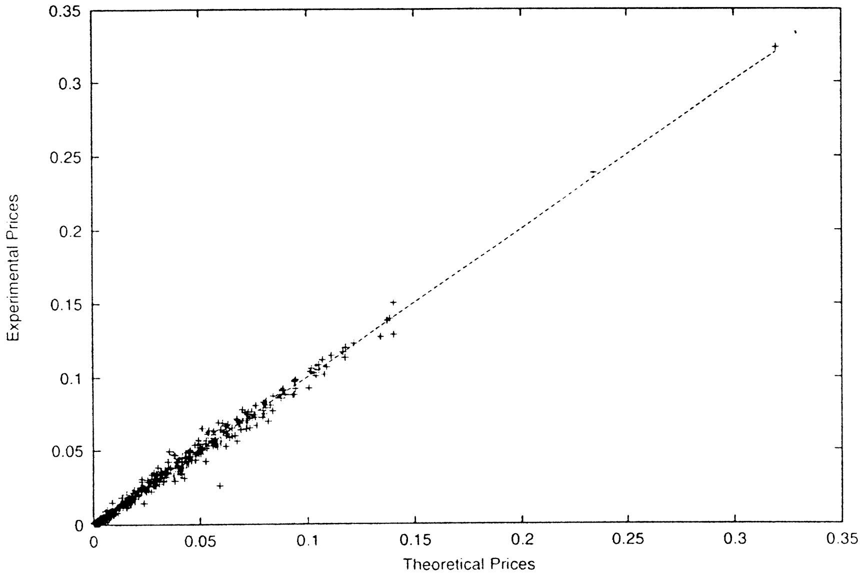

Fig. 4. Experimental option prices against theoretical ones for the call options of OMX with maturity date less or equal then 10 working days in the period between the 2nd of January and the 22nd of September 1995. The theoretical prices are generated with the probability $Q_T(S_T/S_0)$ using the Monte Carlo discussed in this section with 10^5 trials. The linear fit with $ax+b$ gives $a = 1.003 \pm 0.004$ and $b = 0 \pm 10^{-4}$. As discussed in the text the major source of errors is on the estimation of l^* due to the low data frequency considered.

The features of the stock price movements are given by

$$u_i^{(k)} = \frac{S_{i+1}^{(k)}}{S_i^{(k)}}, \quad k = 1,2 . \tag{30}$$

The random variable $u_i^{(k)}$ is assumed to be correlated on other stocks.

An investor of this market commits a fraction of his capital l_k in stock k and a fraction $d_{j,k}$ in the derivative j. The exponential growth rate of the capital is

$$\lambda(l_1,l_2,d_{1,1},d_{1,2}) = E^p\left[\log\left(1 + l_1(u^{(1)}-1) + l_2(u^{(2)}-1) + d_{1,1}\left(\frac{c_1^{(1,1)}}{c_0^{(1,1)}} - 1\right) + d_{1,2}\left(\frac{c_1^{(1,2)}}{c_0^{(1,2)}} - 1\right)\right)\right] . \tag{31}$$

According to the general equilibrium argument, one obtains the two equations:

$$E^q[(u^{(k)}-1)] = 0, \quad k = 1,2 \tag{32}$$

and

$$C_i^{(j,k)}(S_i^{(k)}) = E^q[C_{i+1}^{(j,k)}(S_{i+1}^{(k)})], \quad j = 1,2 , \tag{33}$$

where we have defined, as in Section 3, a new probability measure q:

$$q_h^{(2)} = \sum_m \frac{p_{hm}}{1 + l_1^*(u^{(h,1)} - 1) + l_2^*(u^{(m,2)} - 1)}\,, \tag{34}$$

where the index h and m stands for some realization of the random variables $u^{(1)}$ and $u^{(2)}$ with the joint probability p_{hm}.

The solution of Eq. (32) gives the optimal fraction l_k^* to invest in each of the different stocks S^k and Eq. (33) give the option prices as the expected value under measure (34). The martingale measure may be compounded so that the option is priced by using the pay out function at expire date:

$$C_i^{(j,k)}(S_i^{(k)}) = E^Q[C_T^{(j,k)}(S_T^{(k)})]\,, \tag{35}$$

where the Q denotes the compounded martingale measure of q.

The case of independent stock price movements is of particular interest from a theoretical point of view. The price of the derivative on an underlying stock and even the hedge position do not depend on the other stocks in the 'classical models', i.e., complete markets and risk minimization pricing procedure. This result is a consequence of the property

$$q_h^{(2)} = q_h\,, \tag{36}$$

where q_h and $q_h^{(2)}$ are the martingale measures of one and two shares, respectively.

In the case of a complete market it is a known fact that the martingale measure q_h is unique [45], then it cannot be anything else but the measure (6) found by optimizing the growth rate for just that particular stock

$$q_h = \frac{p_h}{1 + l^*(u^{(h,1)} - 1)}\,. \tag{37}$$

Property (36) is also preserved in the risk minimization pricing procedure but not in the general case of an incomplete market. In this situation, the associated martingale measure (34) depend on the number of stocks traded in the market

$$q_h^{(2)} = p_h \cdot \sum_m \frac{p_m}{1 + l_1^*(u^{(h,1)} - 1) + l_2^*(u^{(m,2)} - 1)} \neq q_h\,. \tag{38}$$

In Fig. 5 we show this difference on the option prices for the trichotomic case.

This fact is clear in an incomplete market with large number L of stocks and the returns of these stocks are independent identically distributed random processes. The optimal fraction invested in each stock is $O(1/L)$ and the excess rate of return becomes almost sure. To avoid arbitrage opportunities the expected excess rate of return μ must vanish and the measure pricing the option becomes close to the observed probability i.e., $C \cong E^P[C_T]$. Let us notice that this corresponds to a generalization to a non-Gaussain $p(u)$ of the Bachelier theory [46].

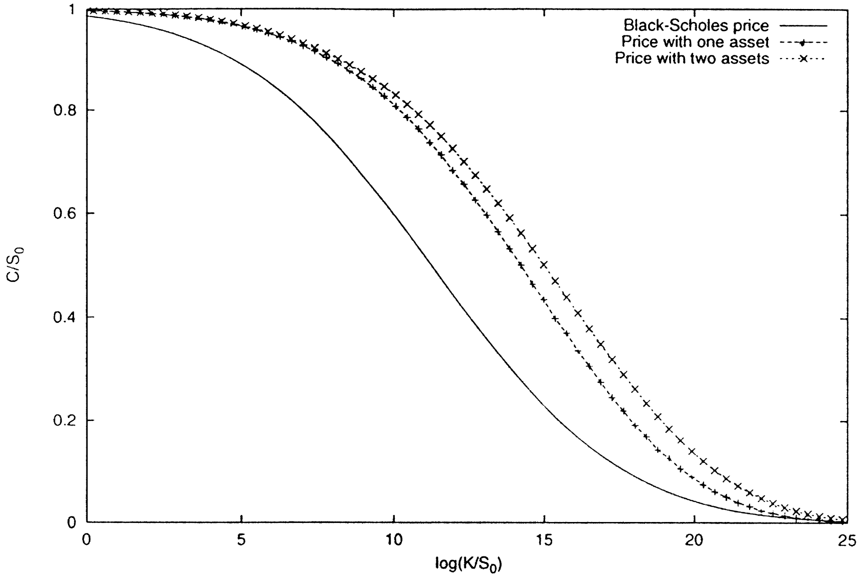

Fig. 5. Option prices C rescaled by the initial value of the stock S_0 vs. $\log(K/S_0)$ for $T = 30$ and the returns of Fig. 1. We compare the exact prices with one and two asset in the market with the Black–Scholes prescription.

8. Conclusion

We have in this paper introduced a new criterion to price derivatives in incomplete markets derived from the theory of optimal gambling strategies in repeated games. The criterion say that one should not be able to construct a portfolio using derivatives that grows almost surely at a faster exponential rate than using only the underlying security.

A corner-stone in modern derivative pricing theory is that discounted prices (in friction-less markets) can be expressed as expected pay-off, expectations taken with respect to equivalent martingale measures. We use the criterion to decide which out of many possible measures to choose. We therefore do not need to assume a complete market to fix unambiguously a price.

In Section 4 we compared our approach to other procedures proposed in the literature, and in Section 5 we discussed the lognormal limit. In our construction we determined first the equivalent martingale measure over one elementary time step. By composition we can then get the risk-neutral distribution at finite times. In a large class of models this compounded probability tends to the lognormal form, at the points close to the maximum, where probability is high. One could then conclude that we just rederive the Black–Scholes model, except for rare events. This is however not true, because, for us, the condition that the risk-neutral distribution is a martingale involves also the distribution of rare and large events. These are not lognormal. In fact, the best lognormal approximation does not by itself have to be a martingale, and in general differs from Black–Scholes.

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