# Multifractality and heteroscedastic dynamics: An application to time series analysis

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**Abstract** – An increasingly important problem in physics concerns scale invariance symmetry in diverse complex systems, often characterized by heteroscedastic dynamics. We investigate the nature of the relationship between the heteroscedastic and fractal aspects of the dynamics of complex systems, by analyzing the sensitivity to heteroscedasticity of the scaling properties of weakly nonstationary time series. By using multifractal detrended fluctuation analysis, we study the singularity spectra of currency exchange rate fluctuations, after partially or completely eliminating $n$-point correlations via data shuffling techniques. We conclude that heteroscedasticity can significantly increase multifractality and interpret these findings in the context of self-organizing and adaptive complex systems.

Complex phenomena [1] as diverse as music waveforms [2,3], heart rate rhythms [4,5], seismic signals [6] and stock market dynamics [7–10] have in common at least two quantifiable features: i) aspects of their dynamics lack a well-defined characteristic scale and instead possess scale invariance symmetry; and ii) in each case, such signals do not have stationary second moments and behave heteroscedastically. With suitable manipulation, these signals can be rendered weakly stationary, with (almost) vanishing first moments, but the second moments typically remain stubbornly nonstationary. What is the relationship between these two aspects of the dynamics of complex systems? One clue to answer this question comes from the observation, obtained from previous studies, that heteroscedastic phenomena follow multifractal [11] rather than monofractal dynamics (*e.g.*, see ref. [10]). Monofractals refer to the case when a single scaling exponent or fractal dimension characterizes the entire system. In contrast, multifractals have nonunique scaling exponents. Here, we investigate the degree to which multifractality and heteroscedasticity are related and whether this relationship is of a causal nature.

This question has relevance in the wider context of self-organizing, adaptive and evolutionary phenomena, especially those found in biological and economic systems. In the latter cases, for a system to "survive" (*i.e.*, not enter extinction or an equivalent absorbing state) it must have at its disposition the widest possible range of paths to follow or choices to make. Previous studies [12] have explored the advantages that scale invariance (hence, multifractality and power laws in general [13]) might confer on adaptive systems, by exploiting self-organization and by allowing greater variability in response to stimuli [14]. It is not inconceivable that the same line of reasoning applies to heteroscedasticity. We will return to address this secondary question towards the end of this paper. If heteroscedasticity is shown to affect multifractality, this finding might have implications in the study of scaling phenomena in complex systems (*e.g.*, Zipf's and Pareto's laws). Multifractality and heteroscedasticity are particularly important in the study of economics [15–17].

Without loss of generality and due to the admitted social importance of economic phenomena, we have chosen high-frequency currency exchange rate time series as our data set. Such series have classic heteroscedastic behavior, with local variances that deviate significantly from the global value. Moreover, the behavior of financial markets has recently become an area of active physics research
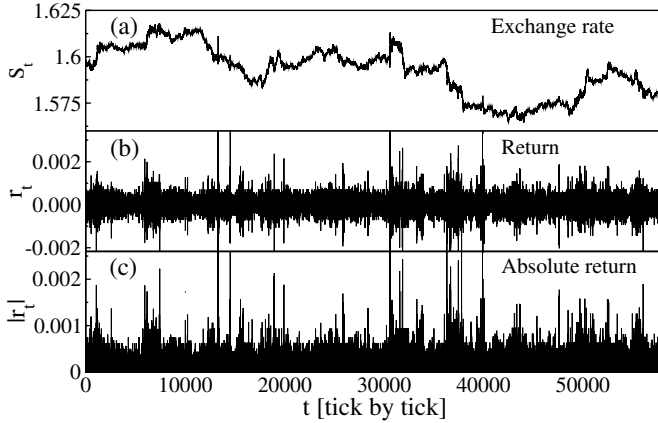
Fig. 1: (a) A sample of $6 \cdot 10^4$ out of the total of 1472240 data points contained in the high-frequency currency exchange rate series of German *Deutsche Marks* (DEM) *vs.* United States Dollars (USD), covering the time period from 1 October 1992 to 30 September 1993. (b) The corresponding log-returns, and (c) absolute returns. Note the heteroscedasticity apparent in both (b) and (c).

because of its rich and complex dynamics [7] and also due to the large amount of recorded data and its easy accessibility [18,19]. Let $S_t$ represent the market price at a given instant $t$. Since $S_t$ does not have stationary moments, we define the financial log-return $r_t = \ln S_{t+1} - \ln S_t$. Such returns have approximately stationary mean but nonstationary second moment. The local variance $\sigma_t^2 \equiv \langle r_t^2 \rangle_{local}$ is an important quantity in finance and is related to volatility. The study on returns and volatilities are of great importance for financial markets [20]. Some open questions are related to i) the presence of fat-tails in the probability density functions (PDF) for both returns and volatility [7,21,22], ii) the functional form and dynamical origin of the fat-tails and iii) the role played by memory and correlations [9]. Here we investigate the multifractal properties of DEM/USD tick-by-tick exchange rates fluctuations and the influence of the temporal organization.

Our study is based on DEM/USD tick-by-tick exchange rates taken from Reuters EFX (provided by Olsen & Associates) during a period of 1 year from 1 October 1992 to 30 September 1993. This period corresponds to a total of 1472240 data points (fig. 1), or one data point for approximately every 20 s. The large amount of data renders greater quality to our analysis.

We generate a control data set by shuffling the order of the returns. This control data set has no correlations and is therefore nonheteroscedastic. We generate a second control data set by shuffling the order of the signs $s_t = \pm 1$ of the returns $r_t = s_t |r_t|$. This second control data set has no sign correlations but retains heteroscedasticity because of the volatility correlations.

In order to quantify the multifractal properties of the time series, we have studied their multifractal spectra. The multifractal spectrum of a time series contains information about $n$-point correlations [23] and thus provides more information compared to two-point correlation functions.

We use the MF-DFA method to study the multifractal properties and also to obtain the multifractal spectrum [15,24]. We briefly summarize the MF-DFA method for a given time series $r_t$ containing $N$ data points as follows. In step: i) Integrate the time series $r_t$ to generate a profile $y(t) \equiv \Sigma_{k=1}^{t}[r_t - \langle r \rangle]$, $t = 1, \ldots, N$. ii) Divide $y(t)$ into non-overlapping segments of size $s$, so that $N \equiv s N_s$. iii) Apply a linear regression for $\nu$-th segment ($\nu = 1, \ldots, N_s$), to calculate a best-fitting "trend" $y_\nu(i)$, where $i = 1, \ldots, s$. iv) Determine the mean square fluctuation, *i.e.*, the measure of the second moment for the $\nu$-th segment as $F_2(\nu, s) \equiv \frac{1}{s} \Sigma_{i=1}^{s} |y((\nu - 1)s + i) - y_\nu(i)|^2$. v) Evaluate the $q$-th order fluctuation function $F_q(s) \equiv \frac{1}{N_s} \Sigma_{\nu=1}^{N_s} F_2(\nu, s)^{q/2}$.

The scaling of the fluctuation function for the moment $q$ (in Landau notation) follows

$$F_q(s) \sim s^{qh(q)}, \qquad (1)$$

where $h(q)$ represents a generalized Hurst exponent; while positive values of $q$ weight large fluctuations, negative moments weight small fluctuations. Monofractal time series have a unique Hurst exponent $h(q) = H$, while for multifractal series the value of $h(q)$ depends nonlinearly on $q$. It is possible to relate $h(q)$ defined in eq. (1) in the MF-DFA method to the $\tau(q)$ exponent conventionally used to quantify the scaling of the partition function in the standard textbook formalism [11,24,25]. The multifractal singularity spectra $f(\alpha)$ can be obtained from a Legendre transform of the $\tau(q)$ exponent:

$$\alpha = \mathrm{d}\tau(q)/\mathrm{d}q, \qquad (2)$$

$$f(\alpha) = q\alpha - \tau(q). \qquad (3)$$

Here, $f$ gives the dimension of the subset of the series characterized by the singularity strength $\alpha$. The literature also refers to $\alpha$ as the Hölder exponent, which represents a measure of the number of continuous derivatives that the underlying signal possesses.

Figures 2(a), (b) show the generalized Hurst exponent and exponent $\tau(q)$ for the returns (open diamond), shuffled returns (closed diamond), and shuffled sign returns (plus symbols). Note the effect of temporal organization on the returns: $h(q)$ for shuffled returns is approximately constant, but shows greater variation for the shuffled sign returns. Figure 2(c) shows the estimated singularity spectra of the returns, shuffled returns, and shuffled signs returns. The temporal organization of the returns is reflected on the fact that the spectrum for shuffled returns is narrower than for the original returns.

Similar observations can be made for the volatility. Figure 3 shows that eliminating heteroscedasticity in the absolute returns (*i.e.*, volatility) leads to a significant drop in multifractality. Part of the significance of fig. 3 lies in the fact that the estimated singularity spectrum for the returns centers approximately around $\alpha = 1/2$ (fig. 2(c)), as expected from the Efficient Market Hypothesis (EMH), whereas for the volatility the center of the spectrum has a
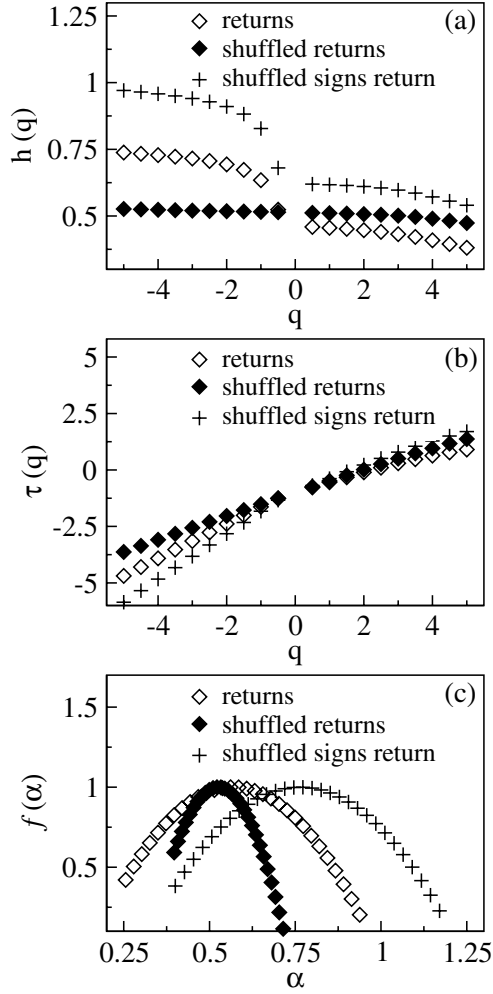
Fig. 2: Multifractal detrended fluctuation analysis (MF-DFA) of the heteroscedastic returns, of the nonheteroscedastic shuffled returns, as well as of the returns with only the order of their signs shuffled (see text). The latter retain heteroscedasticity and volatility autocorrelations due to its unchanged modulation of volatility, but have vanishing return autocorrelations. (a) Generalized Hurst exponent. (b) Exponent $\tau$ of the partition function. (c) Quadratic fits of singularity spectra. Notice that the nonheteroscedastic series has the least degree of multifractality.

higher value of $\alpha$, consistent with the existence of long-range correlations. We suspect that heteroscedasticity possibly causes multifractality by introducing long-range correlations in the absolute value of the random variable.

Our results show that the effect of the temporal organization is stronger on volatility than on return fluctuations. We also find that shuffling the returns reduces multifractality, but shuffling only the order of the signs does not reduce multifractality. The drastic effect of shuffling only the order of the signs, where its dominant Hölder coefficient $\alpha$ (for $f = 1$) shifts away from $\alpha = 0.5$, probably arises due to the known asymmetric return-volatility correlations (*i.e.,* the "leverage effect"). Shuffling the order of the signs allows the volatility correlations to become visible in the returns.
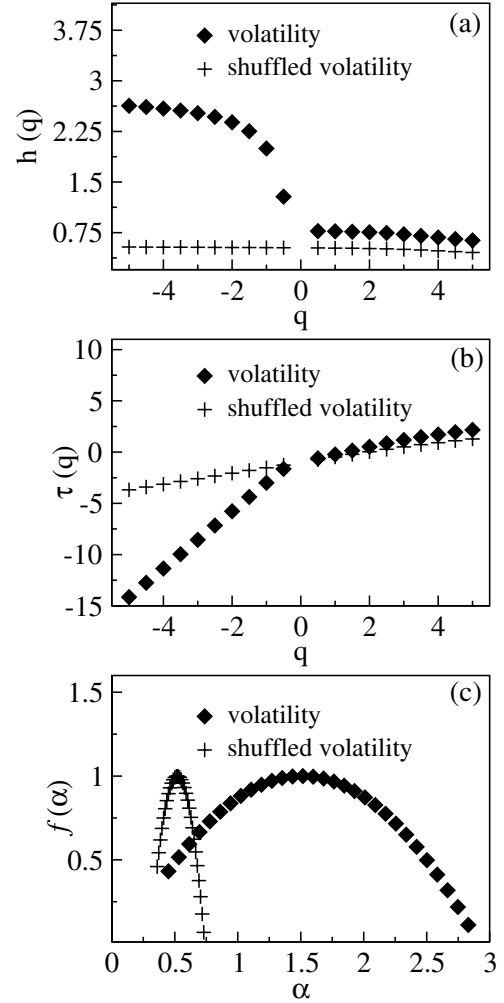


Fig. 3: MF-DFA of the absolute returns, analogous to fig. 2. Elimination of heteroscedasticity leads to a nearly monofractal spectrum.

We test the hypothesis that multifractality and heteroscedasticity are related by applying MF-DFA to another data set. We choose to study a very different kind of system: audio time series [2] of the first movement of Ludwig van Beethoven's Symphony No. 5 in C minor, Op. 67 (fig. 4(a)). Note its strong heteroscedastic behavior. Figures 4(b), (c) show the shuffled data and the multifractal properties. As expected, multifractality decreases when heteroscedasticity vanishes.

Finally, we comment on the advantages that heteroscedasticity might confer on adaptive complex systems. A fixed second moment (*i.e.,* volatility in the financial case) can allow an adversary to gain an advantage over the system. In contrast, heteroscedasticity allows a degree of flexibility similar to that conferred by scale invariance symmetry. This line of reasoning leads to a prediction: heteroscedasticity should be nearly as ubiquitous as scale invariance in adaptive and evolutive systems [26,27] (see also ref. [13]). Although beyond the scope of the present paper, an investigation of the prevalence of
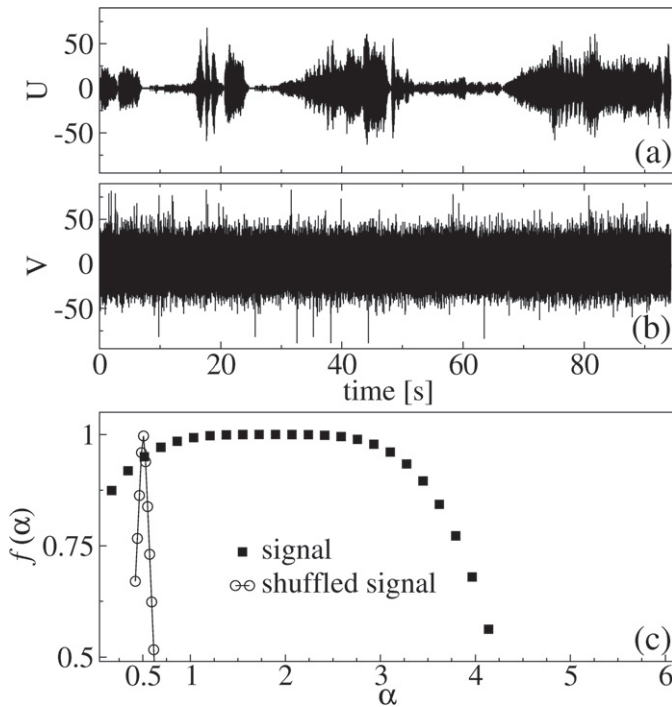
Fig. 4: Approximately 20% of the first movement of Beethoven's Fifth Symphony. (a) The original signal $U_i$, (b) the shuffled signal $V_i$. (c) 4th order (closed square) and interpolated (open circle) fits for singularity spectra. The multifractality practically disappears when heteroscedasticity vanishes.

heteroscedasticity in biological systems would make an interesting topic for a future study.

In summary, we have shown that a reduction in heteroscedasticity leads to a reduction in multifractality. But a reduction in the autocorrelation time without a corresponding reduction in the heteroscedasticity does not lead to a similar reduction. These findings indicate the importance of heteroscedasticity to multifractality.

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