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Magnetization densities as replica parameters: The dilute ferromagnet

Maurizio Serva*

Dipartimento di Matematica, Università dell'Aquila, I-67010 L'Aquila, Italy

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ABSTRACT

In this paper we compute exactly the ground state energy and entropy of the dilute ferromagnetic Ising model. The two thermodynamic quantities are also computed when a magnetic field with random locations is present. The result is reached in the replica approach frame by a class of replica order parameters introduced by Monasson (1998) [5]. The strategy is first illustrated considering the SK model, for which we will show the complete equivalence with the standard replica approach. Then, we apply to the diluted ferromagnetic Ising model with a random located magnetic field, which is mapped into a Potts model.

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1. Introduction

The theoretical modeling of statistical systems in many areas of physics makes use of random Hamiltonians. Assuming self-averaging, observable quantities may be evaluated by the logarithmic average of the partition function $\langle \log(Z) \rangle$ over the quenched random variables. In this way, the technically difficult task of computing Z for a given realization of the disorder is avoided. However, the mathematical operation of directly computing $\langle \log(Z) \rangle$ is also difficult, and can be done only using replica trick, which implies the computation of $\langle Z^n \rangle$.

Despite its highly successful application to the treatment of some disordered systems (the most celebrated is the SK model [1,2]), the replica trick encounters serious difficulties when applied to many other models. The main reason is the unbounded proliferation of replica order parameters, as for example the multi-overlaps in dilute spin glass [3,4].

In this paper we use a class of replica order parameters firstly introduced by Monasson [5], which, in principle can be used for all models. We preliminarily observe that the computation of $\langle Z^n \rangle$ implies the sum over all the realizations of the spin variables $\sigma_i = (\sigma_i^1, \sigma_i^2, \dots, \sigma_i^n)$, any of them may take 2^n values. Assume that $x(\sigma)N$ is the number of vectors σ_i which equal the given vector $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^n)$, then, $\langle Z^n \rangle$ can be re-expressed in terms of a sum over the possible positive values of the 2^n magnetization densities $x(\sigma)$, with the constraint $\sum_{\sigma} x(\sigma) = 1$. In practice, this is equivalent to maximizing with respect to these order parameters.

The strategy proposed here will be illustrated, considering the SK model, in the next section, where we will also show the complete equivalence with the standard approach. Nevertheless, all models can be described in terms of the 2^n order parameters $x(\sigma)$. We will tackle the dilute ferromagnetic Ising model in Section 3 and in Section 4 we will compute exactly the ground state energy and entropy. The results coincide with those found in Ref. [6] where the approach is not based on replicas. In Section 5 we extend the scope by considering the same model in the presence of a magnetic field with random locations, and, finally, in Section 6 we are able to compute the ground state energy and entropy also in this case. Conclusions and outlook are in the final section.

* Tel.: +39 3204399439.

E-mail address: serva@univaq.it.

2. SK spin glass

In this section we try to illustrate our approach considering the SK model. We do not have new results concerning SK, but we just show the complete equivalence with the standard replica approach.

The partition function is

$$Z = \sum_{\#} \exp \left(\frac{\beta}{\sqrt{N}} \sum_{i>j} J_{ij} \sigma_i \sigma_j \right) \quad (1)$$

where the sum $\sum_{\#}$ goes over the 2^N realizations of the σ_i , the sum $\sum_{i>j}$ goes over all $N(N-1)/2$ pairs ij and the J_{ij} are independent random variables with 0 mean and variance 1. Then, by replica approach, neglecting terms which vanish in the thermodynamic limit, we have

$$\langle Z^n \rangle = \sum_{\#} \exp \left(\frac{\beta^2}{4N} \sum_{i,j} \left(\sum_{\alpha} \sigma_i^{\alpha} \sigma_j^{\alpha} \right)^2 \right) \quad (2)$$

where the sum $\sum_{\#}$ goes over the 2^{nN} realizations of the σ_i^{α} and the sum $\sum_{i,j}$ goes over all N^2 possible values of i and j .

Assume that $Nx(\sigma)$ is the number of vectors $(\sigma_i^1, \sigma_i^2, \dots, \sigma_i^n)$ which equal the given vector $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^n)$, then, according to Ref. [5], we can write

$$\sum_{i,j} \left(\sum_{\alpha} \sigma_i^{\alpha} \sigma_j^{\alpha} \right)^2 = N^2 \sum_{\sigma, \tau} x(\sigma) x(\tau) (\sigma \cdot \tau)^2 \quad (3)$$

where $\sum_{\sigma, \tau}$ goes over the 2^{2n} possible values of the variables σ and τ and where $\sigma \cdot \tau$ is the scalar product $\sigma \cdot \tau = \sum_{\alpha} \sigma^{\alpha} \tau^{\alpha}$. Then, observe that the number of realizations corresponding to a given value of the 2^n magnetization densities $x(\sigma)$ is

$$\exp \left(-N \sum_{\sigma} x(\sigma) \log(x(\sigma)) \right) \quad (4)$$

where \sum_{σ} is the sum over the 2^n possible values of the variable σ . Indeed, in the above expression we neglected terms which are in-influent in the thermodynamic limit.

We can now define Φ_n as the large N limit of $\frac{1}{N} \log \langle Z^n \rangle$, then

$$\Phi_n = \max_x \left[\frac{\beta^2}{4} \sum_{\sigma, \tau} x(\sigma) x(\tau) (\sigma \cdot \tau)^2 - \sum_{\sigma} x(\sigma) \log(x(\sigma)) \right]. \quad (5)$$

The maximum is taken over the possible values of the 2^n order parameters $x(\sigma)$ provided that $\sum_{\sigma} x(\sigma) = 1$. The constraint can be accounted by adding the Lagrangian multiplier $\lambda(\sum_{\sigma} x(\sigma) - 1)$ to expression (5), then the maximum is given by

$$\frac{\beta^2}{2} \sum_{\tau} x(\tau) (\sigma \cdot \tau)^2 - \log(x(\sigma)) - 1 = \lambda \quad (6)$$

where λ has to be chosen in order to have $\sum_{\sigma} x(\sigma) = 1$. From this equation we get that the maximum is realized for the set of the 2^n order parameters $x(\sigma)$ which satisfy

$$x(\sigma) = \frac{1}{A} \exp \left(\frac{\beta^2}{2} \sum_{\tau} x(\tau) (\sigma \cdot \tau)^2 \right) \quad (7)$$

where the sum \sum_{τ} goes over the 2^n possible values of the variable τ and where A is

$$A = \sum_{\sigma} \exp \left(\frac{\beta^2}{2} \sum_{\tau} x(\tau) (\sigma \cdot \tau)^2 \right) \quad (8)$$

and where \sum_{σ} is the sum over the 2^n possible values of the variable σ .

The explicit solution could be found, in principle, by a proper choice of the parametrization of the $x(\sigma)$, but it can be easily seen that this solution coincides with the standard solution of the SK model. In fact, if one defines the symmetric matrix $q_{\alpha\beta}$ as

$$q_{\alpha\beta} = \sum_{\tau} x(\tau) \tau^{\alpha} \tau^{\beta} \quad (9)$$

one gets

$$x(\sigma) = \frac{1}{\tilde{A}} \exp \left(\beta^2 \sum_{\alpha > \beta} q_{\alpha\beta} \sigma^\alpha \sigma^\beta \right) \quad (10)$$

where

$$\tilde{A} = \sum_{\sigma} \exp \left(\beta^2 \sum_{\alpha > \beta} q_{\alpha\beta} \sigma^\alpha \sigma^\beta \right) \quad (11)$$

the sum $\sum_{\alpha > \beta}$ goes over all $n(n-1)/2$ pairs α, β and the diagonal terms of the matrix $q_{\alpha\beta}$ disappeared since they cancel out in the expressions (7), (8).

Then, after some work, the expression (5) is rewritten as

$$\Phi_n = \max_q \left[\frac{\beta^2}{4} n - \frac{\beta^2}{2} \sum_{\alpha > \beta} (q_{\alpha\beta})^2 + \log \left(\sum_{\sigma} \exp \left(\beta^2 \sum_{\alpha > \beta} q_{\alpha\beta} \sigma^\alpha \sigma^\beta \right) \right) \right] \quad (12)$$

where the maximum is over the variables $q_{\alpha\beta}$. This is the standard solution of Ref. [1] provided the proper maximum of $q_{\alpha\beta}$ is found [2]. Then $\Phi = \lim_{n \rightarrow 0} \frac{\Phi_n}{n} = S - \beta E$ where S is the entropy and E the energy.

3. Dilute ferromagnet

In this section, we show how our approach works for the dilute ferromagnetic system. This model is much less studied than SK, and much information about its phenomenology is still missing.

The partition function is

$$Z = \sum_{\#} \exp \left(\beta \sum_{i>j} K_{ij} \sigma_i \sigma_j \right) \quad (13)$$

where K_{ij} are quenched variables which take the value 1 with probability $\frac{\gamma}{N}$ and 0 otherwise. The dilution coefficient γ may take any positive value and the number of bonds is about $\frac{\gamma N}{2}$. This model has been recently studied in a recent paper [4], while models with lesser dilution [7,8] (number of bonds of order N^ϵ with $\epsilon > 1$) have been also considered.

We can rewrite the above expression as

$$Z = \exp \left(\frac{\gamma N}{2} \log(\cosh(\beta)) \right) \sum_{\#} \prod_{i>j} (1 + \tanh(\beta) K_{ij} \sigma_i \sigma_j) \quad (14)$$

where the equality holds in the sense that $\frac{1}{N} \log(Z)$ coincide in the thermodynamic limit for (13) and (14) because $\sum_{i>j} K_{ij} = \frac{\gamma}{2} N + o(N)$.

Let us define

$$G = \sum_{\#} \prod_{i>j} (1 + \tanh(\beta) K_{ij} \sigma_i \sigma_j) \quad (15)$$

than we can compute $\langle G^n \rangle$

$$\langle G^n \rangle = \sum_{\#} \prod_{i>j} \left(1 - \frac{\gamma}{N} + \frac{\gamma}{N} \prod_{\alpha} (1 + \tanh(\beta) \sigma_i^\alpha \sigma_j^\alpha) \right) \quad (16)$$

which is thermodynamically equivalent to

$$\langle G^n \rangle = \exp \left(-\frac{\gamma N}{2} \right) \sum_{\#} \exp \frac{\gamma}{N} \sum_{i>j} \prod_{\alpha} (1 + \tanh(\beta) \sigma_i^\alpha \sigma_j^\alpha). \quad (17)$$

By introducing the $x(\sigma)$ with identical meaning as in previous section, and defining Ψ_n as the large N limit of $\frac{1}{N} \log \langle G^n \rangle$ we get

$$\Psi_n = \max_x \left[-\frac{\gamma}{2} + \frac{\gamma}{2} \sum_{\sigma, \tau} x(\sigma) x(\tau) \prod_{\alpha} (1 + \tanh(\beta) \sigma^\alpha \tau^\alpha) - \sum_{\sigma} x(\sigma) \log(x(\sigma)) \right] \quad (18)$$

where the maximum is taken with respect to the 2^n order parameter $x(\sigma)$. To obtain an explicit expression one should find their parametric expression. This will be done in next section only for the zero temperature case. Formally, the maximum is reached for

$$x(\sigma) = \frac{1}{A} \exp \left(\gamma \sum_{\tau} x(\tau) \prod_{\alpha} (1 + \tanh(\beta) \sigma^{\alpha} \tau^{\alpha}) \right) \quad (19)$$

where

$$A = \sum_{\sigma} \exp \left(\gamma \sum_{\tau} x(\tau) \prod_{\alpha} (1 + \tanh(\beta) \sigma^{\alpha} \tau^{\alpha}) \right). \quad (20)$$

Then, according to (14), (15), (17) and (18), we get

$$\Phi = \lim_{n \rightarrow 0} \frac{\Psi_n}{n} + \frac{\gamma}{2} \log(\cosh(\beta)) = S - \beta E \quad (21)$$

where S is the entropy and E the energy.

4. Dilute ferromagnet: zero temperature

Let us consider the simpler case of vanishing temperature. In this limit $\tanh(\beta) = 1$ and $\prod_{\alpha} (1 + \tanh(\beta) \sigma^{\alpha} \tau^{\alpha})$ becomes $\prod_{\alpha} (1 + \sigma^{\alpha} \tau^{\alpha})$. This term equals 2^n when all σ^{α} equal τ^{α} and vanishes otherwise. Therefore, expression (18) becomes

$$\Psi_n = \max_x \left[-\frac{\gamma}{2} + \frac{\gamma}{2} 2^n \sum_{\sigma} x(\sigma)^2 - \sum_{\sigma} x(\sigma) \log(x(\sigma)) \right] \quad (22)$$

which is a standard 2^n -components Potts model [9]. The solution is known and can be found assuming that $2^n - 1$ quantities $x(\sigma)$ take the value $\frac{1-\theta}{2^n}$ and one takes the value $\frac{1+(2^n-1)\theta}{2^n}$. The state with different value can be any of the possible 2^n , we assume that is the one with $\sigma^{\alpha} = 1$ for all α . We can write:

$$x(\sigma) = \frac{1-\theta}{2^n} + \frac{\theta \prod_{\alpha=1}^n (1 + \sigma^{\alpha})}{2^n} \quad (23)$$

which obviously satisfies the constraint $\sum_{\sigma} x(\sigma) = 1$. Inserting the above expression in (22) we obtain

$$\Psi_n = \max_{\theta} \left[\frac{\gamma}{2} (2^n - 1) \theta^2 + B_n \right] \quad (24)$$

with

$$B_n = - \left(\frac{1 + (2^n - 1)\theta}{2^n} \right) \log(1 + (2^n - 1)\theta) - (2^n - 1) \frac{1 - \theta}{2^n} \log(1 - \theta) + n \log(2). \quad (25)$$

If we expand to the first order in n the above expression, we obtain

$$\Psi_n = \max_{\theta} \left[n \log(2) \frac{\gamma}{2} \theta^2 + n \log(2) (1 - \theta) (1 - \log(1 - \theta)) \right]. \quad (26)$$

Then, if we take into account (21) and we also take into account that for large β one has $\log(\cosh(\beta)) = \beta - \log(2)$, we have that the energy at zero temperature equals

$$E = -\frac{\gamma}{2} \quad (27)$$

while the entropy S is $-\frac{\gamma}{2} \log(2) + \lim_{n \rightarrow 0} \frac{\Psi_n}{n}$. Therefore:

$$S = \log(2) \frac{\gamma}{2} (\theta_c^2 - 1) + \log(2) (1 - \theta_c) (1 - \log(1 - \theta_c)) \quad (28)$$

where θ_c is given by the equation

$$\exp(-\gamma \theta_c) = 1 - \theta_c. \quad (29)$$

This equation has a single solution $\theta_c = 0$ if $\gamma \leq 1$ and one more non trivial solution if $\gamma > 1$, which corresponds to the maximum. Therefore, at zero temperature, for $\gamma < 1$ the system is in a paramagnetic phase while for $\gamma > 1$ is in a disordered ferromagnetic phase. The transition corresponds to the percolation transition generated by the ferromagnetic links. The entropy S and the order parameter θ_c are plotted in Fig. 1 as a function of the dilution coefficient γ . At the transition value $\gamma = 1$, the first derivatives of both entropy and θ_c are discontinuous.

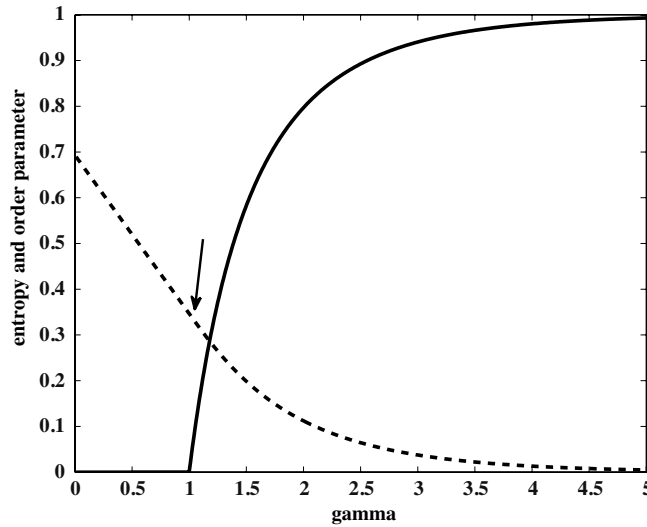


Fig. 1. Entropy (dashed line) and order parameter θ_c (full line) as a function of the dilution coefficient γ at zero temperature and zero magnetic field. The transition is at $\gamma = 1$ where the first derivatives of both entropy and θ_c are discontinuous. An arrow points to the derivative discontinuity of the entropy.

5. Dilute ferromagnet in a magnetic field

In this section, we show how our approach works for the dilute ferromagnetic system in a magnetic field with random locations.

The partition function of this model is

$$Z = \sum_{\#} \exp \left(\beta \sum_{i>j} K_{ij} \sigma_i \sigma_j + \beta \sum_i h_i \sigma_i \right) \quad (30)$$

where K_{ij} are the previously defined quenched variables and the h_i take the value h with probability δ and 0 otherwise. We can rewrite the above expression as $Z = PG$ where

$$G = \sum_{\#} \prod_{i>j} (1 + \tanh(\beta) K_{ij} \sigma_i \sigma_j) \prod_i (1 + \tanh(\beta h_i) \sigma_i) \quad (31)$$

and where

$$P = \exp \left(\frac{\gamma N}{2} \log(\cosh(\beta)) \right) \exp(\delta N \log(\cosh(\beta h))) \quad (32)$$

the equality holds in the sense that $\frac{1}{N} \log(Z)$ coincide in the thermodynamic limit when computed from (30) and from $Z = PG$ with G and P given by (31) and (32).

If one takes into account that

$$\left\langle \prod_{i,\alpha} (1 + \tanh(\beta h_i) \sigma_i^\alpha) \right\rangle = \prod_i \left(1 - \delta + \delta \prod_{\alpha} (1 + \tanh(\beta h) \sigma_i^\alpha) \right) \quad (33)$$

one has that Ψ_n , with respect to (18), contains the extra term

$$\sum_{\sigma} x(\sigma) \log \left(1 - \delta + \delta \prod_{\alpha} (1 + \tanh(\beta h) \sigma^\alpha) \right) \quad (34)$$

which is associated to the magnetic field.

Then, if the correct maximum is found we get

$$\Phi = \lim_{n \rightarrow 0} \frac{\Psi_n}{n} + \frac{\gamma}{2} \log(\cosh(\beta)) + \delta \log(\cosh(\beta h)) = S - \beta E. \quad (35)$$

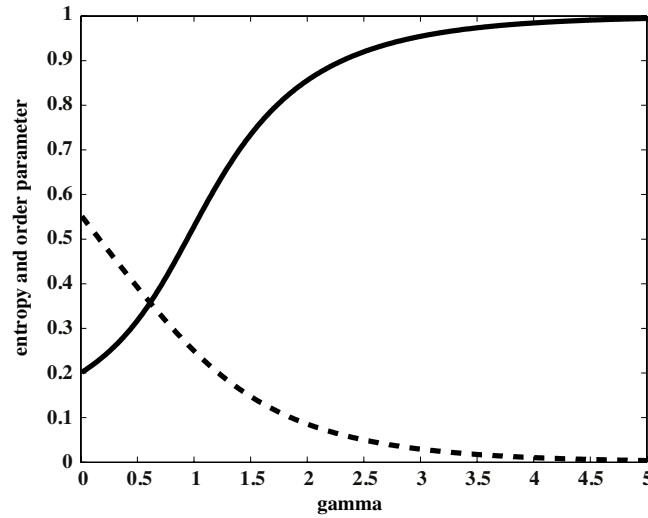


Fig. 2. Entropy (dashed line) and order parameter θ_c (full line) as a function of the dilution coefficient γ at zero temperature and magnetic field concentration $\delta = 0.2$. The transition disappears and the derivatives of both entropy and θ_c are continuous everywhere.

6. Dilute ferromagnet in a magnetic field: zero temperature

In the vanishing temperature limit one has $\tanh(\beta) = \tanh(\beta h) = 1$ and expression Ψ_n becomes

$$\Psi_n = \max_x \left[-\frac{\gamma}{2} + \frac{\gamma}{2} 2^n \sum_{\sigma} x(\sigma)^2 + \sum_{\sigma} x(\sigma) \log \left(1 - \delta + \delta \prod_{\alpha} (1 + \sigma^{\alpha}) \right) - \sum_{\sigma} x(\sigma) \log(x(\sigma)) \right]. \quad (36)$$

The solution can be again found assuming that $2^n - 1$ quantities $x(\sigma)$ take the value $\frac{1-\theta}{2^n}$ and the state with $\sigma^{\alpha} = 1$ for all α takes the value $\frac{1+(2^n-1)\theta}{2^n}$ as in formula (23).

We can compute as usual and expand to the first order in n , then Ψ_n is the maximum over θ of

$$n \log(2) \frac{\gamma}{2} \theta^2 + n \log(2) (1 - \theta) \log(1 - \delta) + n \log(2) \delta + n \log(2) (1 - \theta) (1 - \log(1 - \theta)). \quad (37)$$

If we take into account (35) and the large β equalities $\log(\cosh(\beta)) = \beta - \log(2)$ and $\log(\cosh(\beta h)) = \beta h - \log(2)$, we can write the energy E as

$$E = -\frac{\gamma}{2} - \delta h \quad (38)$$

while the entropy S is

$$S = \log(2) \frac{\gamma}{2} (\theta_c^2 - 1) + \log(2) (1 - \theta_c) \log(1 - \delta) + \log(2) (1 - \theta_c) (1 - \log(1 - \theta_c)) \quad (39)$$

where θ_c is given by the equation

$$(1 - \delta) \exp(-\gamma \theta_c) = 1 - \theta_c. \quad (40)$$

At variance with the zero magnetic field case, this equation has a single solution θ_c . The optimum parameter θ_c is always positive and the transition disappears since the derivatives with respect to γ of both entropy and θ are continuous everywhere. In Fig. 2 we plot entropy and order parameter θ_c as a function of the dilution coefficient γ at magnetic field concentration $\delta = 0.2$.

In Fig. 3 we plot the order parameter θ as a function of the magnetic field concentration δ for 5 different values of γ : $\gamma = 0$, $\gamma = 0.5$, $\gamma = 1$, $\gamma = 1.5$, $\gamma = 2$. At $\delta = 0$, the order parameter θ vanishes only for $\gamma \leq 1$ where spontaneous symmetry is broken.

Finally, in Fig. 4 we plot the entropy as a function of the magnetic field concentration δ for the same 5 different values of γ . The entropy is always smaller than $\log(2)$ except when both γ and δ vanish. Furthermore, the entropy always vanishes when $\delta = 1$ since all spins are oriented along the magnetic field.

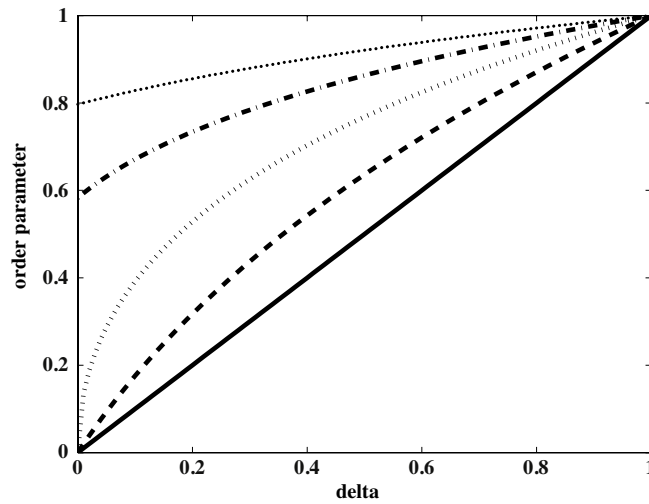


Fig. 3. Order parameter θ_c as a function of the magnetic field concentration δ . The different curves corresponds, starting from below, to dilution coefficient $\gamma = 0, \gamma = 0.5, \gamma = 1, \gamma = 1.5, \gamma = 2$. At $\delta = 0$, the order parameter θ_c vanishes only for $\gamma \leq 1$.

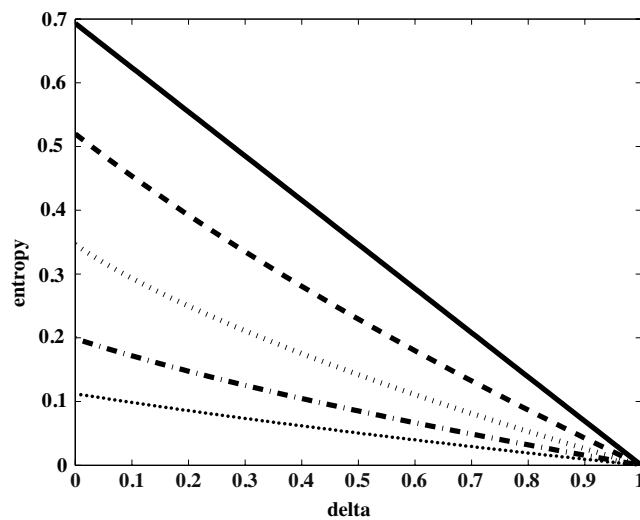


Fig. 4. Entropy as a function of the magnetic field concentration δ . The different curves corresponds, starting from above, to dilution coefficient $\gamma = 0, \gamma = 0.5, \gamma = 1, \gamma = 1.5, \gamma = 2$. At $\delta = 1$ entropy is always vanishing since all spins are oriented along the magnetic field.

7. Discussion

In this paper, following Ref. [5], we used magnetization densities as replica order parameters. The method is tested against the dilute ferromagnet model even in the case in which a randomly located magnetic field is present. We are able to compute exactly the ground state energy and entropy.

When temperature is not vanishing, we only write down the formal solution, while the effective one asks for a correct parametrization of the magnetization densities $x(\sigma)$. In this case in fact, the problem is not mapped into a simple Potts model. Some work in this direction is in progress.

The method can be straightforwardly applied to the dilute spin glass, where we hope to find analogous results. Since, in principle, it can be used for all models, without proliferation of order parameters, we propose it as a general tool in the replica trick context.

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