Exact and approximate solutions for the dilute Ising model

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A B S T R A C T

The ground state energy and entropy of the dilute mean field Ising model is computed exactly by a single order parameter as a function of the dilution coefficient. An analogous exact solution is obtained in the presence of a magnetic field with random locations. Results lead to a complete understanding of the geography of the associated random graph. In particular, we give the size of the giant component (continent) and the number of isolated clusters of connected spins of all given size (islands). We also compute the average number of bonds per spin in the continent and in the islands. Then, we tackle the problem of solving the dilute Ising model at strictly positive temperature. In order to obtain the free energy as a function of the dilution coefficient and the temperature, it is necessary to introduce a second order parameter. We are able to find out the exact solution in the paramagnetic region and exactly determine the phase transition line. In the ferromagnetic region we provide a solution in terms of an expansion with respect to the second parameter which can be made as accurate as necessary. All results are reached in the replica frame by a strategy which is not based on multi-overlaps.

1. Introduction

The study of the dilute mean field Ising model can be tackled using replica approach which has been successfully applied to many disordered systems, the most celebrated being the SK model [1,2]. Nevertheless, the method encounters serious difficulties when applied to many other systems because of a proliferation of replica order parameters as, for example, the multi-overlaps in dilute models [3–5].

In this paper we use a strategy proposed by Monasson [6], which we already applied to the dilute Ising model [7], for which we computed the exact entropy and energy in the vanishing temperature case. The Monasson strategy is very general and shows how $\langle Z^n \rangle$ (Z is the partition function of a generic spin model with disorder) can be expressed in terms of a maximum with respect to the possible values of 2^n positive densities $x(\sigma)$ (where $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$) with the constraint $\sum_\sigma x(\sigma) = 1$.

In the next section, following Ref. [7], we derive the free energy of the dilute Ising model in terms of the densities $x(\sigma)$ for a generic temperature and also in the presence of an external magnetic field with random locations. In Section 3, we consider the zero temperature case, in the absence of magnetic field and we show that the model is formally equivalent to a 2^n Potts model. This observation allows us to express the $x(\sigma)$ in terms of a single order parameter and the exact solution is easily obtained. The results coincide with those found in Ref. [8] where the approach is not based on replicas. We also show that the exactly computed internal energy and entropy allows us to draw conclusions about the geography of the associated random graph. In particular, we determine the size of the giant component (continent), the average size of the isolated clusters of connected spins (islands) and the average number of bonds per spin in the continent and in the islands. Therefore, the solution of the dilute Ising model provides a new approach to these well-studied aspects of random graphs [9].

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In Section 4, we compute exactly the ground state energy and entropy in the presence of a randomly located magnetic field. This allows us for a more complete description of the geography, in particular, the number of isolated clusters of connected spins of all given size (islands) is determined. In Section 5, we extend the scope by considering the same model in the case of a strictly positive temperature, where a second order parameter is needed. We are able to find out the exact solution in the paramagnetic region and exactly determine the phase transition line. In the ferromagnetic region we provide a solution in terms of an expansion with respect to the second order parameter which can be made as accurate as necessary. Conclusions and outlook are in the final section.

2. The dilute ferromagnet and the replica solution

The dilute ferromagnetic Ising model is characterized by a number M of order N of non-vanishing (and positive) bonds connecting pairs of spins. The partition function can be written as

$$Z = \sum_{\#} \exp\left(\beta \sum_{i>j} K_{ij}\sigma_i\sigma_j\right) \tag{1}$$

where the sum $\sum_{\#}$ goes on the 2^N realizations of the N spin variables and the K_{ij} are quenched variables which take the value 1 with probability $\frac{\gamma}{N}$ and 0 otherwise. The dilution coefficient γ may take any positive value and the number M of bonds is $\frac{\gamma}{2}N \pm o(\sqrt{N})$. This is at variance with the fully connected model where the number of non-vanishing bonds is $N(N-1)/2$ and that is the reason why the dilute ferromagnetic has a richer behavior.

An alternative definition of the partition function can be obtained by taking a number of bonds exactly equal to M where M can be both a deterministic number proportional to N or a quenched random variable with average proportional to N and fluctuations of order $\sqrt{N}$. In this context the pair of spin to be connected to a given bond is chosen randomly [4,8]. All these models and model (1) are thermodynamically equivalent, i.e. the intensive thermodynamical quantities are the same.

Models with lesser dilution [10,11] (number of bonds of order N^ϵ with $1 < \epsilon < 2$) have been also considered, but in this case the thermodynamics is the same of the fully connected model.

The problem of evaluating (1) via the replica method, can be approached in many different ways. A possibility is to follow the same path which led to the solution of the SK model using multi-overlaps as replica order parameters. This strategy leads to a proliferation of parameters and it is unable to give exact answers [5]. In Ref. [7], we considered a different strategy, which is based, instead, on the densities $x(\sigma)$ which are 2^n parameters ($\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$ may assume 2^n values) with the constraints $0 \leq x(\sigma) \leq 1$ and $\sum_\sigma x(\sigma) = 1$.

According to Refs. [6,7], $x(\sigma)N$ is the number of spins $\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^n)$ in the configuration $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$. In the $n = 1$ case (single replica), $x(1)$ is the density of positive spins, $x(-1) = 1 - x(1)$ is the density of negative spin and $x(1) - x(-1)$ is the ordinary magnetization.

In Ref. [7] we computed the partition function and, therefore, the free energy F obtaining

$$-\beta F = \lim_{n\to 0} \frac{\Psi_n}{n} + \frac{\gamma}{2}\log(\cosh(\beta)) \tag{2}$$

where Ψ_n is given by

$$\Psi_n = \max_x\left[-\frac{\gamma}{2} + \frac{\gamma}{2}\sum_{\sigma,\tau} x(\sigma)x(\tau)\prod_\alpha(1 + \tanh(\beta)\sigma^\alpha\tau^\alpha) - \sum_\sigma x(\sigma)\log(x(\sigma))\right] \tag{3}$$

where the sum $\sum_{\sigma,\tau}$ goes on the 2^n possible values of the variable σ and the 2^n possible values of the variable τ. The maximum is taken with respect to the 2^n densities $x(\sigma)$ with the constraints $0 \leq x(\sigma) \leq 1$ and $\sum_\sigma x(\sigma) = 1$. The highly non-trivial problem is the maximization of the $x(\sigma)$ which, in principle, can be found by a proper parametrization.

The simplest case turns out to be the high temperature paramagnetic region where the maximum is reached when $x(\sigma) = \frac{1}{2^n}$ for all σ, so that

$$-\beta F = \log(2) + \frac{\gamma}{2}\log(\cosh(\beta)). \tag{4}$$

We will show in Section 5 that the paramagnetic region is the high temperature region given by $\gamma\tanh(\beta) \leq 1$ (see also Refs. [4,8]).

In Ref. [7] we also considered the dilute ferromagnetic system in a magnetic field with random locations. The partition function of this model is

$$Z = \sum_{\#}\exp\left(\beta\sum_{i>j}K_{ij}\sigma_i\sigma_j + \beta\sum_i h_i\sigma_i\right) \tag{5}$$

where K_{ij} are the previously defined quenched variables and the h_i takes the positive value h with probability δ and 0 otherwise.

In this case we get

$$-\beta F = \lim_{n \to 0} \frac{\Psi_n(\delta)}{n} + \frac{\gamma}{2} \log(\cosh(\beta)) + \delta \log(\cosh(\beta h)) \tag{6}$$

where $\Psi_n(\delta)$ is obtained by maximizing an expression which is the same as that in (3) plus the extra term

$$\sum_\sigma x(\sigma) \log \left(1 - \delta + \delta \prod_\alpha (1 + \tanh(\beta h)\sigma^\alpha) \right) \tag{7}$$

where the sum $\sum_\sigma$ goes on the 2^n possible values of the variable σ. Obviously, $\Psi_n(0) = \Psi_n$.

Before trying to compute $\Psi_n(\delta)$ in the general case, we focus on the zero temperature case, which is exactly solvable both in the vanishing (next section) and non-vanishing (Section 4) magnetic field cases. In both cases many conclusions may be drawn concerning the structure of the random graph, both for what concerns the giant component (continent) and the small islands of connected spins.

3. Zero temperature and zero magnetic field

Let us start by considering the case of vanishing temperature and vanishing magnetic field ($\delta = 0$). In the vanishing temperature limit one has $\tanh(\beta) = 1$ and, therefore, $\prod_\alpha (1 + \tanh(\beta)\sigma^\alpha \tau^\alpha)$ becomes $\prod_\alpha (1 + \sigma^\alpha \tau^\alpha)$. This term equals 2^n when all σ^α equals τ^α and vanishes otherwise. Therefore, expression (3) becomes

$$\Psi_n = \max_x \left[-\frac{\gamma}{2} + \frac{\gamma}{2} 2^n \sum_\sigma x(\sigma)^2 - \sum_\sigma x(\sigma) \log(x(\sigma)) \right] \tag{8}$$

which is a standard 2^n-component Potts model [12]. The solution is known and can be found by assuming that $2^n - 1$ quantities $x(\sigma)$ take the value $\frac{1-\theta}{2^n}$ and one takes the value $\frac{1+(2^n-1)\theta}{2^n}$. The state with a different value can be any of the possible 2^n, we assume that is the one with all $\sigma^\alpha = 1$ for all α. We can write

$$x(\sigma) = \frac{1-\theta}{2^n} + \frac{\theta \prod\limits_{\alpha=1}^n (1 + \sigma^\alpha)}{2^n} \tag{9}$$

which satisfy the constraint $\sum_\sigma x(\sigma) = 1$. The order parameter θ has the meaning of a magnetization since $\langle \sigma^\alpha \rangle = \sum_\sigma x(\sigma)\sigma^\alpha = \theta$ for any α and any n (and, therefore, also in the $n \to 0$ limit).

In Ref. [7] we used this solution in order to find the exact expression for the ground state internal energy E

$$E = -\frac{\gamma}{2} \tag{10}$$

while the entropy S was found to be

$$S = \log(2) \max_\theta \left[\frac{\gamma}{2}(\theta^2 - 1) + (1 - \theta)(1 - \log(1 - \theta)) \right]. \tag{11}$$

The maximum is reached in θ_c given by the equation

$$\exp(-\gamma\theta_c) = 1 - \theta_c. \tag{12}$$

This equation has a single non-negative solution $\theta_c = 0$ if $\gamma \le 1$ and one more non-trivial positive solution if $\gamma > 1$ which corresponds to the maximum. Therefore, at 0 temperature, for $\gamma \le 1$ the system is in a paramagnetic phase ($\theta_c = 0$) while for $\gamma > 1$ is in a ferromagnetic phase ($\theta_c > 0$). We will show later that the transition line between the paramagnetic and the ferromagnetic region is $\gamma \tanh(\beta) = 1$ which at vanishing temperature ($\beta = \infty$) gives $\gamma = 1$ here found.

The parameter θ_c is the magnetization of the system, but we will see that it has a simple interpretation in terms of underlying random graph and that its discontinuity corresponds to the percolation transition generated by the ferromagnetic links.

Using Eq. (12) we can rewrite the entropy (11) as

$$S = \log(2) \left[\left(1 - \frac{\gamma}{2}\right) + (\gamma - 1)\theta_c - \frac{\gamma}{2}\theta_c^2 \right]. \tag{13}$$

The entropy S and the order parameter θ_c are plotted in Fig. 1 as a function of the dilution coefficient γ. At the transition value $\gamma = 1$, the first derivative of θ_c and the third derivative of the entropy are discontinuous.

Indeed, these results have an interesting interpretation in terms of random graph geography. In fact, at zero temperature, the magnetization is entirely due to the giant set (continent) of spins all connected, while the small sets of a single or a few interconnected spins isolated from the others (islands) cannot contribute to the magnetization. Since all spins in the
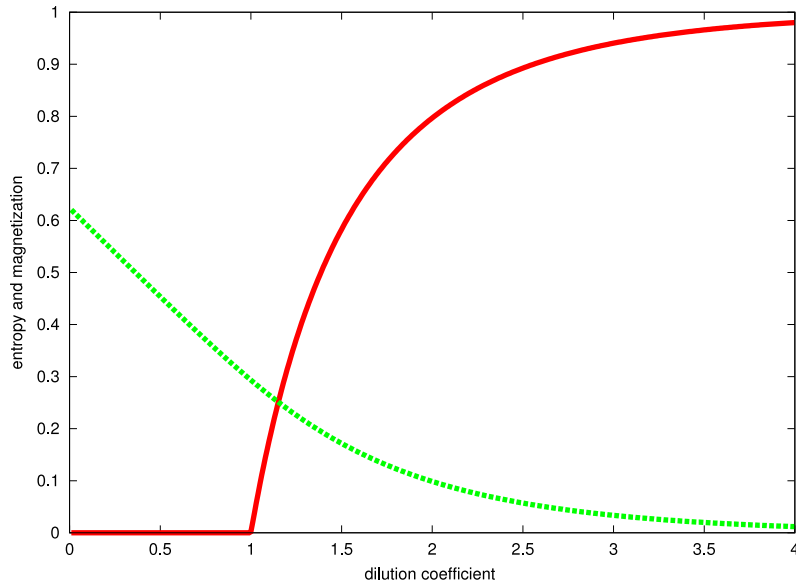
Fig. 1. Entropy (dashed line) and order parameter θ_c (full line) as a function of the dilution coefficient γ at 0 temperature and 0 magnetic field. The transition is at $\gamma = 1$ where the first derivative of θ_c and the third derivative of the entropy are discontinuous.

continent must be directed in the same direction in order to minimize the ground state energy, their magnetization must be 1 and, therefore, their number must be $\theta_c N$. This also means that the total number of spins in the islands must be $(1 - \theta_c)N$. In particular, when $\gamma \leq 1$ there is no continent ($\theta_c = 0$) and all the spin are in islands, on the contrary, all spins are in the continent only when $\gamma \to \infty$ (in this limit $\theta_c = 1$).

It is also possible to count the number of islands, in fact, any of the islands of connected spins can live only in two configurations: all spins up or all spins down. Therefore, any island contribute by a $\log(2)/N$ to the (intensive) entropy. The continent also contributes by a single $\log(2)/N$ but this is irrelevant in the thermodynamical limit. Therefore, the number of islands is simply $\frac{S}{\log(2)}N$.

Let us call $\alpha(l)N$ the number of islands of size l (islands made by a number $l \geq 1$ of interconnected spins which are isolated from all the others). Then, the total number of islands can be written as $\sum_{l \geq 1} \alpha(l) N$. Furthermore, since an island of size l contains by definitions l spins, the total number of spins in the islands can be written as $\sum_{l \geq 1} \alpha(l) l N$.

We have seen that the contribution of any island to the entropy is $\log(2)/N$, therefore, we can write

$$S = \log(2) \sum_{l \geq 1} \alpha(l). \tag{14}$$

Analogously, since the total number of spins in all the islands is $(1 - \theta_c)N$, we can write

$$1 - \theta_c = \sum_{l \geq 1} \alpha(l)l. \tag{15}$$

We can also easily compute the average size $\langle l \rangle$ of the islands (the average number of spins per island) since it is given by the ratio between the total number of spins in all the island $\sum_{l \geq 1} \alpha(l) l N$ divided by the number of islands $\sum_{l \geq 1} \alpha(l)N$. We obtain

$$\langle l \rangle = \frac{(1 - \theta_c) \log(2)}{S}. \tag{16}$$

The average size $\langle l \rangle$ of the islands is plotted in Fig. 2, notice that it has a maximum at the transition $\gamma = 1$ where it equals 2. This means that the average dimension reaches a maximum when the continent begins to exist and then decreases because larger islands are more easily absorbed by the continent.

Another simple consideration allows us to compute the number of bonds ϕN in the islands. In fact, a moment of reflection shows that the number of islands forming loops is irrelevant in the thermodynamical limit. Therefore the spins of an island of size l must be connected by $l - 1$ bonds which implies

$$\phi = \sum_{l \geq 1} \alpha(l) (l - 1) = 1 - \theta_c - \frac{S}{\log(2)}. \tag{17}$$

Since the total number of bonds is $\frac{\gamma}{2}N$, we also have that number of bonds in the continent is $(\gamma/2 - \phi)N$. Finally, we can compute the average number of bonds per spin in the islands as $\phi/(1 - \theta_c)$, while the average number of bonds per spin in the continent is $(\gamma/2 - \phi)/\theta_c$. These two quantities are plotted in Fig. 2. The average number of bonds per spin in the islands
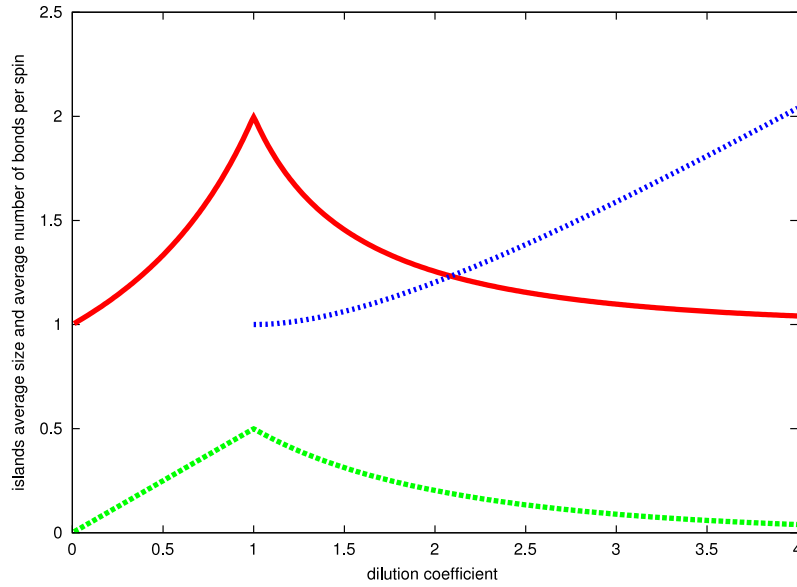
Fig. 2. Average size $\langle l \rangle$ of the islands (full line), average number of bonds per spin in the islands $\phi/(1 - \theta_c)$ (dashed line) and average number of bonds per spin in the continent $(\gamma/2 - \phi)/\theta_c$ (dotted line). The average number per spin in the continent is computed only when the continent exists ($\theta_c > 0$), that is why the dotted line begins when the dilution coefficient γ equals 1.

follows the average dimension of the island and it has a maximum in $\gamma = 1$ where it has the value 0.5. This is coherent with the fact that an island of size two has a single bond. The average number of bonds per spin in the continent is defined only when $\gamma > 1$ because this is the region of parameters where the continent exists. At $\gamma \to 1$ it has its minimal value 1 coherently with the fact that the any of the spin must be connected at least to another spin, which is, in turn connected to a third one and so on.

In the next section we will consider again the case of vanishing temperature, but in the presence of a magnetic field. This will lead to a better understanding of the random graph geography. In fact, we will be able to compute all the $\alpha(l)$, for the moment we just remark that the number $\alpha(1)N$ of islands of size l (isolated spins) is $N \exp(-\gamma)$. This can be computed by considering that $\exp(-\gamma)$ is the thermodynamical limit of $\left(1 - \frac{\gamma}{N}\right)^{N-1}$ which, in turn, is the probability that a spin has no positive bonds connecting to all the others.

4. Zero temperature and non-zero magnetic field

In the vanishing temperature limit one has $\tanh(\beta) = \tanh(\beta h) = 1$ and $\Psi_n(\delta)$ becomes

$$\Psi_n(\delta) = \max_x \left[-\frac{\gamma}{2} + \frac{\gamma}{2} 2^n \sum_\sigma x(\sigma)^2 + \sum_\sigma x(\sigma) \log \left(1 - \delta + \delta \prod_\alpha (1 + \sigma^\alpha) \right) - \sum_\sigma x(\sigma) \log(x(\sigma)) \right]. \tag{18}$$

This is again a Potts model whose solution can be found by assuming that the 2^n densities $x(\sigma)$ have the form (9). With some simple algebra [7] we can compute the ground state internal energy $E(\delta)$

$$E(\delta) = -\frac{\gamma}{2} - \delta h \tag{19}$$

while the entropy $S(\delta)$ is

$$S(\delta) = \log(2) \max_\theta \left[\frac{\gamma}{2}(\theta^2 - 1) + (1 - \theta)\log(1 - \delta) + (1 - \theta)(1 - \log(1 - \theta)) \right]. \tag{20}$$

These two quantities obviously coincide with those computed in the previous section when $\delta = 0$, i.e. $E(0) = E$ and $S(0) = S$. The maximum is reached in $\theta_c(\delta)$ given by the solution of equation

$$(1 - \delta)\exp(-\gamma\theta_c(\delta)) = 1 - \theta_c(\delta). \tag{21}$$

At variance with the zero magnetic field case, this equation has a single non-negative solution $\theta_c(\delta)$ for any value of γ and, therefore, the transition disappears. Also for this quantity one has by definition $\theta_c(0) = \theta_c$ where θ_c is the zero magnetic field magnetization computed in the previous section. Using $\theta_c(\delta)$ given by (21), the entropy (20) can be rewritten as

$$S(\delta) = S(\delta, \theta_c(\delta)) = \log(2) \left[\left(1 - \frac{\gamma}{2}\right) + (\gamma - 1)\theta_c(\delta) - \frac{\gamma}{2}\theta_c(\delta)^2 + (1 - \theta_c(\delta))\log(1 - \delta) \right]. \tag{22}$$

In Fig. 3 we plot the magnetization $\theta_c(\delta)$ as a function of the dilution coefficient γ at three different values of the magnetic field concentration: $\delta = 0$, $\delta = 0.1$ and $\delta = 0.2$. One can easily remark the absence of transition for strictly positive values of δ.
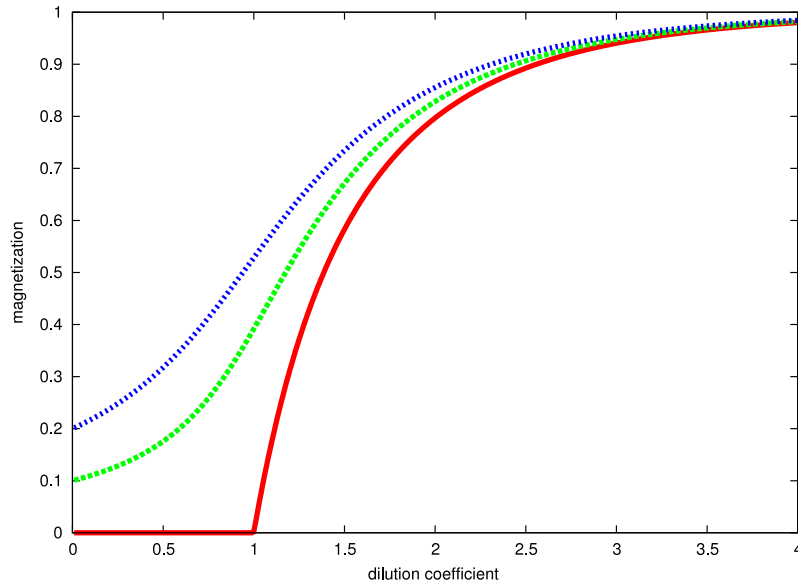
Fig. 3. Magnetization $\theta_c(\delta)$ at zero temperature as a function of the dilution coefficient γ. Three different values of the magnetic field concentration are considered; from below: $\delta = 0$, $\delta = 0.1$ and $\delta = 0.2$. One can easily remark the absence of transition for strictly positive values of δ.

Finally, we compute the entropy response $\sigma(\delta)$ to magnetic concentration δ as

$$\sigma(\delta) = -\frac{dS}{d\delta} = -\frac{\partial S}{\partial \delta} - \frac{\partial S}{\partial \theta_c}\frac{\partial \theta_c}{\partial \delta} = -\frac{\partial S}{\partial \delta} \tag{23}$$

which gives

$$\sigma(\delta) = \frac{1 - \theta_c(\delta)}{1 - \delta}. \tag{24}$$

In order to use the above results to obtain more information concerning the geography of the random graph, we preliminarily observe that the difference between $\theta_c(\delta)$ and θ_c is only due to the effect of the magnetic field on the islands while the magnetization of the spins of the continents remains unchanged and is equal to 1. The same can be said for what concerns the entropy, in fact, the contribution to the entropy coming from the continent remains equal to zero.

It is easy to understand the effect of the magnetic field on a given island: if none of the component spins get a positive magnetic field, the island can still live in two configurations: all spins up or all spins down. For this reason its contribution to the entropy remains $\log(2)/N$ and the contribution to the magnetization remains equal to zero. But if one or more spins of the island get a positive magnetic field, then all spins of the island must be up, which implies that the contribution to the entropy vanishes and the contribution to the magnetization rises from 0 to l/N where l is the size of the island.

The interesting point is that while magnetization $\theta_c N$ has a geometrical interpretation, simply representing the number of spins in the continent, the magnetization $\theta_c(\delta)N$ is the sum of the number of spins in the continent plus the number of all spins in the islands frozen by the magnetic field. The entropy $S(\delta)N$, on the contrary, is simply the number of unfrozen islands multiplied by $\log(N)$.

We notice that the probability that none of the spins of an island of size l get a magnetic field is $(1 - \delta)^l$ while the probability that at least one has a magnetic field is $1 - (1 - \delta)^l$. The above arguments allow us to say that the number of islands of size l which are not frozen is $\alpha(l)(1 - \delta)^l N$. Then, we are allowed to write

$$S(\delta) = \log(2) \sum_{l \geq 1} \alpha(l)(1 - \delta)^l \tag{25}$$

$$1 - \theta_c(\delta) = \sum_{l \geq 1} \alpha(l)(1 - \delta)^l l \tag{26}$$

which coincide with the analogous quantities in the previous section when $\delta = 0$.

It is easy to verify that (25) and (26) are coherent with Eq. (24) for the entropy response. Furthermore, they give much more information, in fact, expanding $1 - \theta_c(\delta)$ for small values of $1 - \delta$ we have that the coefficients of the expansion are $\alpha(l)\, l$. As a consequence, we can have in principle all the density numbers $\alpha(l)$ of islands of size l.
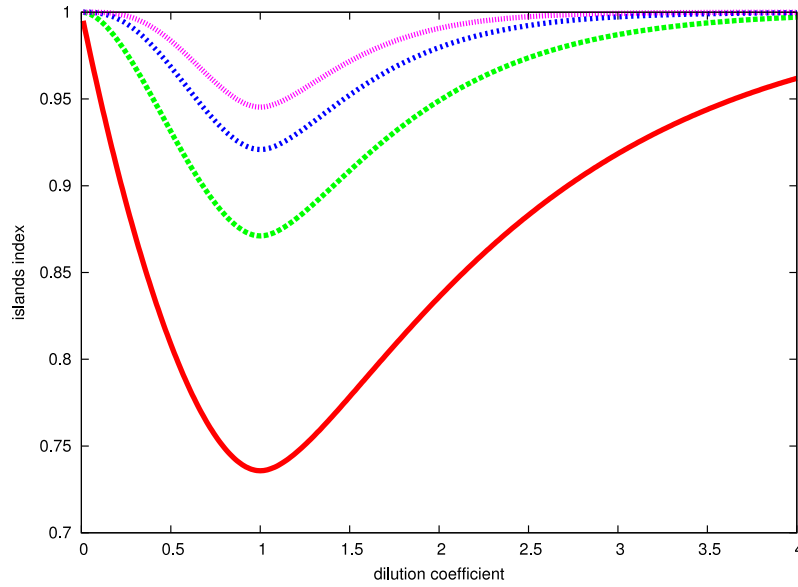
Fig. 4. Number of islands $r(m)$ of size $l \leq m$ divided by the total number of islands for $m = 1, 2, 3$ and 4 (from below). The index approaches 1 for larger values of m. The lower values of $r(m)$ at $\gamma = 1$ mean that the largest islands appear around the transition.

In order to perform this expansion we can use iteratively Eq. (21) and we obtain

$$
\begin{aligned}
\alpha(1) &= \exp(-\gamma) \\
\alpha(2) &= \frac{1}{2}\gamma \exp(-2\gamma) \\
\alpha(3) &= \frac{1}{2}\gamma^2 \exp(-3\gamma) \\
\alpha(4) &= \frac{2}{3}\gamma^3 \exp(-4\gamma)
\end{aligned}
\tag{27}
$$

while, in general, for any $l \geq 2$, one can deduce from (21) the recursive formula

$$
\alpha(l) = \frac{1}{l}\exp(-\gamma)\sum_{k=1}^{l-1}\frac{1}{k!}\gamma^k \sum_{\{s_i\}}\prod_{i=1}^{k}\alpha(s_i)s_i
\tag{28}
$$

where the second sum goes on all the possible values of the k strictly positive integer numbers s_i with the constraint $s_1 + s_2 + \cdots + s_k = l - 1$. This recursive formula gives

$$
\alpha(l) = \frac{l^{l-2}}{l!}\gamma^{l-1}\exp(-\gamma l)
\tag{29}
$$

indeed, we are not able to rigorously prove (29) from (28), nevertheless, we checked that it is exact for all l between 2 and 10. Much more important, we were able to numerically compute $\sum_{l\geq 1}\alpha(l)$ and $\sum_{l\geq 1}\alpha(l)\,l$ and verify that they coincide, respectively, with S and $1 - \theta_c$. In particular, $\sum_{l\geq 1}\alpha(l)\,l$ equals 1 for any $\gamma \leq 1$.

Notice that the expression $\alpha(1) = \exp(-\gamma)$ was already found in the previous section using very simple arguments. Also notice that in the $\gamma \to 0$ limit all $\alpha(l)$ vanish except $\alpha(1)$ which goes to 1. This is simply due to the fact that in this limit all spins are isolated which means that only islands of size 1 exist.

The description of the geography of the random graph is now completed, since we are able, for any value of γ, to give both the dimension of the continent θ_c and the number $\alpha(l)$ of islands of any fixed size. Some relevant intuition about this geography can be obtained considering the island index $r(m)$ defined as

$$
r(m) = \frac{\sum_{l=1}^{m}\alpha(l)}{\sum_{l\geq 1}\alpha(l)} = \frac{\log(2)}{S}\sum_{l=1}^{m}\alpha(l)
\tag{30}
$$

which is the number of islands of size $l \leq m$ divided by the total number of islands. The index $r(m)$ goes to 1 for large m. In Fig. 4 we plot this quantity for $m = 1, 2, 3$ and 4. The index approaches 1 for large values of m but, for $m = 4$ it is still about 0.94 when $\gamma = 1$. The lower values of $r(m)$ at $\gamma = 1$ mean that the largest islands appear around the transition.

5. Positive temperature

For a positive temperature, in the absence of magnetic field, the free energy is given by (2) once the proper maximum in (3) is found. We have seen that in the case of vanishing temperature the exact solution can be found because the problem is equivalent to a Potts model, on the contrary, for positive temperature, there is not an analogous equivalence. Therefore, we have to find the correct parametrization of the $x(\sigma)$ making a decision motivated by the physics. First of all we remark that for positive temperature, the islands cannot contribute to the magnetization, for the same reason why they do not contribute in the zero temperature case. In fact, being isolated clusters of spins, they can be orientated in both directions. So, we can already conclude that only the spins in the continent may contribute to the magnetization. Let us assume that the magnetization of a spin in the continent is λ, then it is natural to assume that

$$x(\sigma) = \frac{1 - \theta_c}{2^n} + \frac{\theta_c \prod_{\alpha=1}^{n} (1 + \lambda \sigma^\alpha)}{2^n} \tag{31}$$

where θ_c is the continent size given by (12) and $0 \le \lambda \le 1$ is the newly introduced second order parameter to be maximized. Notice that the size of the continent is determined only by the geometry induced by the dilution coefficient, that is why it equals θ_c as found in (12) when the zero temperature case was discussed. Also notice that the total magnetization is $m = \theta_c \lambda$, being simply the product of the magnetization of a spin in the continent λ and their density θ_c. In fact,

$$m = \sum_\sigma x(\sigma) \sigma^\alpha = \theta_c \lambda \tag{32}$$

for any of the replicas, i.e. any $1 \le \alpha \le n$.

It is clear, at this point, that the really crucial ansatz is that the magnetization is a self-averaging quantity, otherwise we should chose an expression analogous to (31) but with a different λ_α correspondingly to any σ^α. This breaking of the symmetry would have implied a different magnetization m_α for any possible $1 \le \alpha \le n$.

With the choice (31) we have $\Psi_n = \max_\lambda \Phi_n(\lambda)$, where

$$\Phi_n(\lambda) = \frac{\gamma}{2} \theta_c^2 [(1 + \lambda^2 \tanh(\beta))^n - 1] - \sum_\sigma x(\sigma) \log(x(\sigma)) \tag{33}$$

and where $x(\sigma)$ is given by (31) and the sum goes on all the 2^n possible realizations of σ.

In the zero temperature case ($\tanh(\beta) = 1$) and vanishing n, we can easily check that the maximum is reached in $\lambda = 1$ and we recover the zero temperature solution of Section 2. This check can be performed by simply verifying that the derivative with respect to λ of $\Phi_n(\lambda)$ equals 0 when $\lambda = 1$, $\tanh(\beta) = 1$ and $n \to 0$.

To find the proper maximum and perform the $n \to 0$ limit is much more difficult when the temperature is positive, i.e. $\tanh(\beta) < 1$. The problem is that we were unable to find an analytic expression for any n of the sum $\sum_\sigma x(\sigma) \log(x(\sigma))$ when $0 < \lambda < 1$. Nevertheless, in order to find the behavior of the free energy in the paramagnetic region where $\lambda = 0$ and just around the transition where $\lambda \ll 1$ it is sufficient to expand Φ_n with respect to λ. The minimum order of the expansion is the fourth, in fact, lower orders would be unable to provide a maximum.

Let us rewrite $x(\sigma) = \frac{1}{2^n}(1 + y(\sigma))$ where $y(\sigma) = \theta_c \left(\prod_{\alpha=1}^{n}(1 + \lambda \sigma^\alpha) - 1 \right)$, then notice that $y(\sigma)$ is a polynomial of λ with no terms of order 0. Then we can expand $\sum_\sigma x(\sigma) \log(x(\sigma))$ to the fourth order in $y(\sigma)$, the approximate expression can be easily computed in a compact analytic form in terms of n and λ and coincides with $\sum_\sigma x(\sigma) \log(x(\sigma))$ for all terms of order 4 or less of the expansion in λ.

This analytic approximation can be inserted in (33) by replacing $\sum_\sigma x(\sigma) \log(x(\sigma))$ and we obtain an expression whose expansion coincides with that of $\Phi_n(\lambda)$ up to terms of order 4 in λ. At this point, if we retain only these coinciding terms and we expand to first order in n, we obtain

$$\Phi_n(\lambda) \simeq n\theta_c^2 \left[\frac{\lambda^2}{2} (\gamma \tanh(\beta) - 1) - \frac{\lambda^4}{12} (3\gamma \tanh(\beta)^2 - 3 + 6\theta_c - 2\theta_c^2) \right] + n \log(2). \tag{34}$$

The maximum is reached in $\lambda = 0$ for $\gamma \tanh(\beta) \le 1$ and in $\lambda = \lambda_c$ given by

$$\lambda_c^2 = \frac{\gamma \tanh(\beta) - 1}{\gamma \tanh(\beta)^2 - 1 + 2\theta_c - \frac{2}{3}\theta_c^2} \tag{35}$$

when $\gamma \tanh(\beta) > 1$. Notice that in this region $\gamma \tanh(\beta)^2 \ge \frac{1}{\gamma}$, therefore the denominator in (35) is larger than $\frac{1-\gamma}{\gamma} + 2\theta_c - \frac{2}{3}\theta_c^2$ which, in turn, is larger than 0.

The conclusion is that the magnetization vanishes when $\gamma \tanh(\beta) \le 1$ and equals $m_c = \theta_c \lambda_c$ when $\gamma \tanh(\beta) > 1$ which implies that we have exactly determined the transition line $\gamma \tanh(\beta) = 1$.

Inserting (35) in (34) we obtain

$$-\beta F = \frac{\gamma}{2} \log(\cosh(\beta)) + \log(2) + \frac{\theta_c^2}{4} \frac{(\gamma \tanh(\beta) - 1)^2}{\gamma \tanh(\beta)^2 - 1 + 2\theta_c - \frac{2}{3}\theta_c^2} \tag{36}$$

which gives the free energy in the low (but not too low) temperature region $\tanh(\beta) > 1$. On the contrary, inserting $\lambda = 0$ in (34), we obtain

$$- \beta F = \frac{\gamma}{2} \log(\cosh(\beta)) + \log(2) \tag{37}$$

which gives the exact free energy in the high temperature region $\tanh(\beta) \leq 1$ where the magnetization vanishes.

The free energy in the ferromagnetic region can be computed at any necessary precision in a straightforward manner, it is sufficient in fact to expand $\sum_\sigma x(\sigma) \log(x(\sigma))$ to the higher orders in λ. In fact, at any order, the expansion is a compact analytic expression of λ and n.

Finally, we remark that in the limit $\beta \to 0, \gamma \to \infty$ and $\gamma \beta = \tilde{\beta}$ one recovers the fully connected mean field Ising model. In this limit, our expansion gives $\theta_c = 1, m_c = \lambda_c, m_c^2 = 3(\tilde{\beta} - 1)$ and $-\beta F = \frac{3}{4}(\tilde{\beta} - 1)^2 + \log(2)$ which is the mean field Ising model solution in the same approximation.

6. Discussion and outlook

In this paper we obtained the exact solution of the dilute Ising Model in the case of a vanishing temperature even in the presence of a magnetic field. The solution gives a complete description of the geography of the associated random graph, not only the size of the continent but also the number of isolated clusters of connected spins of all given size (islands) and the average number of bonds per spin both in the continent and in the islands. We were also able to find out the exact solution in the paramagnetic region and exactly determine the phase transition line. On the contrary, in the positive temperature ferromagnetic region, our solution (31) is the correct one only if the magnetization λ_c of the spins in the continent is self-averaging in the thermodynamic limit. This assumption is not proven and the possibility that the magnetization has symmetry-broken characterization remains open, nevertheless, since at zero temperature there is no breaking and since the system is not frustrated, we are quite confident that our choice is correct. Furthermore, the ferromagnetic free energy for positive temperature is only given in terms of an expansion with respect to the order parameter λ and, although the expansion can be made as accurate as necessary, the full and explicit free energy is still to be computed, if this is possible.

Finally, we would like to mention that the present strategy can be straightforwardly applied to the dilute spin glass, but, unfortunately, it seems much more difficult to find analogous results.

Acknowledgments

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