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Critical properties of the SIS model dynamics on the Apollonian network

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Abstract. We present an analysis of the classical SIS (susceptible–infected–susceptible) model on the Apollonian network which is scale free and displays the small word effect. Numerical simulations show a continuous absorbing-state phase transition at a finite critical value λ_c of the control parameter λ. Since the coordination number k of the vertices of the Apollonian network is cumulatively distributed according to a power-law $P(k) \propto 1/k^{\eta-1}$, with exponent $\eta \simeq 2.585$, finite size effects are large and the infinite network limit cannot be reached in practice. Consequently, our study requires the application of finite size scaling theory, allowing us to characterize the transition by a set of critical exponents $\beta/\nu_\perp$, $\gamma/\nu_\perp$, $\nu_\perp$, β. We found that the phase transition belongs to the mean-field directed percolation universality class in regular lattices but, very peculiarly, is associated with a short-range distribution whose power-law distribution of k is defined by an exponent η larger than 3.

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Keywords: critical exponents and amplitudes (theory), critical exponents and amplitudes (experiment), epidemic modelling

Contents

1. Introduction 2
2. The Apollonian network 3
3. The SIS model and its numerical implementation 4
4. Critical behavior 6
5. Discussion and conclusions 10

Acknowledgments 11

References 11

1. Introduction

In the last few years the scientific community has recognized that many real-world networks show complex topological properties such as the small word effect, related to a very short minimal path between vertices, and the scale-free property, which is related to the power-law nature of the cumulative distribution of the number k of contacts ($P(k) \propto 1/k^{\eta-1}$). These complex properties have important implications in the study of real processes in these networks, such as virus spreading in computers, sharing of technological information and diffusion of epidemic diseases, to name just a few. For this reason, there is a great deal of interest in experimental, analytical and numerical study of the dynamical processes in these networks. For instance, in [1]–[3] the problem has been studied through the analysis of real data (such as real computer virus spreading), and compared with analytical and numerical studies of theoretical processes on scale-free networks such as Watts–Strogatz and Barabási–Albert [4, 5].

Among complex networks, the Apollonian network is a particularly useful theoretical tool, since it belongs to a particular class of networks which are scale free, display the small-world effect, can be embedded in a Euclidean lattice and show space filling as well as matching graph properties. Therefore, in spite of its deterministic nature, it shares the most relevant characteristics of real-world networks.

In this work we focus our attention on the statistical properties of the SIS (susceptible–infected–susceptible) model [6]–[9], performing an extensive numerical simulation of its dynamics on the Apollonian network (see [10]–[14] for recent literature on the SIS model in scale-free networks).

Since the coordination number of the vertices of the Apollonian network is distributed according to a power-law with exponent $\eta \simeq 2.585$, finite size effects are large and the infinite network limit cannot be reached in practice. Therefore, our study required the

J. Stat. Mech. (2013) P05003

application of finite size scaling theory, which has allowed us to determine the threshold λ_c of the continuous phase transition between a stationary active state and an absorbing one, characterizing the transition by a set of critical exponents $\beta/\nu_\perp$, $\gamma/\nu_\perp$, $\nu_\perp$, β. We found that the phase transition belongs to the mean-field directed percolation universality class in regular lattices but, very peculiarly, corresponds to a power-law distribution of k with an exponent η larger than 3 (or to a short-range distribution). Recalling that, according to the mean-field approximation, the critical behavior of the system depends on the distribution exponent η, three regions are here identified: for $\eta < 2$, there is no transition for a finite λ; for $2 < \eta < 3$, the critical exponents are $\beta = 1/(\eta - 2)$ and $\nu_\perp = (\eta - 1)/(\eta - 2)$; and for $\eta > 3$, $\beta = 1$ and $\nu_\perp = 2$ [15]–[19]. These last critical exponents also hold if the coordination number is short-range distributed. The results in this paper, concerning the SIS model confirms previous results [20] concerning the CP (contact process) on the Apollonian network.

This paper is organized as follows. In section 2 the Apollonian network model is described. Section 3 deals with the SIS model and explains the numerical procedure that we have implemented to realize the associated dynamics. The results of our simulations are shown in section 4, where the information concerning the critical parameter and the critical exponents using finite scaling analysis is extracted and analyzed. Finally, section 5 is devoted to some concluding remarks.

2. The Apollonian network

Apollonius of Perga was a Greek geometer and astronomer who lived around the year 200 BC. Very little is known about his life, but his contributions to geometry and astronomy are, on the contrary, very well known [21]. One of the problems of Apollonius was to find a circle in a plane that is tangent to three objects: points, lines, or circles, with special attention to the case where the three objects are all circles. Apollonius of Perga posed and solved this problem in his work called *Tangencies*. Unfortunately, *Tangencies* has been lost, and only a report of his work by Pappus of Alexandria is left [22].

An immediately related problem is the space filling packing of circles in a two-dimensional space. This packing is constructed by starting with three mutually touching circles, whose interstice is a curvilinear triangle to be filled. Then a circle is inscribed, touching all the sides of this curvilinear triangle. We call this stage the zero generation ($n = 0$); for subsequent iterations we indefinitely repeat the process for all the newly generated curvilinear triangles. In the limit of infinite iterations, the well-known two-dimensional Apollonian packing is obtained.

From the problem of Apollonian packing, Andrade *et al* [23] introduced the Apollonian network, also independently proposed by Doye and Massen [24]. In the Apollonian network, vertices are associated to the centers of the circles and two vertices are connected if the corresponding circles are tangential. The resulting graph physically corresponds to the force network of a dense granular packing (see figure 1), similar to that introduced by Dodds [25] for random packing which has been also used in the context of porous media [26]. Its main properties are summarized below.

(a) It is *scale free*. The number of vertices at generation n is $N_n = 3 + (3^{n+1} - 1)/2$. According to this definition, the generation $n = 0$ corresponds to a network with four vertices. The number of sites with coordination k is $m(k, n)$, which equals 3^{n-s}

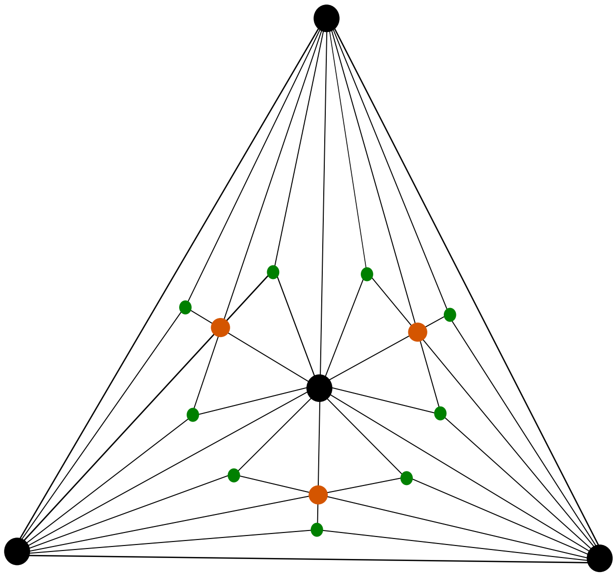

Figure 1. Apollonian network construction for generations $n = 0$ (black, larger rims), $n = 1$ (orange, intermediate rims) and $n = 2$ (green, smaller rims).

if $k = 3 \times 2^s$ with $s = 0, \ldots, n$; equals 3 (the three corners) if $k = 1 + 2^{n+1}$; and equals 0 otherwise. According to these facts, the cumulative distribution $P(k) = \sum_{k' \geq k} m(k, n)/N_n$ exhibits, for large values of N_n, a power-law behavior, i.e., $P(k) \propto 1/k^{\eta-1}$, with $\eta = 1 + \ln(3)/\ln(2) \simeq 2.585$ [27].

(b) It displays the *small-world effect*. The average length $\langle \ell \rangle$ of the shortest path between two vertices grows as $\langle \ell \rangle \propto [\ln(N_n)]^{\phi}$, with $\phi \simeq 3/4$ [23]. Moreover, the clustering coefficient, in the limit of large N, is close to $C = 0.828$ [23], which is a large value when compared to the clustering coefficient of sport groups sharing on Facebook ($C = 0.715$), and to the movie actors collaboration (0.79). Therefore, since $\langle \ell \rangle$ grows sub-logarithmically and C is close to unity, the Apollonian network exhibits the small-world effect [5].

Because of the coincidence of the scale-free properties and the small word effect, the Apollonian network has been shown to be useful to describe biological and physical systems (see, for example, [28]–[31]). Very recently, it has also been used to describe epidemic processes [20], being, in many respects, very close to the real situation of human networks.

3. The SIS model and its numerical implementation

The SIS model is probably the simplest process able to describe the propagation of an infection in a population when transmission is by contact, and infected individuals may spontaneously recover after a certain time. What can be defined as a contact depends on the disease: it can be a sexual contact, as in HBV (hepatitis B virus), or simply physical proximity, as in SARS (severe acute respiratory syndrome). The typical assumption concerning transmission is that an individual may move from the healthy to the infected group when they come into contact with one or more infected persons. Contacts result in infection only at a given rate; for instance, only a small fraction of sexual contacts result in the transmission of HBV. Once someone is infected, it takes some random time to move back to the healthy group, and the individual can eventually be infected again by contact (no immunization is assumed).

In this paper we will assume that in a time step each individual occupies a site (vertex) of the network. If they are healthy (inactive) and one or more connected sites in the network are occupied by infected (active) persons, they will be infected with probability λ (control parameter). On the other hand if they are infected, they will simply turn healthy. The competition leads to a dynamics characterized by a critical point λ_c: if $\lambda > \lambda_c$, the infection propagates and the fraction of infected individuals will tend to be positive constant. Otherwise, if $\lambda < \lambda_c$, the infection disappears (absorbing state) and the fraction of infected individuals goes to zero. Therefore, the critical control parameter λ_c is the epidemic threshold from the active to the absorbing state, i.e., at the λ_c the system exhibits a phase transition. The stationary density of infected individuals ρ_D is the order parameter which vanishes in the absorbing state and is strictly positive in the active one.

The simulation dynamics has been realized according to the follows steps.

(a) It is assumed that each vertex is occupied by a single individual. Therefore, at each time step t, a vertex i is unequivocally associated to the variable $\sigma_i(t)$, which takes either the value 0 if the individual is inactive, or the value 1 if they are active.

(b) At the initial time $t = 0$, half of the vertices on the Apollonian network, randomly chosen, are infected, while the remaining half are left healthy.

(c) At each time step, one vertex is randomly chosen and its state σ is verified. If $\sigma = 0$ and the vertex is connected to one or more infected vertices, it is updated to $\sigma = 1$ with probability λ. Otherwise, if $\sigma = 1$, it is updated to $\sigma = 0$ with probability 1.

(d) At any time t, we compute the density

$$\rho_D(t) = (1/N)\sum_{i=1}^{N}\sigma_i(t). \tag{1}$$

If the system reaches the absorbing state, i.e, $\rho_D(t) = 0$, we replace a randomly chosen healthy individual with an infected one in order that the simulation may continue (see [32] for rationale and details).

For any value of the network size N and the infection probability λ the dynamics is iterated (items (c) and (d) are repeated) as long as we observe that the system relaxes to a steady state. We find that $2000 \times N$ steps is always large enough, where a step is here defined as the updating of a single vertex randomly chosen. After relaxation, the dynamics continues and we compute the quantities as time averages on a lapse of $2000 \times N$ steps. We repeat the procedure for n_s samples, where n_s equals at least 500. More precisely, n_s is 16 000, 8000, 4000, 2000, 1000, 500 for $N = 16$, 43, 124, 367, 1096, 3283, respectively, and our quantities are then further averaged over all samples (in section 4, the final averages will be indicated). The largest size $N = 9844$ ($n_s = 500$) is only considered for the precise determination of the critical point λ_c (figure 5) and critical exponent $1/\nu_\perp$ (figure 6).

J. Stat. Mech. (2013) P05003

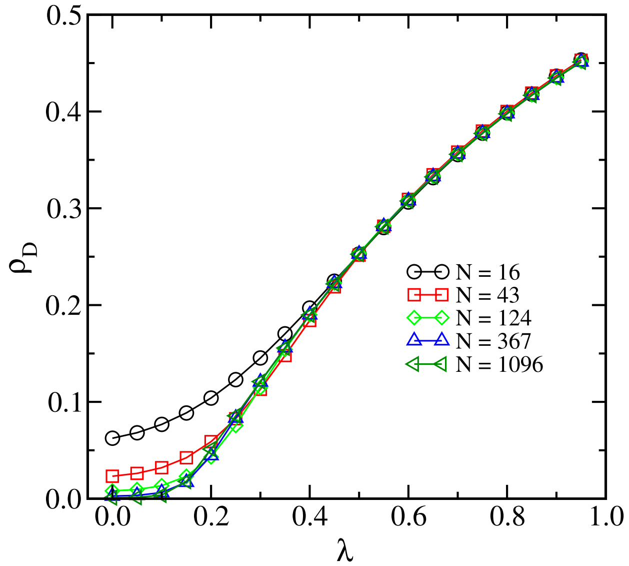

Figure 2. The density of infected individuals ρ_D against the infection probability λ for different network sizes. A clear transition appears when N is increased.

4. Critical behavior

In this section we show the result of our simulations of the SIS model on the Apollonian network. Finite size scaling analysis is performed on different quantities in order to estimate the critical point and the critical exponents $\beta/\nu_\perp$, $\nu_\perp$, $\gamma/\nu_\perp$ and β.

In figure 2 we show a typical plot of the average density $\rho_D = \langle \rho_D \rangle$ against the infection probability λ for different network sizes N. The set of curves indicates a clearly continuous phase transition with the threshold of transition identified by the critical point λ_c when ρ_D approaches zero. This phase transition may be also observed in figure 3, where the semi-log plot shows a typical sigmoidal behavior of a continuous phase transition characterized by a curve with the critical point located at its inflection point [33].

For a continuous phase transition, the critical point λ_c of the system can be computed [34] by a finite size scaling analysis of the variance of the order parameter given by

$$\chi = N(\langle \rho_D^2 \rangle - \langle \rho_D \rangle^2), \tag{2}$$

which is plotted in figure 4 against λ for different network sizes N. For any N, the function χ has a maximum at λ^*. When N tends to infinity the maximum diverges and λ^* approaches the critical point λ_c. Note that for large values of λ all χ curves coincide.

Figure 5 shows how λ^* depends on the system size. More precisely, the first panel shows λ^* against the inverse of the system size $1/N$, while the second panel shows $\ln(\lambda^*)$ against $\ln(1/N)$. The critical threshold can then be obtained by the asymptotic limit $1/N \to 0$.

The exponent λ_c was best determined by the fit $\lambda = a + cN^{-b}$, where a represents λ_c and b the exponent $1/\nu_\perp$. The best fit values of these three parameters are $a = \lambda_c = 0.099(15)$, $c = 1.95$ and $b = 1/\nu_\perp = 0.46$. The fit was performed using data from the first panel (λ^*

J. Stat. Mech. (2013) P05003

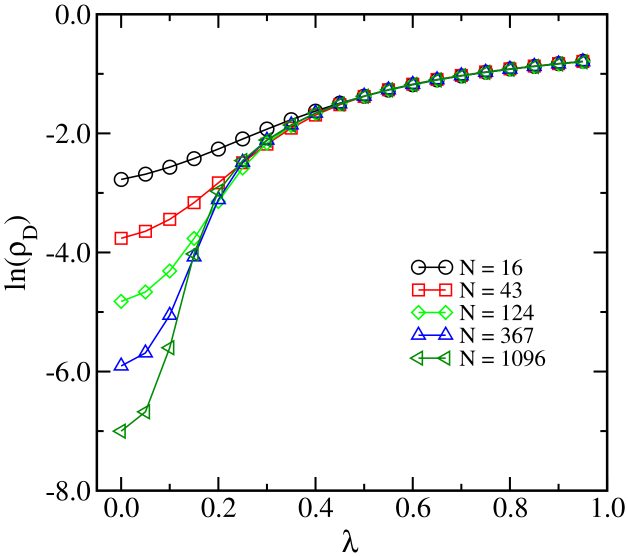

Figure 3. The density of infected individuals ρ_D versus the infection probability λ for several network sizes, showing a sigmoidal shape and a marked inflection point characterizing the critical behavior.

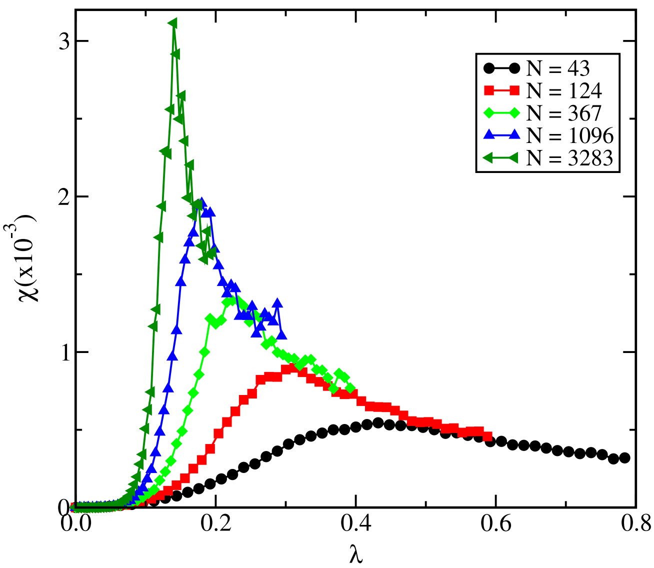

Figure 4. The behavior of the density fluctuation χ versus the infection probability λ for several network sizes. The value of χ is multiplied by a factor of 10^{-3}.

against the inverse of the system size $1/N$) and then depicted in the second panel to confirm the result.

In order to better estimate the critical exponent $\nu_\perp$ we considered again the relation [35]

$$\lambda^* - \lambda_c \sim \left(\frac{1}{N}\right)^{1/\nu_\perp}, \tag{3}$$

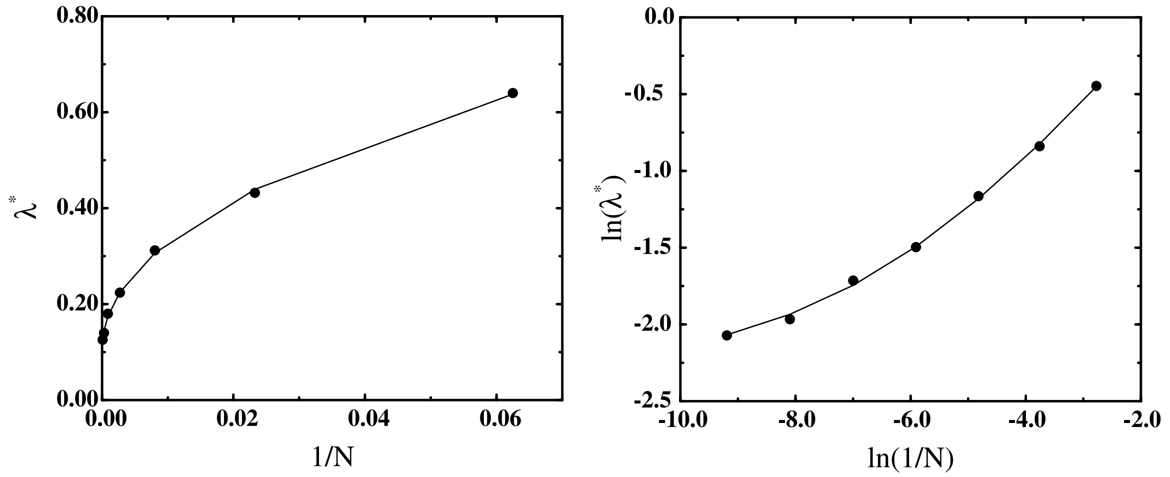

Figure 5. The profile of the recovery probability λ as a function of the inverse network size. In the limit $1/N \to 0$ we obtain, by a second order polynomial fit of data in the first panel, $\lambda_c = 0.099(15)$. In the second panel, the best fit is tested against data on a log–log scale.

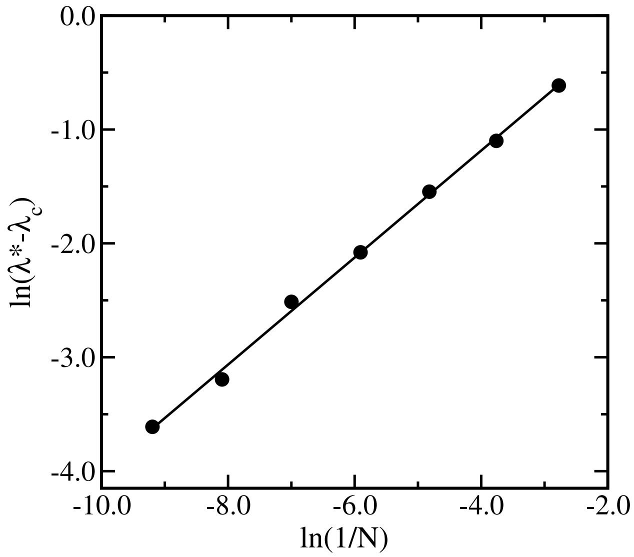

Figure 6. The scaling behavior of $\lambda^* - \lambda_c$ as a function of the inverse network size, allowing us to estimate $1/\nu_\perp$. The linear fit yields $1/\nu_\perp = 0.47(1)$.

using the value of λ^* previously computed (figure 5). In figure 6 we plot our data, the linear fit of which yields $1/\nu_\perp = 0.47(1)$.

Figure 7 shows the best data collapse according to the following critical parameters: $1/\nu_\perp = 0.48(2)$, $\gamma/\nu_\perp = 0.39(4)$ and $\lambda_c = 0.099(15)$. To obtain this data collapse we have used our previous result concerning λ_c, and we have determined best values of 0.39(4) for $\gamma/\nu_\perp$ and 0.48(2) for $1/\nu_\perp$. We remark that this last value is in complete agreement with the previous result obtained by the linear fit depicted in figure 6.

In figure 8 we show the collapse of the order parameter data. In this case, we obtain $1/\nu_\perp = 0.46(6)$, $\beta/\nu_\perp = 0.56(7)$ and $\lambda_c = 0.099(13)$. These new values for $1/\nu_\perp$ and λ_c are

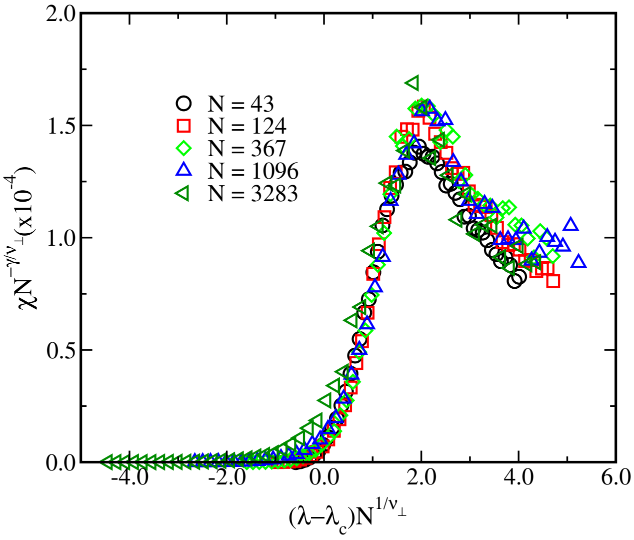

Figure 7. Data collapse of fluctuations of the order parameter data. Using $\lambda_c = 0.099(15)$ previously computed, the best data collapse is given by the following critical parameters: $1/\nu_\perp = 0.48(2)$, $\gamma/\nu_\perp = 0.39(4)$.

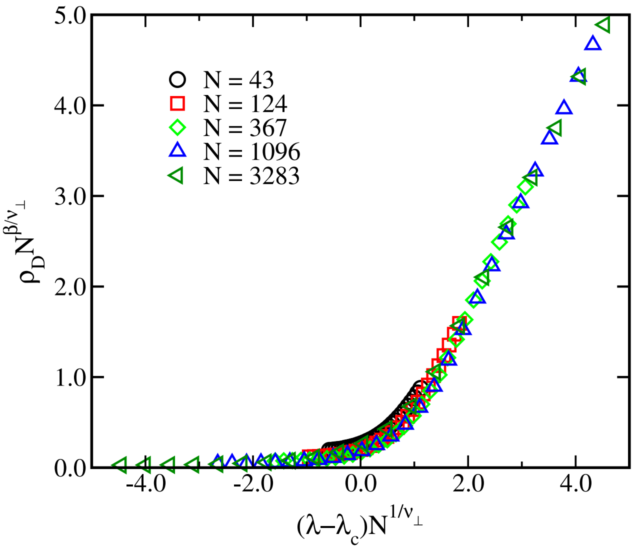

Figure 8. Data collapse of the order parameter. The best data collapse is given by the following critical parameters: $1/\nu_\perp = 0.46(6)$, $\beta/\nu_\perp = 0.56(7)$ and $\lambda_c = 0.099(13)$.

in complete agreement with the values previously estimated. Therefore, we have the final values $\lambda_c = 0.099(14)$ and $\nu_\perp = 0.47(4)$, as average values of all the independent estimates.

Finally, the exponent β is obtained by the scaling relation $\rho_D \sim (\lambda - \lambda_c)^\beta$ [36], which holds near the critical point. A linear fit of our data gives $\beta = 1.3(3)$. The log–log plot of figure 9 depicts the power-law behavior of the order parameter as a function of $\lambda - \lambda_c$ for a network of size $N = 3283$.

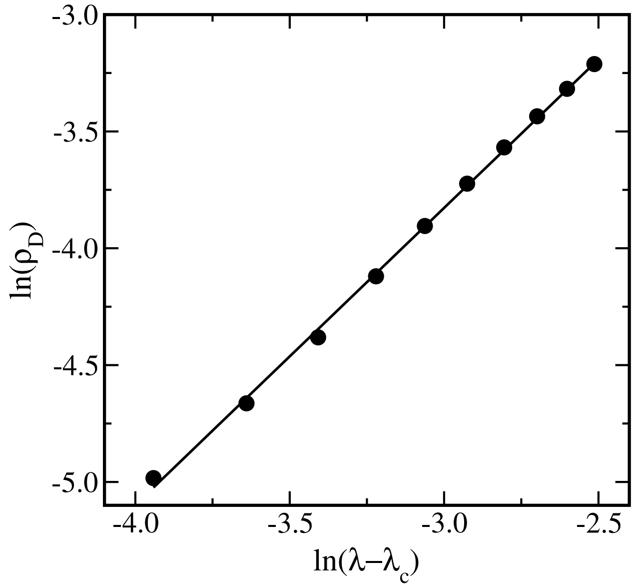

Figure 9. Power-law behavior of the order parameter as a function of $\lambda - \lambda_c$ for a network of size $N = 3283$. A linear fit of these data gives $\beta = 1.3(3)$.

Table 1. Critical properties of the SIS model and the contact process (CP) on the Apollonian network.

	λ_c	$\beta/\nu_\perp$	$\gamma/\nu_\perp$	$1/\nu_\perp$	β
CP	0.700(4)	0.54(2)	—	0.51(2)	1.06(7)
SIS	0.099(14)	0.56(7)	0.39(4)	0.47(4)	1.3(3)

The critical behavior pointed out in this paper seems to fall in the same universality class as the contact process (CP) on the Apollonian network [20], where the authors show that there exists a continuous phase transition to an absorbing state with an epidemic threshold $\lambda_c = 0.700(4)$ and critical exponents $\beta/\nu_\perp = 0.54(2)$, $\nu_\perp = 0.51(2)$ and $\beta = 1.06(7)$.

Table 1 compares the critical exponents and critical parameters in [20] with those in this paper.

Both the present model and the CP process on the Apollonian network are in agreement with the exponents $\beta = 1$ and $\nu_\perp = 2$ associated with regular lattices with contacts distributed according to a power-law with exponent larger then 3 (short-range distributed). These results seem to indicate that, although for the Apollonian network $\eta = 1 + \ln(3)/\ln(2) \simeq 2.585$, their profiles are closer to the mean-field model with $\eta > 3$ or short-range distribution. This confirms that epidemic processes on scale-free networks may have peculiar properties, as already pointed out in [37].

5. Discussion and conclusions

In this paper we have used the SIS model to mimic epidemic spreading on the Apollonian network. Finite size scaling analysis was performed in order to estimate the critical properties of the system. In this way, we have shown that a continuous phase transition takes place between the active and inactive states at a finite critical point λ_c. The critical exponents have also been computed to a high precision, and the associated critical properties are different from those found in similar epidemic processes, like Watts–Strogatz and scale-free networks. Our results indicate that the universality class of the SIS model on the Apollonian network seems to be closely related to the mean-field directed percolation in regular lattices [38] with $\eta > 3$ or short-range distributions, although the Apollonian network predicts the critical exponent $\eta = 1 + \ln(3)/\ln(2) \simeq 2.585$! To justify this discrepancy, one can argue that this peculiar behavior is associated with the particular nature of the cumulative distribution $P(k) \propto 1/k^{\eta-1}$ of the coordination number in the Apollonian network, the peculiarities of which make it, in some respects, similar to a short-range distribution.

Let us explain this point starting from the $N_n \to \infty$ limit of the ratio $m(k,n)/N_n$, which gives the probability $p(k)$ that a randomly chosen vertex has k contacts. Using the expressions presented in section 2, one obtains that $p(k)$ equals $(2/3) \times 3^{-s}$ if $k = k_s = 3 \times 2^s$, with $s = 0, \ldots, n$, and equals 0 otherwise. Accordingly, the cumulative distribution $P(k) = \sum_{k' \geq k} p(k)$ equals 3^{-s} if $3 \times 2^{s-1} < k \leq 3 \times 2^s$.

For large values of k, $P(k)$ decays with a power-law behavior, i.e., $P(k) \propto 1/k^{\eta-1}$, with $\eta = 1 + \ln(3)/\ln(2) \simeq 2.585$. Nevertheless, the distance $k_{s+1} - k_s = 2^s = k_s$ between two allowed values of k grows proportionally to k itself, i.e. the probability $p(k)$ becomes exponentially sparse for large k. This has the following important consequence: $r(k) \equiv \max_{k' \geq k} [p(k)/P(k)] = 2/3$, which also holds in the limit $k \to \infty$. The maximum is reached when k' assumes the lowest of the possible k_s values in the range $k' \geq k$.

This behavior is at strong variance with the standard power-law distribution, for which $r(k) \propto k^{-1}$ and, therefore, $r(k)$ vanishes for large k. In contrast, this is the typical behavior of short-range distributions for which $r(k)$ is of order 1.

The meaning is that in the Apollonian distribution there is always a relatively high number of vertices with lowest allowed value in the range $[k, \infty)$. The same is true for short-range distributions, while, for ordinary power-law distributions, the number of vertices with lowest allowed value in the range $[k, \infty)$ vanishes for large k as $1/k$.

As a consequence some democracy is restored, i.e., the contribution of the vertices with small values of k is larger than one would simply expect from $P(k) \propto 1/k^{\eta-1}$. Our ansatz is that the mean-field, short-range values of the critical exponents are a direct consequence of this particular structure of the distribution of vertices.

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J. Stat. Mech. (2013) P05003