Ising models on the regularized Apollonian network

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We investigate the critical properties of Ising models on a regularized Apollonian network (RAN), here defined as a kind of Apollonian network in which the connectivity asymmetry associated with its corners is removed. Different choices for the coupling constants between nearest neighbors are considered and two different order parameters are used to detect the critical behavior. While ordinary ferromagnetic and antiferromagnetic models on a RAN do not undergo a phase transition, some antiferrimagnetic models show an interesting infinite-order transition. All results are obtained by an exact analytical approach based on iterative partial tracing of the Boltzmann factor as an intermediate step for the calculation of the partition function and the order parameters.

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Many real-world networks exhibit complex topological properties as the small-world effect, related to a very short minimal path between nodes, and the scale-free property, related to the power-law nature of the connectivity distribution. These properties have important implications in real phenomena as virus spreading in computers, sharing of technological information, and diffusion of epidemic diseases, to name just a few.

In this context, the Apollonian network [1] is a particularly useful theoretical tool since it is scale free, displays the small-world effect, can be embedded in a Euclidean space, and shows space filling as well as matching graph properties. Therefore, in spite of its deterministic nature, it shares the most relevant characteristics of real-world networks.

Phase transitions have been detected for a number of different physical models on Apollonian networks. For example, the ideal gas undergoes a Bose-Einstein condensation [2–6] and epidemics exhibit a transition between an absorbing state and an active state [7,8]. In particular, in [6] an analytical strategy has been adopted that is similar to that in this paper.

In this work we focus on the infinite-order transition exhibited by some Ising models on the Apollonian network. Previous studies of similar Ising models [9,10] and the Potts model [11] have not detected critical properties due to the fact that infinite-order transitions are elusive in the sense that it is very difficult to find them with a numerical approach. In contrast, a second-order phase transition, as a function of the noise parameter, has been detected for a majority vote model [12].

Ising models on different hierarchical fractals have been studied [13] and in the case of the diamond fractal they have been exactly solved by exact renormalization [14,15]. These models, differently from the present model, show a second-order phase transition. Indeed, the standard Ising behavior is a second-order transition in plane models [16,17], nevertheless, it may be very intricate, with many phases [18], when the interactions are made more complicated.

We start by regularizing the standard Apollonian network (AN) in order to remove the connectivity asymmetry associated with its corners, which consistently simplifies the analytical computation of the thermodynamics of the Ising models. The regularized Apollonian network (RAN) is defined starting from a $g=0$ generation network with four nodes all connected, forming a tetrahedral structure with six bonds. Each of the four triples of nodes individuates a different triangle. At generation $g=1$ a new node is added inside each of the four triangles and is connected with the surrounding three nodes, creating 12 new triangles. Then the procedure is iterated at any successive generation inserting new nodes in the last created triangles and connecting each of them with the three surrounding nodes.

The connectivity of a node is defined as the number of connections to other nodes. In the RAN the connectivity of any of the already existing nodes (so-called old nodes) is doubled when generation is updated, while the connectivity of the newly created nodes (the new nodes) always equals 3, leading to the following relevant property: The connectivity at generation g of a node depends on only its age. More explicitly, its connectivity is $3\times 2^{g-g'}$, where g' is the generation in which it was created. In addition, the RAN has the following properties.

(i) The total number of nodes is $N_g=(4\times 3^g+4)/2$.

(ii) The number of new nodes created at generation $g\geqslant 1$ is $4\times 3^{g-1}\simeq \frac{2}{3}N_g$ (equality for large g).

(iii) The average connectivity is $C_g=2U_g/N_g\simeq 6$ (for large g).

(iv) The total number of bonds is $U_g=2\times 3^{g+1}$.

(v) The number of new bonds created at generation $g\geqslant 1$ is $4\times 3^g=\frac{2}{3}U_g$.

(vi) The number of nodes having connectivity k is $m(k,g)$, which equals $4\times 3^{g-g'-1}$ if $k=3\times 2^{g'}$ with $g'=0,\ldots,g-1$; equals 4 if $k=3\times 2^g$; and equals 0 otherwise.

According to property (vi), the cumulative distribution $P(k)=\sum_{k'\geqslant k} m(k,g)/N_g$ exhibits, for large values of g, a power-law behavior, i.e., $P(k)\propto 1/k^\eta$, with $\eta=\ln(3)/\ln(2)\simeq 1.585$.

Analogously to the AN, the RAN is scale-free and displays the small-world effect. Furthermore, since the RAN can be decomposed into four ANs cutting a finite number of couplings, the thermodynamics of the two models is the same.

We stress that the deep reason for considering the RAN is that the connectivity of the nodes depends on only their age. The asymmetry associated with the corners of the AN model (which have a variant connectivity) is thus removed. This property allows for exact iterative calculations that in the AN can be done only approximately. Therefore, since the thermodynamics of the RAN and the AN is the same, our choice is mainly technical (but also aesthetic).

Ising models are defined according to the following Hamiltonian:

$$H_g = -\sum_{i,j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i - q\sum_{i,j,k}\sigma_i\sigma_j\sigma_j, \quad (1)$$

where the first sum is over all U_g connected pairs of nodes of the RAN of generation g, the second sum is over all N_g nodes, and the third sum is over only the 4×3^g triangles of the last generation g (one of the nodes i, j, or k must be generated last). The constants J_{ij} and h_i may depend on the connectivities (of the age) of the involved nodes. The couplings J_{ij} can be either positive (ferromagnetic) or negative (antiferromagnetic). The constant q is introduced only for technical reasons, the relevant physics corresponding to $q=0$.

The partition function is

$$Z_g = \sum_{\#}\exp(-\beta H_g), \quad (2)$$

where the sum is over all 2^{N_g} configurations and β is the inverse temperature, i.e., $\beta = 1/T$ (we consider a unitary Boltzmann constant k_B). Then the thermodynamical variables can be obtained from

$$\Phi = \lim_{g\to\infty}(1/N_g)\ln(Z_g). \quad (3)$$

Our strategy consists in performing a partial sum in (2) with respect to the $4\times 3^{g-1}$ spin variables over nodes created at the last generation g. This sum creates a new effective interaction between all remaining spins and new magnetizations and a new value for the parameter q. In other words, we exactly map the g generation model in the same $g-1$ model with new parameters. This technique works for any possible choice of the parameters J_{ij}, h_i, and q, but we will consider here only some simple cases.

We stress that our approach is different from the transfer matrix technique [9,10] and it gives exact expressions for the thermodynamical variables in the $g\to\infty$ limit. While it confirms the absence of a transition in the ordinary ferromagnetic and antiferromagnetic models, it detects an infinite-order phase transition in a simple antiferrimagnetic model occurring at a finite temperature, in contrast with what was found in [9,10], where no critical behavior at a finite temperature was identified for this kind of model.

In order to illustrate our strategy we start with the simplest case in which all interactions J_{ij} are equal ($J_{ij}=J$), $q=0$, and all $h_i=0$. Without loss of generality, one can chose $J=1$ (ferromagnetism) or $J=-1$ (antiferromagnetism).

Since a new node is linked to only three older surrounding nodes (it is created inside a triangle), the summation of the spin over a new node creates an extra interaction among the three surrounding spins on the older nodes. Therefore, the partial sum of spins over new nodes in (2) yields, after some lengthy but straightforward calculations, the equality

$$\ln[Z_g(\beta,J)] = \ln[Z_{g-1}(\beta,J_1)] + 4\times 3^{g-1}A(\beta J), \quad (4)$$

where

$$J_1 = J + \frac{1}{2\beta}\ln[2\cosh(2\beta)-1] \quad (5)$$

and

$$A(\beta J) = \tfrac{1}{4}\ln[2\cosh(2\beta J)-1] + \ln[2\cosh(\beta J)] \quad (6)$$

[note that $J=\pm 1$ implies $A(\beta J) = A(\beta)$]. In (4), $Z_{g-1}(\beta,J_1)$ is the partition function of the same model at generation ($g-1$) with a different value J_1 (J_1 has the same sign of J). Note that in the RAN (contrary to the AN) equality (4) is exact.

We can iterate the procedure and obtain, as a final result, the partition function of the finite-size model (generation g) in terms of a sum of explicit analytical functions. Since we are interested in the thermodynamical limit we do not follow this route. In contrast, we directly exploit the infinite-size properties by performing the thermodynamical limit $g\to\infty$ of both sides of (4):

$$\Phi(\beta,J) = \tfrac{1}{3}\Phi(\beta,J_1) + \tfrac{2}{3}A(\beta), \quad (7)$$

where we have used $N_{g-1}/N_g \to \frac{1}{3}$ and $4\times 3^{g-1}/N_g \to \frac{2}{3}$. We have thus reexpressed the thermodynamical function $\Phi(\beta,J)$ in terms of $\Phi(\beta,J_1)$, proving the absence of a transition. In fact, since $\Phi(\beta,J)$ depends on only the product βJ and since the above equation can be iterated, a single nonanalytical point would imply an infinite number of nonanalytical points.

Iteration of (7) gives

$$\Phi(\beta,J) = \frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{3^k}A(\beta J_k), \quad (8)$$

where

$$J_k = J_{k-1} + \frac{1}{2\beta}\ln[2\cosh(2\beta J_{k-1})-1], \quad (9)$$

with $J_0 = J$ and A given by (6). If $J_0 = J = 1$ (ferromagnetism), the positive J_k increase monotonically and diverge for large k. In contrast, if $J_0 = J = -1$ (antiferromagnetism) the negative J_k converge to 0 for large k. In both cases it is easy to verify that the sum (8) converges.

Since we have proven that both constant coupling cases (the ferromagnetic and the antiferromagnetic ones) do not undergo a phase transition, we now extend our scope to consider a different model. We return to the Hamiltonian (1) and assume that the action of the external magnetic field h_i is proportional to the connectivity of the node, i.e., $h_i = hz_i$, where z_i is the connectivity of node i. Accordingly, we can define the following spontaneous magnetization:

$$M = \lim_{g\to\infty}\frac{\sum_i z_i\langle\sigma_i\rangle}{\sum_i z_i} = \frac{1}{6\beta}\left[\frac{\partial\Phi}{\partial h}\right]_{h=q=0^+}, \quad (10)$$

where we have used $\sum_i z_i = 2U_g \simeq 6N_g$. The notation $\langle\cdot\rangle$ indicates an average with respect to the Gibbs measure in such a way that M satisfies $0 \leqslant M \leqslant 1$. The rationale for this choice is that it turns out to be the simplest tool to show the existence of an infinite-order phase transition in an antiferrimagnetic model.

For the same practical reasons, we compute the coordination defined as

$$L = \lim_{g\to\infty}\frac{\sum_{i,j,k}\langle\sigma_i\sigma_j\sigma_j\rangle}{4\times 3^g} = \frac{1}{2\beta}\left[\frac{\partial\Phi}{\partial q}\right]_{h=q=0^+}, \quad (11)$$

where the sum is over only the 4×3^g triangles of last generation g. Note that the factor $\frac{1}{2}$ in the last term comes from the fact that in the limit $g \to \infty$ one has $4 \times 3^g/N_g \to 2$ and, accordingly, $|L| \leqslant 1$.

Since we have proven that a constant value for the couplings J_{ij} leads to the absence of a transition, we will now assume, on the contrary, that they can be different. The symmetry of the model implies that they may depend on only the connectivity of the connected nodes i and j. Models with different values of the couplings are usually denominated ferrimagnetic (or antiferrimagnetic). For ferrimagnetic models defined on a regular lattice, different couplings usually correspond to different Euclidean distances of the connected nodes, while for models defined on hierarchical networks they may depend on connectivity (as in the present model) or on other specific indices associated with the nodes.

Since in the RAN the connectivity of a node depends only on its age, the couplings only may depend on the age of connected nodes. The simplest age dependence for an antiferrimagnetic model is $J_{i,j} = -u$ with $u > 1$ if the connectivity of at least one of the two nodes i or j is 3 and $J_{i,j} = -1$ otherwise. This is the same as assuming that $J_{i,j} = -u$ for bonds involving nodes of last generation and $J_{i,j} = -1$ otherwise. Then, following the same procedure, we take a partial sum with respect to the spins over the last created nodes and we obtain the exact equality

$$\Phi(\beta,h,q,u) = \tfrac{1}{3}\Phi(\beta,J_1,h_1,q_1) + (2/3)A(\beta,h,q). \qquad (12)$$

We stress that, while the initial antiferromagnetic model [described by $\Phi(\beta,h,q,u)$] had two possible values for the couplings ($-u$ and -1), the effective model after a partial sum [described by $\Phi(\beta,J_1,h_1,q_1)$] has the single value J_1 for all of them. The new parameters J_1, h_1, q_1, and A can be again explicitly computed in terms of β, h, q, and u. Since we are interested in only thermodynamical potentials, such as the spontaneous magnetization M and coordination L in the absence of external magnetic fields, what we really use are only the values of J_1, h_1, and q_1 for $q = h = 0$ and their derivatives with respect to h and q, in the limit $h \to 0$ and $q \to 0$.

Assuming $h = 0$ and $q = 0$ we have that h_1 and q_1 are also equal to zero and we obtain again (8) with A given by (6). The difference is that $J_0 = -u$,

$$J_1 = -1 + \frac{1}{2\beta}\ln[2\cosh(2\beta u) - 1], \qquad (13)$$

while the remaining J_k for $k \geqslant 2$ are again obtained by (9). The resulting entropy and specific heat, as a function of the temperature T (in units of the Boltzmann constant k_B), are depicted in Figs. 1 and 2 for various values of u. Note that the case $u = 1$ is the regular antiferromagnet with nonvanishing zero-temperature entropy. Interestingly, whenever $u > 1$, as a consequence of the larger value of the coupling constants involving new nodes, frustration is removed and the zero-temperature entropy drops to zero.

Let us call $\beta_c = \frac{1}{2}\ln[2\cosh(2u\beta_c) - 1]$ the (nonvanishing) value of β for which J_1 vanishes, which depends on only u; then there are three cases. (i) In the case $\beta = \beta_c$, $J_1 = 0$ and one immediately obtains $\Phi = \frac{1}{3}\ln(2) + \frac{1}{6}\ln[2\cosh(2u\beta_c) - 1] + \frac{2}{3}\ln[2\cosh(u\beta_c)]$. (ii) In the case $\beta > \beta_c$, $J_1 > 0$, i.e., the initial antiferrimagnetic model is mapped onto a ferromagnetic model. The J_k increases monotonically and diverges for large k. (iii) In the case $\beta < \beta_c$, $J_1 < 0$, i.e., the initial antiferrimagnetic model is mapped onto an antiferromagnetic model. The J_k converge monotonically to zero for large k.

Is β_c a critical point? The answer to this question is not easy since all thermodynamical functions seem to behave regularly at β_c (see Figs. 1 and 2). To address this question we have to compute the spontaneous magnetization M and the coordination L.

Given $A(\beta,h,q)$ in (12), it turns out that $\partial A/\partial h$ and $\partial A/\partial q$, as well as $\partial J_1/\partial h$ and $\partial J_1/\partial q$, calculated at $h = q = 0$ vanish. Then, given (10) and (11), we obtain from (12) $M = T_{11}\,M(J_1) + T_{12}\,L(J_1)$ and $L = T_{21}\,M(J_1) + T_{22}\,L(J_1)$, where $M(J_1)$ and $L(J_1)$ are the magnetization and coordination

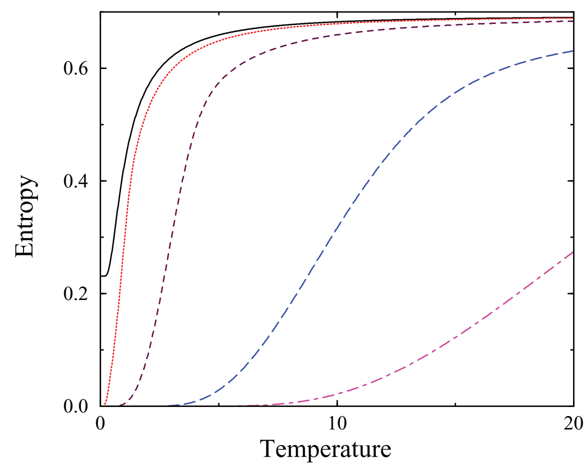

FIG. 1. (Color online) Entropy versus the temperature T (in units of the Boltzmann constant k_B) for the following values of u (from bottom to top): $u = 10,5,2,1.2,1$. The case $u = 1$ is the regular antiferromagnet with nonvanishing zero-temperature entropy [the entropy is equal to ln(2)/3]. Interestingly, whenever $u > 1$, the zero-temperature entropy drops to zero. All curves behave regularly and do not show any sign of phase transition.

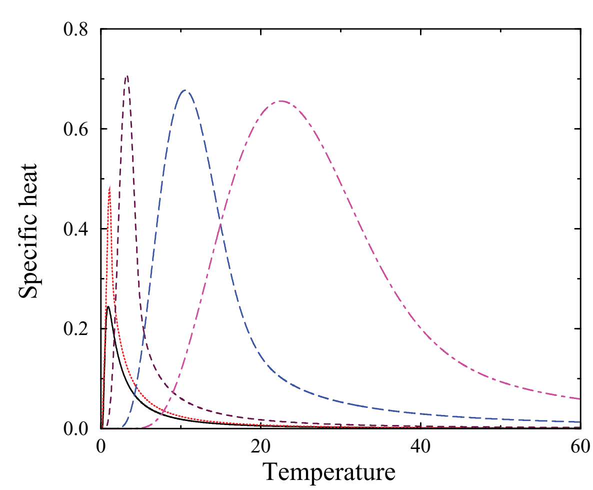

FIG. 2. (Color online) Specific heat versus the temperature T (in units of the Boltzmann constant k_B) for the following values of u (from right to left): $u = 10,5,2,1.2,1$. The curves behaves regularly and do not show any sign of phase transition.

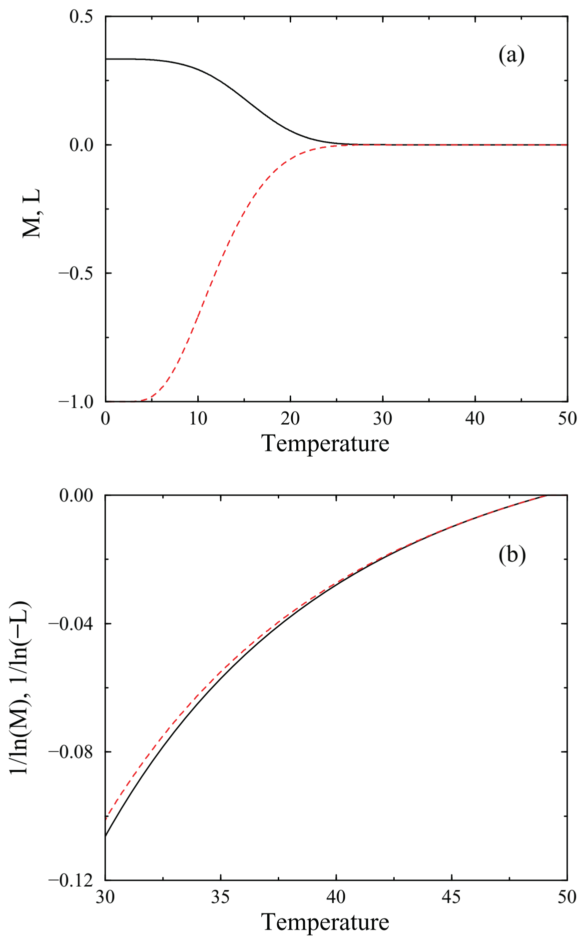

FIG. 3. (Color online) (a) Plot of the spontaneous magnetization M (solid line) and the coordination L (dashed line) versus the temperature T (in units of the Boltzmann constant k_B) for the $u = 5$ model. (b) Plot of $1/\ln(M)$ (solid line) and $1/\ln(-L)$ (dashed line) versus the temperature T (in units of the Boltzmann constant k_B). In (a) it seems that the transition occurs at a temperature around 26, but this is a wrong perception. (b) shows in fact that the critical temperature is about $T_c = 49.16$, which is the correct transition temperature that we found analytically.

of the model with couplings J_1. Also,

$$
\begin{aligned}
T_{11}(\beta J_0) &= \frac{1}{3}\frac{\partial h_1}{\partial h} = \frac{2}{3} + \frac{1}{4}[\tanh(3\beta J_0) + \tanh(\beta J_0)],\\
T_{12}(\beta J_0) &= \frac{1}{9}\frac{\partial q_1}{\partial h} = \frac{1}{12}[\tanh(3\beta J_0) - 3\tanh(\beta J_0)],\\
T_{21}(\beta J_0) &= \frac{\partial h_1}{\partial q} = \frac{1}{4}[3\tanh(3\beta J_0) - \tanh(\beta J_0)],\\
T_{22}(\beta J_0) &= \frac{1}{3}\frac{\partial h_1}{\partial q} = T_{11}(\beta J_0) - \frac{2}{3},
\end{aligned}
\tag{14}
$$

where all derivative are calculated at $h = q = 0$. This relation can be iterated and one obtains

$$
m = \prod_{k=0}^{\infty} T(\beta J_k) m(J_\infty), \tag{15}
$$

where m is the vector whose two elements are M and L; $T(\beta J_k)$ are the 2×2 matrices with elements $T_{ij}(\beta J_k)$, and $m(J_\infty)$ corresponds to spontaneous magnetization and coordination of the model with couplings J_∞.

If $\beta < \beta_c$, the sequence of J_k converges to 0 and $\prod_{k=0}^{\infty} T(\beta J_k)$ is a vanishing matrix; therefore, $m = 0$ independently of $m(J_\infty)$. If $\beta > \beta_c$, in contrast, one find that $J_\infty = \infty$, with both components of $m(\beta J_\infty)$ equal to one (a ferromagnet with infinitely large couplings). Furthermore, in this case, $\prod_{k=0}^{\infty} T(\beta J_k)$ does not vanish and therefore $m \neq 0$. We have thus proven the existence of a transition at β_c.

The magnetization M and the coordination L for the $u = 5$ model are depicted in Fig. 3(a). Apparently, the transition occurs at a temperature around 26, which is far from the transition temperature $T_c \simeq 49.16$ obtained by the nonvanishing solution of $\beta_c = \frac{1}{2}\ln[2\cosh(2u\beta_c) - 1]$ with $u = 5$ and $T_c = 1/\beta_c$. This is due to the fact that both M and L vanish extremely slowly. In order to make this evident, we have depicted in Fig. 3(b) $1/\ln(M)$ and $1/\ln(-L)$, which clearly show agreement with the expected critical temperature $T_c \simeq 49.16$. The behavior of both magnetization and coordination, close to the critical point, is compatible with the function $\exp[-C/(T_c - T)]$, which implies an infinite-order transition since M and L and all their derivatives are continuous at T_c.

Note that a negative coordination implies that spins are preferentially organized in order that in any triangle one of them points in the opposite direction of the other two. This organization becomes exact at $T = 0$, where the coordination is -1. Observe also that, contrary to the antiferromagnetic model on regular lattices, the magnetization is positive below the critical temperature with a value compatible with $\frac{1}{3}$ at $T = 0$.

Qualitatively identical results are produced whenever $u > 1$, leading to (i) $\beta_c = \frac{1}{2}\ln[2\cosh(2u\beta_c) - 1]$, which individuates the transition temperature, and (ii) the transition being of infinite order. In the limit $u \to 1$ one easily verifies that $T_c = 0$, which confirms the absence of a transition in the ordinary antiferromagnetic model.

Our minimal choice of $J_{i,j} = -u$ for newly created bonds and $J_{i,j} = -1$ otherwise is the simplest, but it is not the only one that leads to a paramagnetic-ferromagnetic infinite-order phase transition. Indeed, there are many possible hierarchical choices for $J_{i,j}$ that lead to the same qualitative behavior.

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