Scaling behavior for random walks with memory of the largest distance from the origin

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We study a one-dimensional random walk with memory. The behavior of the walker is modified with respect to the simple symmetric random walk only when he or she is at the maximum distance ever reached from his or her starting point (home). In this case, having the choice to move farther or to move closer, the walker decides with different probabilities. If the probability of a forward step is higher then the probability of a backward step, the walker is bold, otherwise he or she is timorous. We investigate the asymptotic properties of this bold-timorous random walk, showing that the scaling behavior varies continuously from subdiffusive (timorous) to superdiffusive (bold). The scaling exponents are fully determined with a new mathematical approach based on a decomposition of the dynamics in active journeys (the walker is at the maximum distance) and lazy journeys (the walker is not at the maximum distance).

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I. INTRODUCTION

In 1827, the botanist Robert Brown noticed that pollen grains in water perform a peculiar erratic movement. Many decades later, in 1905, Albert Einstein [1] gave an explanation of the pollen *random walk* in terms of collisions with the water moleculae, relating the diffusion coefficient to observable quantities. Indeed, Einstein was scooped by Louis Bachelier, who, 5 years before, in his 1900 doctoral thesis [2] and in a following paper [3], arrived at similar conclusions. Actually, Bachelier was interested in the motion of prices on the French stock market, but (log-)prices move like pollen in water and their *random walk* can be treated mathematically on the same ground.

This twofold origin of random walk as a probabilistic tool is illuminating; in fact, this tool can be applied everywhere "a walker" (e.g., a particle, a cell, an individual, a price, a language) moves erratically in such a way that its square displacement $x^2(t)$ increases in average according to $\langle x^2(t)\rangle \sim t$.

In its simpler version, the path of a random walk is the output of a succession of independent random steps. In this case, the scaling relation $\langle x^2(t)\rangle \sim t$ is immediate. Nevertheless, in most cases, this relation also holds if memory effect in size and direction of the steps is present. The criterion is that memory is short ranged and steps do no have diverging length.

The scaling relation $\langle x^2(t)\rangle \sim t$ is traditionally associated to the appellative *normal diffusion*, while *anomalous diffusion* corresponds to a scaling $\langle x^2(t)\rangle \sim t^{2\nu}$ with $\nu \neq 1/2$. In particular subdiffusive behavior corresponds to $\nu < 1/2$ and superdiffusive behavior to $\nu > 1/2$.

There is a very large number of phenomena which exhibit anomalous diffusion as well a variety of models which have been used to describe them. We refer to Refs. [4–8] for a review of both.

Broadly, anomalous diffusion may arise via diverging steps length, as in Lévy flights or via long-range memory effects as in fractional Brownian motion and in self-avoiding random walks. Diverging step lengths and long-range memory are two different ways of violating the necessary conditions for the central limit theorem when applied to random walks.

Anomalous diffusion (superdiffusion) in Lévy flights [9] is the simple consequence of the fact that the length of the steps has a heavy-tailed probability distribution. This does not mean that the problem is trivial; see, for example, Ref. [10], where the authors consider the interesting case in which diffusion is *strongly* anomalous ($\langle x^q(t)\rangle \sim t^{q\nu}$ with ν depending on q).

Anomalous diffusion induced by long-range memory is the non-self-evident output of the self-interaction of the walker position at different times; the most celebrated example probably is the "true" self-avoiding random walk introduced quite a long time ago [11] and later rigorously studied (see Refs. [12,13], and references therein). In this model, the exponent ν depends only on dimensionality.

In some cases, the mechanism which gives origin to anomalous scaling can differ, for example, special deterministic or random environments (see, for example, Refs. [14,15]) or multiparticle interactions [16].

Since exact solution of nontrivial models with memory are quite difficult to obtain, some effort has been made in this direction. For example, in the elephant model [17], the walker decides the direction of his step depending on his previous decisions. Unfortunately, this model, given the direction of the first step, can be exactly mapped in a Markovian model, without the necessity of enlarging the phase space, and, more importantly, the anomalous scaling is not a consequence of an anomalous diffusion but of the movement of the center of mass of the probability distribution of the position. Some generalizations of this model have been proposed (see Ref. [18]) which are genuinely non-Markovian but which show the same problem concerning the origin of the anomalous scaling. Another, analytically treatable, model considered the case of a semi-Markovian sub-diffusive processes in which the waiting time for a step is given by a probability distribution with a diverging mean value [19].

Random walks with memory have been also employed to model the spreading of an infection in a medium with a history-dependent susceptibility [20,21]; the focus, in this case, is the time scaling of the survival probability (a trap is collocated somewhere) and not the scaling of diffusion. Moreover, random walks with memory have been used in finance as, for example, in Ref. [22]. This paper describes the strategy of a prudent investor who tries to maximize invested capital while never decreasing his standard of life. In Refs. [20,21] and in Ref. [22], as in the model presented in this paper, the behavior of the walker is modified only when it is at the maximum distance from the origin and Markovianity is recovered only when the phase space is properly enlarged.

Motivated by the scarcity of exact solutions, we present in this paper a model which is treatable, one-dimensional, and genuinely non-Markovian and which shows anomalous scaling ranging from subdiffusion to superdiffusion according to a single continuous parameter.

The paper is organized as follows: in Sec. II we present the model and we expose our results; in Sec. III we describe the decomposition of the dynamics which is at the basis of our mathematical approach; the asymptotic behavior is computed in Sec. IV; and in Sec. V we write and exactly numerically solve the associated forward Kolmogorov equation; Sec. VI contains our conclusions. Some of the calculations whose result is used in Sec. III are postponed in a final Appendix.

II. MODEL AND RESULTS

The model presented in this paper is one-dimensional, steps all have the same unitary length, time is discrete, and the walker can only move left or right at any time step. The behavior of the random walker is modified with respect to the simple symmetric random walk (SSRW) only when he is at the maximum distance ever reached from his starting point (home). In this case, he decides with different probabilities to make a step forward (going farther from home) or a step backward (going closer to home).

More precisely, the model is the following: The walker starts from home $[x(0) = 0]$, then, at any time he or she can make a (unitary length) step to the right or to the left,

$$x(t+1) = x(t) + \sigma(t) \tag{1}$$

with $\sigma(t) = \pm 1$. We define

$$y(t) = \max_{0 \leqslant s \leqslant t} |x(s)|, \tag{2}$$

which is the maximum distance from home the walker ever attained, which obviously implies $-y(t) \leqslant x(t) \leqslant y(t)$. Then we assume

(i) $\sigma(0) = \pm 1$ with equal probability, i.e., the walker chooses with equal probability the direction of the first step,

(ii) $\sigma(t) = \pm 1$ with equal probability if the walker is not at the maximum distance from home, i.e., $|x(t)| < y(t)$,

(iii) $\sigma(t) = \mathrm{sgn}(x(t))$ with probability $p(y(t))$ and $\sigma(t) = -\mathrm{sgn}(x(t))$ with probability $1 - p(y(t))$ if $|x(t)| = y(t)$,

(iv) the probability $p(y)$ depends on y according to $p(y) = y^\gamma/(1 + y^\gamma)$.

Therefore, SSRW holds when $|x(t)| < y(t)$ but when the walker is at the maximum distance from home $[|x(t)| = y(t)]$, he boldly prefers to move farther if $\gamma > 0$ or timorously prefers to move closer if $\gamma < 0$.

Our goal is to find the asymptotic behavior of $y(t)$ and $|x(t)|$. We preliminarily observe that in case $\gamma = 0$ one has $p(y) = 1/2$, which implies SSRW holds everywhere and also if the walker is at maximum distance. In this case, ordinary scaling applies: $\langle y^\alpha(t)\rangle \sim \langle |x(t)|^\alpha\rangle \sim t^{\alpha/2}$ for any real positive α (the sign $\sim$ indicates that the ratio of the two sides asymptotically tends to a strictly positive constant. We use the sign $\simeq$ for the stronger statement that the ratio tends to 1).

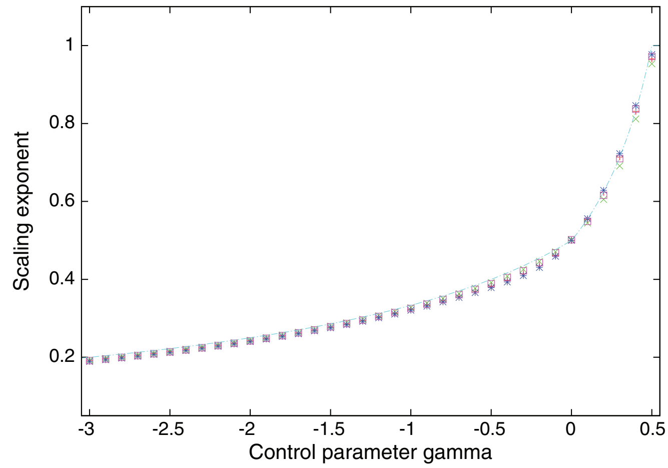

FIG. 1. (Color online) Scaling exponent ν deduced from $\langle |x(t)|\rangle$ (crosses, red), $\langle y(t)\rangle$ (slanted crosses, green), $\langle |x(t)|^2\rangle$ (stars, blue), and $\langle y^2(t)\rangle$ (squares, violet) against prevision (dashed line, light blue).

Results of this paper can be summarized as follows:

(a) $\langle y^\alpha(t)\rangle \simeq \langle y(t)\rangle^\alpha \simeq (t/2\nu)^{\alpha\nu}$ with $\nu = 1/(2-\gamma)$ for $-\infty < \gamma < 0$,

(b) $\langle y^\alpha(t)\rangle \sim t^{\alpha\nu}$ with $\nu = 1/(2-2\gamma)$ for $0 \leqslant \gamma \leqslant 1/2$, and

(c) $\langle y^\alpha(t)\rangle \simeq \langle y(t)\rangle^\alpha \simeq t^\alpha$ for $1/2 < \gamma < \infty$.

Moreover, $\langle |x(t)|^\alpha\rangle \sim \langle y^\alpha(t)\rangle$ for all γ.

In both regions $-\infty < \gamma < 0$ and $1/2 < \gamma < \infty$ relations are indicated with $\simeq$, i.e., the ratio of the two sides tends to 1 in the limit $t \to \infty$, providing both the scaling exponent and the scaling factor for $\langle y^\alpha(t)\rangle$. Furthermore, $\langle y^\alpha(t)\rangle \simeq \langle y(t)\rangle^\alpha$, which implies that the variable $y(t)$ scales deterministically as its average.

In particular, in region $-\infty < \gamma < 0$ the behavior is subdiffusive, and, as a consequence of the propensity, the walker has to step in the home direction when at maximum distance, while in the region $1/2 < \gamma < \infty$, behavior is ballistic, with coefficient 1, as a consequence of the strong propensity to step away from home when at maximum distance.

Finally, in the intermediate region $0 \leqslant \gamma \leqslant 1/2$, only the scaling exponent of $\langle y^\alpha(t)\rangle$ is determined. If $\gamma = 1/2$, ordinary diffusion holds (with standard coefficients), while in the region $0 < \gamma \leqslant 1/2$, we have nonballistic superdiffusive behavior as a consequence of the (not too strong) propensity the walker has to step away from home when at maximum distance.

The behavior of the anomalous scaling exponent ν with respect to the control parameter γ in region $\gamma < 1/2$ is depicted in Fig. 1. In region $\gamma \geqslant 1/2$ the exponent ν equals 1, i.e., behavior is ballistic (notice that, by construction, it cannot be superballistic).

The following three sections are devoted to the validation of the results here presented.

III. DECOMPOSITION OF THE DYNAMICS

Here we outline the decomposition of the dynamics which is at the basis of our new mathematical approach.

Trajectories are decomposed in active journeys and lazy journeys. The lazy journey starts at time t when the walker leaves the maximum and it ends when he or she reaches it again at time $t+m+1$, i.e., $|x(t)| = y(t) = y$, $|x(t+s)| < y$ for $1 \leqslant s \leqslant m$ and $|x(t+m+1)| = y$. The total number of steps of this journey is $1+m$ since the first step is for leaving the maximum and m is the random number of steps necessary to reach it again starting from a position $|x| = y-1$. During all steps of the lazy journey the maximum remains the same. The minimum duration of the lazy journey is two time steps ($1+m=2$ when the walker immediately steps back to the maximum after having left it).

The active journey starts at the time $t+m+1$ when the walker arrives on a maximum and it ends when he or she leaves it at time $t+m+n+1$, i.e., $|x(t+m)| = y-1$, $|x(t+m+1+s)| = y+s$ for $0 \leqslant s \leqslant n$ and $|x(t+m+n+2)| = y+n-1$ (the first step of a new lazy journey). The total (random) number of time steps of this journey is n with a minimum duration of zero steps ($n=0$ when the walker immediately leaves the maximum after being arrived). During the active journey the maximum increases from y to $y+n$.

A cycle journey is composed by a lazy journey followed by an active journey, its duration is $1+m+n$ and the maximum increases of n. A portion of a trajectory, composed by various cycle journeys, is shown in Fig. 2.

Notice that both $n = n(y)$ and $m = m(y)$ are random variables whose distribution only depends on y. In fact, $m(y)$ is the SSRW first hitting time of one of the barriers y or $-y$ starting from position $x = y-1$ or $x = -y+1$, while the statistics of $n(y)$ is determined by y through $p(y)$.

We start by evaluating the probability $\pi(n|y)$ that the walker makes at least n steps during the active journey, i.e., $\pi(n|y) = \text{prob}\,(n(y) \geqslant n)$. Straightforwardly,

$$\pi(n|y) = \prod_{s=0}^{n-1} p(y+s), \tag{3}$$

where $p(y+s) = (y+s)^\gamma/(1+(y+s)^\gamma)$.

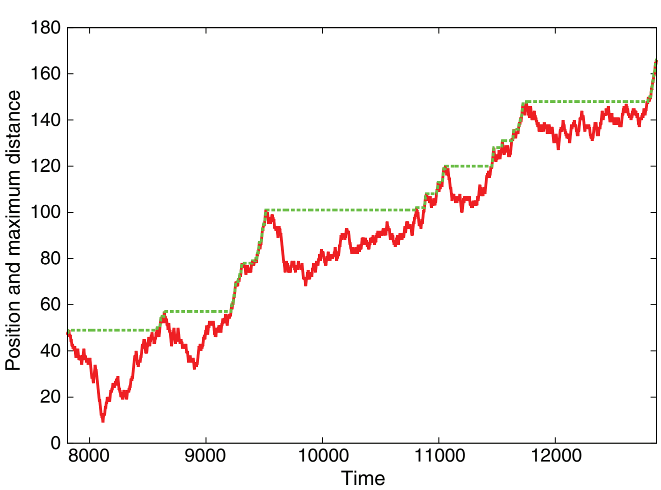

FIG. 2. (Color online) Portion of a trajectory with various cycle journeys, each composed of a lazy journey followed by an active journey. The maximum distance (dashed line, green) remains constant during lazy journeys and increases with the position (full line, red) during active journeys. The trajectory in this figure is a realization of the superdiffusive $\gamma = 0.1$ process.

For large y, we have to distinguish three different ranges of γ,

(i) $-\infty < \gamma < 0$, in this case $\pi(n|y) \leqslant y^{n\gamma}$, where the approximate equality $\pi(n|y) \simeq y^{n\gamma}$ holds for n small with respect to y,

(ii) $0 < \gamma < 1$, in this case $\pi(n = \beta y^\gamma|y) \simeq e^{-\beta}$, which means that $n(y) \simeq \xi y^\gamma$, where ξ is a random variable distributed according to an unitary exponential probability,

(iii) $\gamma > 1$, in this case $\pi(n|y) \simeq e^{-\psi(y,n)}$, where $\psi(y,n) = \sum_{s=0}^{n-1} 1/(y+s)^\gamma$, noticeably, $\pi(\infty|y)$ is finite, which implies that $n(y)$ is infinite with finite probability.

The above relations are derived in the Appendix. In both cases (i) and (ii) the average is $\langle n(y)\rangle \simeq y^\gamma$, while in case (iii) it diverges. Moreover, the standard deviation in case (i) is $\sigma_{n(y)} \simeq y^{\gamma/2}$ while in case (ii) is $\sigma_{n(y)} \simeq y^\gamma$.

The statistical properties of $m(y)$ are well known [3,23]; in fact, as already stressed, $m(y)$ is simply the SSRW time for hitting one of the frontiers of the interval $[-y,y]$ starting from position $y-1$ (or $-y+1$). Using a standard martingale approach, it is easy to compute the average $\langle m(y)\rangle \simeq 2y$ and the standard deviation $\sigma_{m(y)} \simeq (8/3)^{1/2} y^{3/2}$, where the approximations hold for large y (see the Appendix).

We underline that the standard deviation is larger than average and that both diverge when $y \to \infty$, nevertheless, $m(y)$ has a finite probability to be of order 1 [consider that $m(y) = 1$ with probability 1/2 independently on y]. This is reflected in the fact that the averages $\langle m(y)^\beta\rangle$ are of order 1 when β is negative. For m that is not very small, $\theta(m|y) \sim m^{-1/2}$ [3,23] with a cutoff at $m_c \sim y^2$, where $\theta(m|y)$ drops to 0 (m_c is the typical time the walker can "see" the far barrier). This implies that $\theta(m|y)$ is approximately a truncated Lévy distribution.

IV. ASYMPTOTIC ANALYSIS

We have seen in the previous section that in a cycle journey starting from a maximum y, time increases $1 + m(y) + n(y)$ and the maximum increases $n(y)$.

Let us indicate with k (to be not confused with time t) the progressive number identifying cycle journeys, each composed of a lazy journey followed by an active journey. Also, let us indicate with $y(k)$ the value of the maximum when the cycle journey number k starts.

The time t is linked to the progressive number k by the stochastic relation

$$t(k+1) = t(k) + 1 + m(y(k)) + n(y(k)) \tag{4}$$

while the value of the maximum is linked to k by

$$y(k+1) = y(k) + n(y(k)), \tag{5}$$

where $m(y(k))$ and $n(y(k))$ are all independent random variables whose statistical properties we have already described. In principle, one should simply solve the two equations and, by substitution, obtain the scaling behavior of $y(t)$. Obviously, this necessitates some work.

We start our asymptotic analysis by considering the region $-\infty < \gamma < 1$. Let us consider first Eq. (5). In the region of γ we are considering, the variables have average $\langle n(y(k))\rangle \simeq y(k)^\gamma$, then, from Eq. (5), one has $\langle y(k+1)^{1-\gamma}\rangle \simeq \langle y(k)^{1-\gamma}\rangle + (1-\gamma)$. The omitted terms are of lower order in $y(k)$ since the standard deviation of the $n(y(k))$ can be $\sigma_{n(y(k))} \simeq y(k)^{\gamma/2}$ (for $-\infty < \gamma < 0$) or $\sigma_{n(y(k))} \simeq y(k)^{\gamma}$ (for $0 < \gamma < 1$). By integration we obtain $\langle y(k)^{1-\gamma}\rangle \simeq (1-\gamma)k$ and by iteration $\langle y(k)^{l(1-\gamma)}\rangle \simeq (1-\gamma)^l k^l$, where l is a positive integer number. Finally, by analytical continuation we have $\langle y(k)^{\alpha}\rangle \simeq (1-\gamma)^{\alpha/(1-\gamma)} k^{\alpha/(1-\gamma)} \simeq \langle y(k)\rangle^{\alpha}$ for any real α. We have thus proven the relation

$$y(k) \simeq (1-\gamma)^{1/(1-\gamma)}\, k^{1/(1-\gamma)}, \tag{6}$$

which holds deterministically, i.e., fluctuations are comparatively negligible in the large $y(k)$ limit.

Let us now consider Eq. (4), by simple sum we get

$$t(k) = y(k) + k + M((k)), \tag{7}$$

where $y(k)$ is given by (6) and $M(k) = \sum_{i=0}^{k-1} m(y(i))$, which is a sum of independent variables distributed according to truncated Lévy distributions with $\langle m(y(i))\rangle \simeq 2y(i)$ and $\sigma_{m(y(i))} \simeq (8/3)^{1/2} y(i)^{3/2}$. According to (6) we obtain the average

$$\langle M(k)\rangle \simeq \frac{2}{2-\gamma}\,[(1-\gamma)k]^{(2-\gamma)/(1-\gamma)} \tag{8}$$

and the standard deviation $\sigma_{M(k)} \sim k^{(2-\gamma/2)/(1-\gamma)}$.

If $\gamma < 0$, the standard deviation $\sigma_{M(k)}$ is asymptotically negligible with respect to the average $\langle M(k)\rangle$ and we can thus replace $M(k)$ with its average $\langle M(k)\rangle$ in (7). Furthermore, we can neglect the smaller terms $y(k)$ and k and obtain $t(k) \simeq \langle M(k)\rangle$, which is solved with respect to k and, substituted into (6), finally gives the relation

$$y(t) \simeq (1-\gamma/2)^{1/(2-\gamma)}\, t^{1/(2-\gamma)}, \tag{9}$$

which holds deterministically in the region $-\infty < \gamma < 0$.

On the contrary, if $0 < \gamma < 1$ the standard deviation of $M(k)$ is larger than its average. In this case, it is necessary to determine the behavior of its probability distribution. This can be done considering that all the independent $m(y(i))$ in the sum which defines $M(k)$ are distributed according to a truncated Lévy. Then, according to the generalized central limit for leptokurtic variables [9],

$$L(k) = \frac{1}{k^2}\sum_{i=1}^{k} m(y(i)) = \frac{1}{k^2} M(k) \tag{10}$$

is also a truncated leptokurtic variable. Notice, in fact, that the denominator equals the power 2 of the number of the summed variables (Lévy is $\alpha = 1/2$ stable). Also notice that $L(k)$ has average $\sim k^{\gamma/(1-\gamma)}$ and variance $\sim k^{(3/2)\gamma/(1-\gamma)}$, which both diverge in the large k limit (truncation disappears). Accordingly, $L(k)$ is of order 1 with probability 1 and all the averages $\langle L(k)^{\beta}\rangle$ with negative β are of order 1. This property is true for any k and it also holds in the limit $k \to \infty$ where $L(k) \to L$.

Then, in the region $0 < \gamma < 1$, Eq. (7) can be rewritten as

$$t(k) \simeq y(k) + k^2 L, \tag{11}$$

where the term k has been be dropped since it is smaller with respect to $y(k)$ and $k^2 L$.

The region $0 < \gamma < 1$ splits into two subregions; when $1/2 < \gamma < 1$, the term $y(k)$ is larger than $k^2 L$, therefore we can assume $t(k) \simeq y(k)$ and therefore $y(t) \simeq t$ holds deterministically. When $0 < \gamma < 1/2$, on the contrary, $k^2 L$ is larger than $y(k)$ so $t(k) \simeq k^2 L$. This implies $k = (t(k)/L)^{1/2}$, which, substituted into (6), finally gives the relation

$$\langle y^{\alpha}(t)\rangle \simeq \langle L^{-\alpha\nu}\rangle\,(1-\gamma)^{\alpha/(1-\gamma)}\, t^{\alpha\nu} \tag{12}$$

with $\nu = 1/(2-2\gamma)$. The average $\langle L^{-\alpha\nu}\rangle$ is of order 1 since the exponent is negative, but we are unable to determine its exact value in terms of γ and α. So we simply conclude that $\langle y^{\alpha}(t)\rangle \sim t^{\alpha\nu}$.

At this point only the region $1 < \gamma < \infty$ remains, but for these values of γ, active walks of infinite length have a finite probability, therefore, after some excursions away from the maximum the walker decides once and for all to follow the same direction, remaining always on the maximum. Accordingly, the relation $y(t) \simeq |x(t)| \simeq t$ holds deterministically. Our analysis of the scaling behavior of $y(t)$ is thus concluded.

Since the walker spends part of the time on the maximum where $|x(t)| = y(t)$ and part of the time in an ordinary random walk in the interval $-y(t) < x(t) < y(t)$ where, on average, $|x(t)| \geqslant y(t)/2$ (the lazy trek always starts form the frontier), we also conclude that $\langle |x(t)|^{\alpha}\rangle \sim \langle y^{\alpha}(t)\rangle$.

V. FORWARD KOLMOGOROV EQUATION

We would like to test our results against the output of the exact numerical solution of the forward Kolmogorov equation. Indeed, the process is non-Markovian, but it can be rendered Markovian by enlarging the phase space, including the variable $y(t)$, and, therefore, jointly considering the evolution of the variables $x(t)$ and $y(t)$. Accordingly, it is possible to write a forward Kolmogorov equation for $P(x,y,t)$, which is the probability that $x(t) = x$ and $y(t) = y$.

The initial condition $(t = 0)$ is $P(0,0,0) = 1$ while all others $P(x,y,0)$ equal zero. First, notice that $P(x,y,t)$ obviously vanishes when $|x| > y$ and when $y > t$. Moreover, the symmetry of both initial condition and dynamics implies $P(x,y,t) = P(-x,y,t)$ for all x, y, and t. Thus, let us write the forward equation only for $x \geqslant 0$.

At any time $t \geqslant 1$ one has $P(0,0,t) = 0$; furthermore, one has $P(1,1,t) = 0$ at even times, $P(1,1,t) = (1/2)^{(t+1)/2}$ at odd times, $P(0,1,t) = 0$ at odd times, and $P(0,1,t) = (1/2)^{t/2}$ at even times.

Assuming that $y \geqslant 2$, the forward Kolmogorov equation is completed by

$$P(x,y,t+1) = \tfrac{1}{2}P(x+1,y,t) + \tfrac{1}{2}P(x-1,y,t), \tag{13}$$

which holds when $0 \leqslant x \leqslant y-2$. In the case where $x = 0$, we can use the symmetry to replace $P(-1,y,t)$ with $P(1,y,t)$ in the right-hand side of the equation. When $x = y-1$ we have

$$P(y-1,y,t+1) = (1-p(y))\,P(y,y,t) + \tfrac{1}{2}P(y-2,y,t), \tag{14}$$

where $p(y) = y^{\gamma}/(1+y^{\gamma})$ and, finally, when $x = y$ we have

$$P(y,y,t+1) = p(y-1)P(y-1,y-1,t) + \tfrac{1}{2}P(y-1,y,t). \tag{15}$$

We have numerically exactly solved the Kolmogorov equation and computed $\langle y(t)\rangle$, $\langle |x(t)|\rangle$, $\langle y^2(t)\rangle$, and $\langle |x(t)|^2\rangle$.

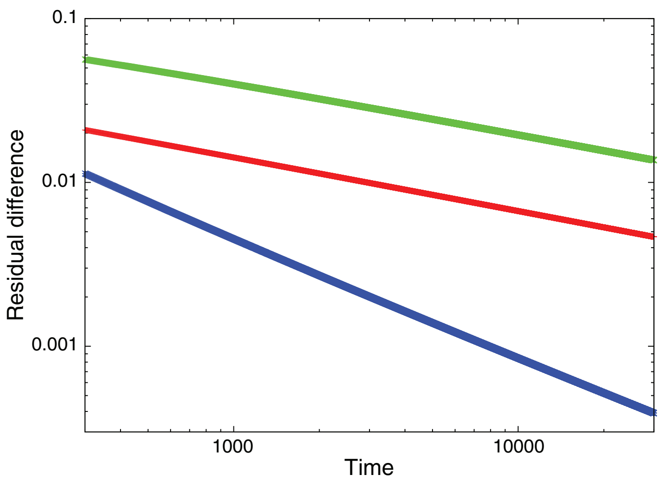

FIG. 3. (Color online) Log-log plot of $\sqrt{\langle y^2(t)\rangle}/\langle y(t)\rangle - 1$ (intermediate line, red), $\langle y(t)\rangle/(t/2\nu)^\nu - 1$ (upper line, green), and $2[\langle y(t+1)\rangle - \langle y(t)\rangle]/(t/2\nu)^{\nu-1} - 1$ (lower line, blue) for $\gamma = -1$.

The scaling exponent can be obtained by the ratio $\log(\langle y(t)\rangle)/\log(t)$ and analogous expressions where $\langle y(t)\rangle$ is replaced by $\langle |x(t)|\rangle$, $\langle y^2(t)\rangle$, or$\langle |x(t)|^2\rangle$. Nevertheless, convergence is much faster if one computes the exponent from $\log(\langle y(t_2)\rangle/\langle y(t_1)\rangle)/\log(t_2/t_1)$ and analogous expressions since the scaling factor is wiped out. In Fig. 1 we plot our results for $t_2 = 11\,000$ and $t_1 = 10\,000$ against prevision. Independently of the use of $\langle y(t)\rangle$, $\langle |x(t)|\rangle$, $\langle y^2(t)\rangle$, or $\langle |x(t)|^2\rangle$, we find excellent agreement. Notice that for $\gamma = 1/2$ the exponent ν equals 1 (ballistic behavior); for larger values of γ necessarily it must have the same value, thus, without loss of information, Fig. 1 ends at $\gamma = 1/2$.

In the region $-\infty < \gamma < 0$ we also have found the explicit scaling factor and the relation $\langle y^\alpha(t)\rangle \simeq \langle y(t)\rangle^\alpha$. In order to confirm the latter we consider the residual difference $\sqrt{\langle y^2(t)\rangle}/\langle y(t)\rangle - 1$ up to 30 000 time steps. This difference converges to 0 according to a power law as shown by the log-log plot in Fig. 3 for two orders of magnitude of time. Moreover, the log-log plot of $\langle y(t)\rangle/(t/2\nu)^\nu - 1$ shows power-law convergence to 0 (Fig. 3), proving that both scaling factor and scaling exponent are correct. A faster power-law convergence (Fig. 3) can be obtained considering $2(\langle y(t+1)\rangle - \langle y(t)\rangle)/(t/2\nu)^{\nu-1} - 1$, since only the differential average at the largest time contributes. The plot in Fig. 3 corresponds to the case $\gamma = -1$ but we have verified the same power-law behavior for various values in the region $-\infty < \gamma < 0$.

VI. CONCLUSIONS

Anomalous diffusion in this model is induced by long-range memory in a conceptually very simple manner; furthermore, the model is one-dimensional and it is controlled by a single parameter. In spite of this conceptual simplicity, the scaling behavior yields all possibilities, varying continuously from subdiffusive to ballistic. More precisely, if the walker timorously prefers to go back when it is at the frontier of unexplored regions, it is subdiffusive, on the contrary, if the walker boldly prefers to go where he or she never has gone before, it is superdiffusive.

The subdiffusive region is below the threshold $\nu = 0$. Above the threshold $\nu = 1/2$ the process is ballistic, and the walker moves uniformly at a constant velocity. Finally, in the region above the threshold $\nu = 0$ but below the threshold $\nu = 1/2$ the process is super-diffusive but subballistic. This region is probably the most interesting since the walker has an intermittent behavior, with bursts of linear growth followed by longer bursts of random motion. This behavior is typical of the transition from laminar to turbulent behavior in chaotic systems [24].

This reach phenomenology can be used, in principle, to model a variety of phenomena. We think, for example, of the problem of foraging strategies, with the walker (animal) changing his or her attitude when he or she is at the frontier of unexplored regions. The aim, in this case, is to evaluate the degree of success of the search in comparison with ordinary random walk search and Lévy search [25]. Also in epidemics, recent focus is on the effects of superdiffusive spreading of an infection, via heavy-tailed distributed jumps [26]. The present model could be alternative, with superdiffusive (or subdiffusive) spreading arising as an effect of memory of infection agents. Moreover, the orthography of languages performs a random walk on the discrete space of possible vocabularies [27,28]. As in the present model, the jump rates differ if changes are in the direction of a radical innovation or if they run on an already treaded territory. Finally, as we already mentioned, the (nonballistic) superdiffusive region of the present model could represent a stochastic counterpart to chaotic systems with intermittent behavior (see also Ref. [29]).

We conclude pointing out that the mathematical characterization of the BTRW in this paper is far from complete, for example, all the scaling factors for the variable $x(t)$ and the scaling factors for the variable $y(t)$ in the region $0 < \gamma \leqslant 1/2$ remain unknown as well as all the correlations at different times among variables.

Finally, we would like to underline that this model could be successfully extended to higher dimensions.

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APPENDIX

In the first part of this Appendix we prove the three relations, (i), (ii), and (iii), of Sec. III.

In the region $-\infty < \gamma < 0$, one has $p(y+s) = (y+s)^\gamma/(1+(y+s)^\gamma) \leqslant y^\gamma$, which implies $\pi(n|y) \leqslant y^{n\gamma}$. If n is small with respect to y, one also has $p(y+s) = (y+s)^\gamma/(1+(y+s)^\gamma) \simeq y^\gamma$ which, using (3), immediately gives the approximated equality $\pi(n|y) \simeq y^{n\gamma}$.

In the region $0 < \gamma < 1$, we directly obtain from (3)

$$[p(y)]^n \leqslant \pi(n|y) = \prod_{s=0}^{n-1} p(y+s) \leqslant [p(y+n)]^n, \qquad (A1)$$

in fact, with ν being positive, $p(y)$ is the smallest among the elements of the product and $p(y+\beta y^\gamma)$ the largest.

Then assume $n=\beta y^\gamma$ (if βy^γ is not an integer, then $n=\beta y^\gamma+\epsilon$, where $0<\epsilon<1$), one immediately gets

$$[p(y)]^{\beta y^\gamma+\epsilon} \leqslant \pi(n=\beta y^\gamma|y) \leqslant [p(y+\beta y^\gamma)]^{\beta y^\gamma+\epsilon}. \quad \text{(A2)}$$

Then, using the definition of $p(y)$ and taking into account that $0<\gamma<1$, it is straightforward to verify that the limit for $y\to\infty$ of both bounds is $e^{-\beta}$ so

$$\pi(n=\beta y^\gamma|y) \simeq e^{-\beta}. \quad \text{(A3)}$$

The above-approximated equality means that $n(y)\simeq \xi y^\gamma$, where ξ is a random variable distributed according to an unitary exponential probability.

Finally, consider the region $\gamma>1$, where we have

$$p(y+s) = 1/(1+(y+s)^{-\gamma}) \simeq e^{-1/(y+s)^\gamma}, \quad \text{(A4)}$$

which implies $\pi(n|y)\simeq e^{-\psi(y,n)}$, where $\psi(y,n)=\sum_{s=0}^{n-1} 1/(y+s)^\gamma$. Noticeably, $\nu>1$ implies that $\pi(\infty|y)$ is finite which, in turn, implies that $n(y)$ is infinite with finite probability.

We now compute the average and standard deviation of $m(y)$ which appear in Sec. III.

We preliminary remark that the process $x(t)$ is a SSRW when it is not on the maximum, therefore $m(y)$ is simply the random time necessary for hitting one of the frontiers of the interval $[-y,y]$ starting from position $y-1$ (or $-y+1$).

Assume that at time $t+1$ the walker is in $y-1$, i.e., $x(t+1)=y-1$ (the choice $x(t+1)=-y+1$ is symmetrical and leads to the same results). Also assume that the walker hits for the first time one of the barriers at time $t+1+m(y)$, i.e., $x(t+1+m(y))=y$ or $x(t+1+m(y))=-y$. The strong martingale property implies that the average of $x(t+1+s)-x(t+1)$ equals zero at any non-negative s also if s is a random time, therefore

$$\langle x(t+1+m(y))\rangle - x(t+1) = 0. \quad \text{(A5)}$$

Given that a is the probability of hitting the barrier y and $1-a$ is the probability of hitting the barrier $-y$, one has that $\langle x(t+1+m(y))\rangle = ay+(1-a)(-y)$. Therefore, equality (A5) rewrites $ay-y(1-a)-y+1=0$, which, in turn, implies $a=1-1/(2y)$.

Then we notice that $[x(t+1+s)-x(t+1)]^2-s$ is also a martingale, therefore

$$\langle [x(t+1+m(y))-x(t+1)]^2 - m(y)\rangle = 0, \quad \text{(A6)}$$

where $\langle [x(t+1+m(y)]^2\rangle = ay^2+(1-a)y^2=y^2$ which implies $\langle m(y)\rangle = a+(1-a)(2y-1)^2\simeq 2y$.

Moreover, $[x(t+1+s)-x(t+1)]^4-3s^2$ is as well a martingale and, therefore,

$$\langle [x(t+1+m(y))-x(t+1)]^4 - 3m^2(y)\rangle = 0, \quad \text{(A7)}$$

where $\langle [x(t+1+m(y)]^2\rangle = ay^4+(1-a)y^4=y^4$, which implies $\langle m^2(y)\rangle = (a+(1-a)(2y-1)^4)/3 \simeq (8/3)y^3$.

The standard deviation can be finally easily computed as $\sigma_{m(y)} = [\langle m^2(y)\rangle - \langle m(y)\rangle^2]^{1/2} \simeq (8/3)^{1/2}y^{3/2}$.

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