Linear and anomalous front propagation in systems with non-Gaussian diffusion: The importance of tails

Maurizio Serva,[1] Davide Vergni,[2] and Angelo Vulpiani[3]
[1]*Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università dell'Aquila, L'Aquila, Italy*
[2]*Istituto per le Applicazioni del Calcolo "Mauro Picone," CNR, Roma, Italy*
[3]*Dipartimento di Fisica, Università di Roma "Sapienza," and ISC-CNR, Roma, Italy*
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We investigate front propagation in systems with diffusive and subdiffusive behavior. The scaling behavior of moments of the diffusive problem, both in the standard and in the anomalous cases, is not enough to determine the features of the reactive front. In fact, the shape of the bulk of the probability distribution of the transport process, which determines the diffusive properties, is important just for preasymptotic behavior of front propagation, while the precise shape of the tails of the probability distribution determines asymptotic behavior of front propagation.

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I. INTRODUCTION

Reaction-diffusion processes appear in a large class of phenomena from combustion to ecology [1]. In the presence of nontrivial geometry (e.g., graphs) [2] or anomalous diffusion processes [3,4], the reaction dynamics is an intriguing and difficult issue [5–7].

In particular, the case of reaction in subdiffusive systems [8–11] is rather interesting for its role in living systems, since anomalous subdiffusion has been reported for different biological transport problems such as two-dimensional diffusion in the plasma membrane and three-dimensional diffusion in the nucleus and cytoplasm (see Ref. [13] and references therein).

The aim of the present work is to show that the behavior of tails of the probability distribution of the pure transport problem has the most important role for the front propagation properties, which can be anomalous even in the case of standard diffusion. On the other hand, one can have the standard linear front propagation also in the presence of subdiffusion. It is important to stress that, as already noted in Refs. [8,9] and clearly stated in Ref. [12], the details of the reaction-diffusion rules can be very important for the front propagation features.

The paper is organized as follows: in Sec. II we briefly summarize some results about nonstandard diffusive systems and front propagation in presence of reaction. Section III is devoted to an analysis of front propagation in a simplified model with subdiffusion. In Sec. IV we show that, even in a more realistic model, the basic ingredient for the asymptotic behavior of front propagation is given by the shape of the tail of the probability distribution, $P(x,t)$, of the pure transport problem. We point out that the shape of the bulk of $P(x,t)$, which determines the diffusive properties, could be important just for the preasymptotic behavior of the front propagation. Section V is dedicated to the conclusions.

II. A SURVEY OF KNOWN RESULTS

Let us indicate with $P(x,t)$ the probability distribution (or the probability density) that a walker is in x at time t. Under rather general hypothesis, $P(x,t)$ satisfies the equation

$$P(x,t+\Delta t) = \int_{-\infty}^{+\infty} P(x-u,t)\rho_{\Delta t}(u,x,t)du, \quad (1)$$

where $\rho_{\Delta t}(u,x,t)$ is the probability density to be in x at time $t+\Delta t$ under the condition to be in $x-u$ at time t. Even if $\rho_{\Delta t}(u,x,t)$ has an explicit dependence on x and t the process is Markovian. When $\rho_{\Delta t}(u,x,t)$ does not depend on x and t and in absence of fat tails, Eq. (1) describes standard diffusion, i.e., at large time $\langle x^2(t)\rangle \sim t$ and $P(x,t)$ is close to a Gaussian. The simplest case is the standard random walk, where $\Delta t = 1$ and $\rho(u) = \frac{1}{2}[\delta(u-1)+\delta(u+1)]$.

In order to introduce a reaction term in the diffusion Eq. (1), we consider [5] a time-discrete reaction process such as, in the absence of diffusion, $\theta(x,t+\Delta t) = G_{\Delta t}(\theta(x,t))$, where $\theta(x,t)$ is the concentration field, and $G_{\Delta t}$, the reaction map, can be approximated by $G_{\Delta t}(\theta) = \theta + \frac{\Delta t}{\tau} f(\theta)$, where $f(\theta)$ is the reaction term, which, in the simple auto-catalytic case, reads $f(\theta) = \theta(1-\theta)$ and τ is the characteristic time associated to the reaction. In any case, we are interested in pulled reactions (i.e., $f''(\theta) < 0$ and $f'(0) = 1$, as in the case of auto-catalytic reactions) for which the detailed shape of $f(\theta)$ is not important.

Following Ref. [5] the concentration evolution satisfies the equation

$$\theta(x,t+\Delta t) = \int_{-\infty}^{+\infty} \rho_{\Delta t}(u,x,t)G_{\Delta t}(\theta(x-u,t))du. \quad (2)$$

The above equation extends, for general diffusive process Eq. (1) in a discrete time version, the usual reaction-diffusion equation $\partial_t\theta = D\nabla\theta + f(\theta)/\tau$.

For Eq. (2) it is possible to apply the maximum principle [6,14], obtaining

$$\theta(x,t) \leqslant P(x,t)\, e^{\max_\theta\{f(\theta)/\theta\}t/\tau}. \quad (3)$$

In the case of pulled reaction, to which we always refer in this paper, the previous equation becomes (see Ref. [15])

$$\theta(x,t) \sim P(x,t)\, e^{t/\tau}. \quad (4)$$

For a standard diffusive process, i.e., when $P(x,t) \sim e^{-x^2/4Dt}$, Eq. (4) reads

$$\theta(x,t) \sim e^{-x^2/4Dt + t/\tau}, \quad (5)$$

where D is the diffusion coefficient. The front position $x_F(t)$ can be easily obtained assuming that the concentration $\theta(x,t)$, as it is given by Eq. (5), is of order 1. It turns out

$$x_F(t) = 2\sqrt{\frac{D}{\tau}}\,t, \tag{6}$$

representing the well-known result of a linear propagating front with velocity $v_f = 2\sqrt{D/\tau}$.

In the case of anomalous diffusion, a simple assumption for the probability distribution that generalizes the Gaussian shape for the standard diffusive case is

$$P(x,t) \sim e^{-C(x/t^{\nu})^{\beta}}. \tag{7}$$

The value of ν discriminates between subdiffusion and superdiffusion, with $\nu < 1/2$ and $\nu > 1/2$, respectively. Equation (7), in fact, implies that

$$\langle x^2(t)\rangle \sim t^{2\nu}. \tag{8}$$

Using Eq. (7) in Eq. (4), one gets

$$\theta(x,t) \sim e^{-C(x/t^{\nu})^{\beta}+t/\tau}, \tag{9}$$

so that, as previously discussed, one can easily obtain the front position $x_F(t)$ that, in this case, behaves as

$$x_F(t) \sim t^{\delta}, \quad \delta = \nu + \frac{1}{\beta}. \tag{10}$$

Let us stress that, also when Eq. (7) holds, the value of the exponent δ depends on the characteristics of the process underlying anomalous diffusion. An interesting case, suggested by an argument due to Fisher [6,16], is when

$$\beta = \frac{1}{1-\nu}. \tag{11}$$

Using Eq. (11) in Eq. (10), one gets that the front always propagates linearly in time, i.e., $\delta = 1$, although diffusion is anomalous ($\nu \neq 1/2$). Such a linear behavior holds both in the subdiffusive case (for example, in the random walk on a comb lattice where $\nu = 1/4$) and in the super-diffusive one (for example, in the random shear flow where $\nu = 3/4$) [6].

In general, when relation Eqs. (7) or (11) do not hold, the front propagation could be nonlinear in time. We refer to these cases as "nonstandard" propagation. We will see that nonstandard propagation can occur also in the case of a standard diffusive process when the tails of the $P(x,t)$ show a nonstandard behavior.

As an example, we report the work of Ref. [5], where the diffusion process is given by Eq. (1) but the probability density of jumps $\rho(u)$ has fat tails, i.e., $\rho(u) \sim 1/|u|^{\alpha+1}$ with α positive. The central limit theorem ensures that for large times t and for $\alpha \geqslant 2$,

$$\frac{|x(t)|}{t^{1/2}} = \frac{1}{t^{1/2}}\sum_{s=1}^{t} u_s \to \omega, \tag{12}$$

where the u_s are independent extractions with probability $\rho(u)$ and ω is a Gaussian variable. Accordingly, the standard diffusion scaling,

$$\langle |x(t)|^q\rangle \sim t^{q/2}, \tag{13}$$

holds (provided that $q < \alpha$ in order to avoid divergence induced by tails). Therefore, one would expect that the associate propagating front scales linearly with time, but, on the contrary, one finds that its behavior is exponential in time. In fact, while the bulk of the $P(x,t)$ becomes Gaussian when t increases, the tails continue to be fat as $1/|x|^{\alpha+1}$, although the frontiers between the Gaussian bulk and the fat tails shift at the extremes when time increases. Using Eq. (4), one obtains

$$\theta(x,t) \sim \frac{1}{|x|^{\alpha+1}}\, e^{t/\tau}, \tag{14}$$

so that for large times the front increases exponentially fast [5]:

$$x_F(t) \sim e^{t/(\alpha+1)\tau}. \tag{15}$$

In conclusion, the shape of the tails and not the scaling of $\langle |x(t)|^q\rangle$ is a crucial ingredient in determining the features of the front propagation.

III. DIFFUSION AND SUBDIFFUSION IN A SIMPLE MODEL

In the previous section we have seen that, in the presence of fat tails of the probability distribution for the pure transport process, standard diffusion are compatible with super-linear behavior for the front propagation. The main goal of the present paper is to confirm that the shape of such tails is the most important element influencing the scaling behavior of the front position. In particular, we are interested in subdiffusive and (more importantly) standard diffusive system when slim tails of the probability distribution induce to a sublinear behavior of the front propagation. In this respect, the present paper is complementary to the result presented at the end of the previous section and in Mancinelli *et al.* [5].

In order to tackle this goal we consider a very simple random walk model, which, according to a single control parameter, can be subdiffusive or diffusive: a walker, starting from home [$x(0) = 0$], at any discrete time can make a (unitary length) step to the right or to the left,

$$x(t+1) = x(t) + \sigma(t), \tag{16}$$

where

(1) $\sigma(t) = \pm 1$ with equal probability if $|x(t)| \leqslant t^{\lambda}$;
(2) $\sigma(t) = -\mathrm{sgn}[x(t)]$ if $|x(t)| > t^{\lambda}$.

This process is confined in a box by two reflecting boundaries situated in L and $-L$ where $L = t^{\lambda}$.

If $0 < \lambda < 1/2$, the process is subdiffusive according to $\langle |x(t)|\rangle \sim t^{\nu}$ with $\nu = \lambda$, as shown in Fig. 1. Notice that in this case the distribution of $z(t) = x(t)/t^{\nu}$ is approximately uniform between -1 and 1 (for large times) as the box dimension increases slowly with respect to the relaxation of the process in the box.

On the contrary, if $1/2 \leqslant \lambda \leqslant 1$, the process behaves as an ordinary diffusion with $\langle |x(t)|\rangle \sim t^{1/2}$ so that $\nu = 1/2 \neq \lambda$, as shown in Fig. 1. In this ordinary diffusive region the effect of the reflecting boundaries is just to kill the tails of the process for $|x| > t^{\lambda}$. The distribution of $z(t) = x(t)/t^{1/2}$, as time t increases, becomes closer to a Gaussian, with the truncation which shifts on the extremes and disappears for large times $t \to \infty$. This is shown in Fig. 2.

Model Eq. (16) is a simplified version of the non-Markovian random walk introduced in Refs. [17,18]. The stochastic

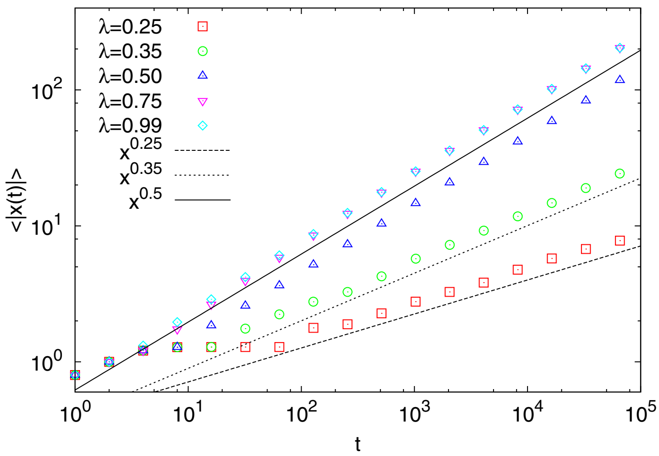

FIG. 1. Diffusion. The random walk process Eq. (16) is subdiffusive when $0 < \lambda < 1/2$, according to $\langle|x(t)|\rangle \sim t^{\nu}$ with $\nu = \lambda$, and is standard diffusive $\langle|x(t)|\rangle \sim t^{1/2}$ when $1/2 \leqslant \lambda < 1$.

process Eq. (16) is a Markovian process, so we can use Eq. (2) to describe reaction-diffusion dynamics.

When a reaction term is involved, it turns out that in all cases the exponent δ of the propagating front equals the exponent λ, which defines the process, i.e., $x_F(t) \sim t^{\delta}$ with $\delta = \lambda$. Therefore, not only the scaling behavior of the front is anomalous in case of subdiffusion (for $0 < \lambda < 1/2$), but also in case of regular diffusion (for $1/2 \leqslant \lambda < 1$). In fact, e.g., in the last case, Eq. (4) reads

$$\theta(x,t) \sim e^{-x^2/4Dt}\Theta(t^{\delta} - |x|)e^{t/\tau}, \tag{17}$$

where $\Theta(x)$ is the Heaviside step function representing the truncation of the distribution tails. Since the truncation overcomes the linear propagation due to the Gaussian diffusion together with the exponential growing of the reaction, one has

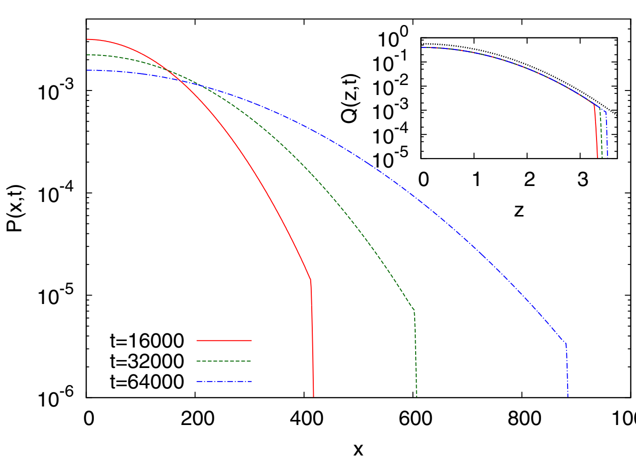

FIG. 2. Probability. The right side of the probability distribution $P(x,t)$ of the process Eq. (16) is shown for $\lambda = 0.55$ and different times. Although the value of γ is close to 1/2 the truncation only involves distribution tails that carry a minimal total probability. Moreover, truncation shifts to the right in time and asymptotically vanishes. This can be better appreciated in the inset where the probability $Q(z,t)$ of $z(t) = x(t)/t^{1/2}$ is shown and compared with a Gaussian (dotted black line).

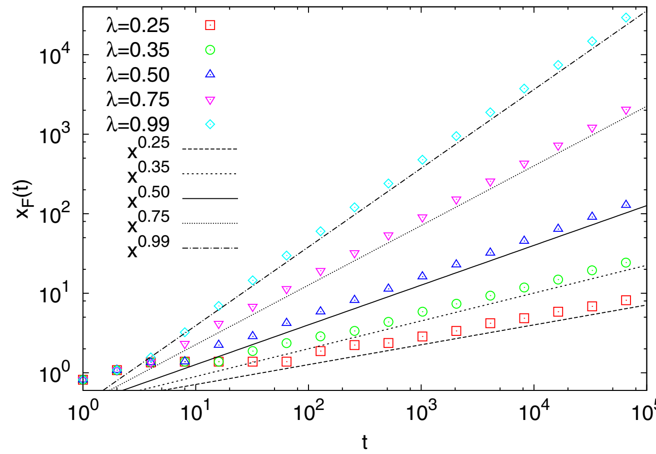

FIG. 3. Reaction. Front position x_F for different values of λ and $\tau = 1$ is shown. In all cases $x_F(t) \sim t^{\delta}$ with $\delta = \lambda$.

$x_F(t) \sim t^{\delta}$, as can be appreciated in Fig. 3, where the front position $x_F(t) = \min\{\,|x|\,;\,\theta(x,t) < 0.5\}$ is shown. However, completely equivalent results are obtained by computing the average position of reaction products or their total quantity.

IV. THE ROLE OF TAILS IN A MODEL WITH SUBDIFFUSION

We now consider a more complex model without the sharp boundaries of the previous one, in order to investigate the role of tails on front propagation. We consider

$$x(t+1) = x(t) + \sigma(t), \tag{18}$$

with $\sigma(t)$ given by

(1) $\sigma(t) = \mathrm{sgn}[x(t)]$ with probability $p(x(t),t)$ and

(2) $\sigma(t) = -\mathrm{sgn}[x(t)]$ with probability $1 - p(x(t),t)$, where

$$p(x,t) = \frac{1}{2 + \left(\frac{|x|}{t^{\lambda}}\right)^{\eta}}. \tag{19}$$

The above model has been motivated by ecological problems. For instance, the problem of foraging strategies, with the walker (animal) changing its attitude when it is at the frontier of unexplored regions [17,18].

Such a process does not present sharp boundaries. In fact, if η is finite, the walker can cross the position $|x| = t^{\lambda}$, but she advances with an increasing difficulty when she is at a larger distance from the origin. However, in the limit $\eta \to \infty$, this model coincides with the model of Sec. III (which has boundaries in $|x| = t^{\lambda}$). In the opposite case, i.e., for $\eta \to 0$, it is worth noting that the process becomes an "asymmetrical" random walk with probabilities 1/3 for $\sigma(t) = \mathrm{sgn}[x(t)]$ and 2/3 for $\sigma(t) = -\mathrm{sgn}[x(t)]$ (in the origin the probability is symmetrical). In this case the walker does not diffuse anymore and the stationary probability $P(x)$ can be easily computed from detailed balance finding $P(x) = \frac{1}{3}2^{-|x|}$.

The average $\langle|x(t)|\rangle$ is shown in Fig. 4 for $\lambda = 0.55$ and different values of η. As a consequence of the previous discussion for η small the diffusion decline (and for $\eta \to 0$ the diffusion is absent, with $\langle|x(t)|\rangle = \text{const.}$), while for η large the process tends to standard diffusion ($\langle|x(t)|\rangle \sim t^{1/2}$ since

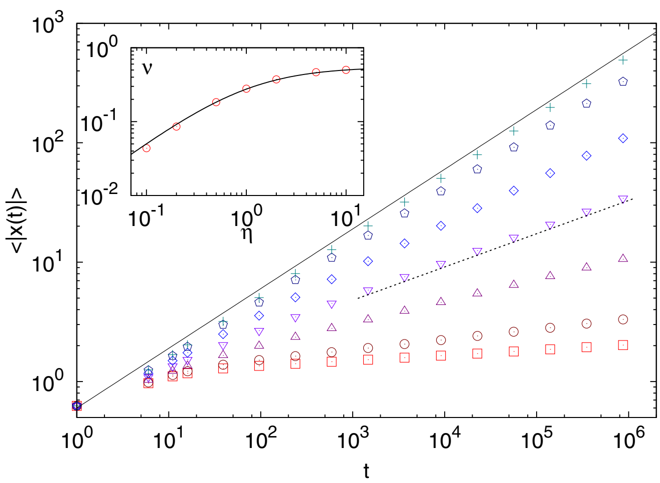

FIG. 4. Diffusion. The average $\langle|x(t)|\rangle$ is shown for $\lambda = 0.55$ for different values of η (from top to bottom $\eta = 10$ (+), $\eta = 5.0$ (⬠), $\eta = 2.0$ (◇), $\eta = 1.0$ (▽), $\eta = 0.5$ (△), $\eta = 0.2$ (○), $\eta = 0.1$ (□)). The dashed line is the scaling behavior $\langle|x(t)|\rangle \sim t^\nu$ for $\eta = 1$, with ν given by Eq. (22). In the inset the measured scaling ν is reported (○) together with the prediction Eq. (22) (solid line) at varying η.

$\lambda \geqslant 1/2$). For intermediate values of η, the process is always subdiffusive. A simple argument to compute the anomalous exponent is as follows. Using Eq. (18) and writing $\langle x(t)\sigma(t)\rangle$ in terms of the conditional expected value with respect to the probability Eq. (19), one obtains the exact evolution rule for $\langle x^2(t)\rangle$,

$$\langle x^2(t+1)\rangle - \langle x^2(t)\rangle = 1 - \left\langle \frac{2|x(t)|\left(\frac{|x(t)|}{t^\lambda}\right)^\eta}{2+\left(\frac{|x(t)|}{t^\lambda}\right)^\eta}\right\rangle. \tag{20}$$

First of all we assume a diffusive dynamics, $\langle|x(t)|\rangle \sim t^\nu$, with monofractal properties, i.e.,

$$\langle|x(t)|^\eta\rangle \sim \langle|x(t)|\rangle^\eta \sim t^{\nu\eta}.$$

Such a conjecture has been checked *a posteriori*. We observe that forcely $\nu < \lambda$, because in the opposite case the righthand side of Eq. (20) becomes negative, leading to a nondiffusive dynamics. Therefore, one can safely conclude that $(|x(t)|/t^\lambda)^\eta$ is vanishing for large times, and Eq. (20) can be approximated by

$$\langle x^2(t+1)\rangle - \langle x^2(t)\rangle \sim 1 - \frac{1}{t^{\lambda\eta}}\langle|x(t)|^{\eta+1}\rangle. \tag{21}$$

The above expression can be analyzed considering three different cases:

$\nu > \lambda\eta/(1+\eta)$: the second term of Eq. (21) is negative, implying absence of diffusion and contradicting the above assumptions.

$\nu = \lambda\eta/(1+\eta)$: Eq. (21) is coherent as far as $\lambda\eta/(1+\eta) \leqslant 1/2$, since the maximum scaling exponent consistent with Eq. (21) is $\nu = 1/2$.

$\nu < \lambda\eta/(1+\eta)$: in this case the second term of Eq. (21) is equal to one (for large times), implying $\nu = 1/2$.

In conclusion,

$$\nu = \min\left[\frac{1}{2}, \frac{\lambda\eta}{1+\eta}\right], \tag{22}$$

and in the case of $0 < \lambda \leqslant 1/2$ one always has $\nu = \lambda\eta/(1+\eta)$. Notice that, for any $0 < \lambda \leqslant 1$ and in the limit $\eta \to \infty$, one recovers the result already found for the model with sharp boundaries. In the inset of Fig. 4 the perfect agreement of the prediction Eq. (22) with the anomalous diffusive exponent $\langle|x(t)|\rangle \sim t^\nu$ is shown.

For the understanding of the features of front propagation we have to determine the shape of the probability distribution, $P(x,t)$, of the process Eq. (18). According to the definition of the process, we have the following conditional expectation with respect to $x(t) = x$:

$$\langle\sigma(t)|x\rangle = -\mathrm{sgn}(x)\frac{\left(\frac{|x|}{t^\lambda}\right)^\eta}{2+\left(\frac{|x|}{t^\lambda}\right)^\eta}. \tag{23}$$

Let us use the standard procedure to continuously approximate the forward Kolmogorov equation (Fokker-Planck equation in the terminology used in physics) for the probability density $P = P(x,t)$ as

$$\frac{\partial P}{\partial t} = -\frac{\partial(bP)}{\partial x} + \frac{1}{2}\frac{\partial^2(aP)}{\partial x^2}, \tag{24}$$

where

$$b(x,t) = \langle\sigma(t)|x\rangle = -\mathrm{sgn}(x)\frac{\left(\frac{|x|}{t^\lambda}\right)^\eta}{2+\left(\frac{|x|}{t^\lambda}\right)^\eta}, \tag{25}$$

and

$$a(x,t) = \langle\sigma^2(t)|x\rangle - \langle\sigma(t)|x\rangle^2 = 1 - b^2(x,t). \tag{26}$$

Moreover, with the change of variable $y = x/t^\nu$ one can define the probability density for y as $\tilde{P}(y,t)dy = P(y\,t^\nu,t)dx$. Since $dx = t^\nu\,dy$, one has $\tilde{P}(y,t) = P(y\,t^\nu,t)t^\nu$. This density satisfies the following equation:

$$\frac{\partial\tilde{P}}{\partial t} = \frac{\nu}{t}\frac{\partial(y\tilde{P})}{\partial y} - \frac{1}{t^\nu}\frac{\partial(\tilde{b}\tilde{P})}{\partial y} + \frac{1}{2t^{2\nu}}\frac{\partial^2(\tilde{a}\tilde{P})}{\partial y^2}, \tag{27}$$

where $\tilde{a}(y,t) = a(y\,t^\nu,t)$ and $\tilde{b}(y,t) = b(y\,t^\nu,t)$.

First we assume that $\lambda\eta/(1+\eta) > 1/2$, which implies $\nu = 1/2$ and consider the above equation with a and b given by Eqs. (26) and (25) with the choice $\nu = 1/2$. Neglecting all terms in Eq. (27) that are of higher order than $1/t$, one obtains

$$\frac{\partial\tilde{P}}{\partial t} = \frac{1}{2t}\left(\frac{\partial(y\tilde{P})}{\partial y} + \frac{\partial^2(\tilde{P})}{\partial y^2}\right). \tag{28}$$

This equation has a stationary solution $\tilde{P} = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$, which implies that the core of the distribution $P(x,t)$ is Gaussian, $P(x,t) = \frac{1}{\sqrt{2\pi\,t}}\,e^{-\frac{x^2}{2t}}$.

Second we assume that $\frac{\lambda\eta}{1+\eta} < 1/2$, which implies $\nu = \frac{\lambda\eta}{1+\eta}$. Neglecting all terms in Eq. (27) that are of higher order than $1/t^{2\nu}$, one obtains

$$\frac{\partial\tilde{P}}{\partial t} = \frac{1}{2t^{2\nu}}\left(\mathrm{sgn}(y)\frac{\partial(|y|^\eta\tilde{P})}{\partial y} + \frac{\partial^2(\tilde{P})}{\partial y^2}\right). \tag{29}$$

Also, this equation has a stationary solution $\tilde{P} = ce^{-\frac{|y|^{\eta+1}}{\eta+1}}$, where $c = (\eta+1)^{\eta/(\eta+1)}/2\Gamma(1/(\eta+1))$, $\Gamma(\cdot)$ being the gamma function. This implies that the core of the distribution

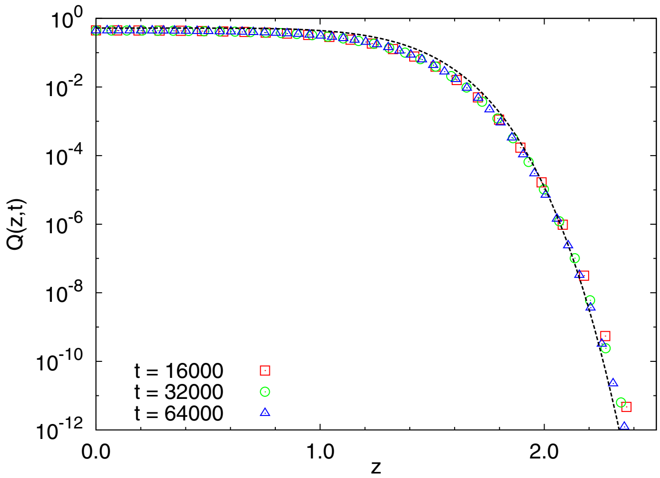

FIG. 5. Probability. The right side of the probability distribution $P(x,t)$ of the diffusive process Eq. (18) is shown using the rescaled variable $z(t) = x(t)/t^\nu$ with ν given by Eq. (22) for $\lambda = 0.55$ and $\eta = 5.0$ at different times. The dashed line is the rescaled distribution $Q(z) = \exp[-z^{\eta+1}/(\eta+1)]$ as from Eq. (30).

$P(x,t)$ is

$$P(x,t) = \frac{c}{t^\nu}\, e^{-\frac{1}{\eta+1}\left|\frac{x}{t^\nu}\right|^{\eta+1}}. \tag{30}$$

In conclusion, for large times, the distribution approaches a Gaussian when $\lambda\eta/(1+\eta) > 1/2$ and $\nu = 1/2$, and approaches the above anomalous density when $\nu = \lambda\eta/(1+\eta) < 1/2$. Let us remark that the transition between these two densities at $\lambda\eta/(1+\eta) = 1/2$ is discontinuous, unless $\eta = 1$.

In Fig. 5 we show the rescaled probability distribution associated to the process Eq. (18) for $\lambda = 0.55$, $\eta = 5.0$ and different times together with the prediction of Eq. (30). The agreement is very good.

Let us remark that in the $\eta \to \infty$ limit we obtain the same results already discussed in Sec. III for the model with sharp boundaries. In particular for $\lambda < 1/2$ one has that the anomalous distribution Eq. (30) is constant in the interval $-t^\nu \leqslant x \leqslant t^\nu$ and vanishes elsewhere, while for $\lambda > 1/2$ the core of the distribution is Gaussian. The effect on tails when $\eta = \infty$ is simply a sharp cut at $|x| = t^\lambda$, which obviously disappears when time diverges. Finally, in the limit $\eta \to 0$, the density Eq. (30) is independent from time with value $P(x,t) \sim e^{-|x|}$, confirming what was obtained directly choosing $\eta = 0$. Nevertheless, the exponential decay is different, considering that for $\eta = 0$ we found $P(x,t) \sim 2^{-|x|}$. This difference is a consequence of the fact that the large time limit and the vanishing η limit cannot be interchanged as it can be easily checked.

Now we are ready to discuss the front propagation behavior when a reaction term is involved. The argument in Eqs. (4), (9), and (10) would give $x_F \simeq \sqrt{2}\,t$ in the Gaussian case (i.e., when $\lambda\eta/(1+\eta) > 1/2$) and $x_F \simeq (1+\eta)^{1/(1+\eta)}\, t^\delta$, with

$$\delta = \frac{1+\lambda\eta}{1+\eta}, \tag{31}$$

for distribution Eq. (30) (i.e., when $\lambda\eta/(1+\eta) < 1/2$). These two scalings can be observed only for a transient time since,

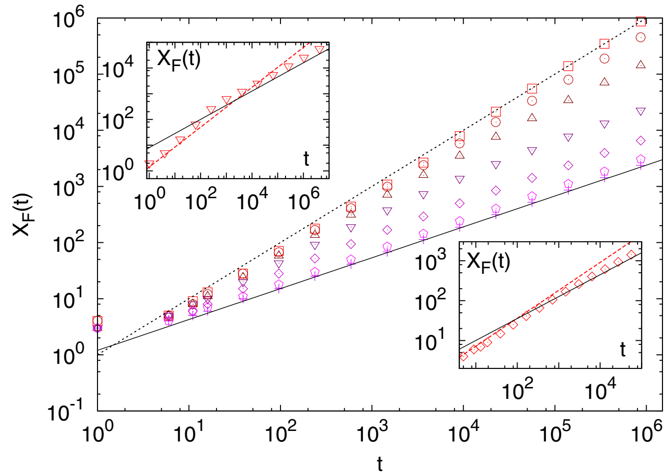

FIG. 6. Reaction. Front position for $\lambda = 0.55$, with reaction parameter $\tau = 1$ and different values of η (from bottom to top, $\eta = 10$ (+), $\eta = 5.0$ (⬠), $\eta = 2.0$ (◇), $\eta = 1.0$ (▽), $\eta = 0.5$ (△), $\eta = 0.2$ (○), $\eta = 0.1$ (□)) is shown. In all cases the scaling is $x_F(t) \sim t^\delta$ with $\delta = \lambda = 0.55$ (solid line), but there is a transient in which a different exponent [see Eq. (31)] is selected. In the limit case $\eta \to 0$ a linear front behavior ($x_F(t) \sim t$ dashed line) is restored. In the inset the scaling behavior in the transient region $x_F(t) = (1+\eta)^{1/(1+\eta)} t^\delta$ with δ given by Eq. (31) (dashed line) is shown together with the asymptotic behavior $x_F(t) = (A)^{1/\eta} t^\lambda$, with $A = 8$ corresponding to a threshold 1/10 for the jump probability Eq. (19) (solid line) for $\eta = 1.0$ (top left) and for $\eta = 2.0$ (bottom right). The matching between the two scalings gives the crossover time [see Eq. (32)].

as we will discuss below, the correct asymptotic scaling is $x_F(t) \sim t^\lambda$, as shown in Fig. 6 where the front position is shown for $\lambda = 0.55$, with reaction parameter $\tau = 1$ and different values of η. In all cases the asymptotic scaling is $x_F(t) \sim t^\delta$ with $\delta = \lambda = 0.55$, but there is a transient in which the exponent is given by Eq. (31). In the top-left inset it is clearly shown for $\eta = 1$ the preasymptotic and asymptotic behavior of the front position.

A simple theoretical explanation to justify the behavior of the asymptotic front properties goes as follows. Let us first remark that the forward jump probability Eq. (19) when $|x|$ is of the order of $t^{\lambda+\epsilon}$ vanishes as $t^{-\epsilon\eta}$, therefore, for large times the support of the distribution $P(x,t)$ vanishes when $|x| \sim t^{\lambda+\epsilon}$. This implies that the front $x_F(t)$ cannot move faster than $x_F(t) \sim t^\lambda$ when t is large. On the other hand, when $|x| \sim t^{\lambda-\epsilon}$, the forward jump probability Eq. (19) tends to 1/2 for large times for any positive ϵ; this implies that the front $x_F(t)$ moves at least as fast as $x_F(t) \sim t^\lambda$ when t is large. In conclusion, $x_F(t) \sim t^\lambda$.

A rough determination of the crossover time, i.e., the time at which the front propagation starts to reach its asymptotic behavior, can be given with a matching between the position of the front in the preasymptotic regime, $x_F(t) \simeq (1+\eta)^{1/(1+\eta)} t^\delta$, and the value $\tilde{x}(t)$ at which the jump probability Eq. (19) is small, $p(\tilde{x},t) = 1/[2 + (|\tilde{x}|/t^\lambda)^\eta] = 1/(2+A) \ll 1$, i.e., when $\tilde{x}(t) \simeq A^{1/\eta} t^\lambda$, where A is an appropriately large constant. The matching between those behaviors gives

$$t^* \simeq \left[\frac{A^{1/\eta}}{(\eta+1)^{(\eta+1)}}\right]^{\frac{(1+\eta)}{(1-\lambda)}}. \tag{32}$$

With $A = 8$, corresponding to a threshold $1/10$ for the jump probability Eq. (19), one has a good agreement with the actual results, as shown in the insets of Fig. 6.

V. CONCLUSIONS

We study a class of modified random-walk processes, whose probability distributions can be analytically determined, which can have, at varying a control parameter, standard or subdiffusive behavior. For reaction-diffusion systems, with a pulled reaction function, the scaling exponent of the diffusive problem is not relevant for the asymptotic behavior of the front: the basic ingredient to determine the front propagation behavior is the shape of tails of the probability distribution. This holds both for standard and subdiffusion.

Our results show, as already discussed in Refs. [8,9,12], that the details of the diffusion and reaction dynamics play a fundamental role in determining the front propagation features.

We conclude mentioning some topics that would be interesting to investigate. In our study we use a macroscopic description in terms of concentration, in addition, we considered a pulled reaction function, which acts even for very small concentration. The macroscopic approach is not appropriate if the density of transported "particles" is not very large, and therefore the discrete nature of the population cannot be neglected. We expect that the discreteness of the population will impact in nontrivial ways on the spatial propagation properties of the population. In addition, a topic to investigate, even in the macroscopic approach, is the role of shape of $f(\theta)$. For instance, we can probe how the use of reaction term in the class of Allee nonlinearity (see, for example, Ref. [1]) impacts on the front propagation behavior.

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