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Observability of market daily volatility

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HIGHLIGHTS

- We give a definition of market volatility.
- We show that it is observable.
- We show that it is consistent with stylized facts.

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ABSTRACT

We study the price dynamics of 65 stocks from the Dow Jones Composite Average from 1973 to 2014. We show that it is possible to define a Daily Market Volatility $\sigma(t)$ which is directly observable from data. This quantity is usually indirectly defined by $r(t) = \sigma(t)\omega(t)$ where the $r(t)$ are the daily returns of the market index and the $\omega(t)$ are i.i.d. random variables with vanishing average and unitary variance. The relation $r(t) = \sigma(t)\omega(t)$ alone is unable to give an operative definition of the index volatility, which remains unobservable. On the contrary, we show that using the whole information available in the market, the index volatility can be operatively defined and detected.

1. Introduction

It is well known that stock market returns are uncorrelated on lags larger than a single day. This is an inescapable consequence of the efficiency of markets. On the contrary, absolute returns have memory for very long times, a phenomenon known as volatility clustering. These phenomena are very well documented in the literature and known as stylized facts [1–4].

Furthermore, there is a large empirical evidence that volatility auto-correlations decay hyperbolically [5–9] and there is also a growing evidence that volatility signals have a multi-fractal nature [10–15].

Daily historical volatility is an unobservable variable and is usually measured by the absolute value of daily returns which are instead observable. This definition gives only an approximation of the real volatility $\sigma(t)$ which can be indirectly defined by $r(t) = \sigma(t)\omega(t)$ where $r(t)$ are the daily returns of the market index and the $\omega(t)$ are i.i.d. random variables with vanishing average and unitary variance. The relation $r(t) = \sigma(t)\omega(t)$ alone is unable to give an operative definition of the index volatility, which remains unobservable. We will show that using the whole information available in the market, the index volatility can be operatively defined and detected, i.e., we will define an observable volatility for a market index which exhibits all the statistical features expected for this variable. In this work we analyse 65 stocks from the 'Dow Jones 65 Composite Average', just to remind it is the composite index that measures changes within the 65 companies that make up

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three Dow Jones averages: the 30 stocks that form the Dow Jones Industrial Average (DJIA), the 20 stocks that make up the Dow Jones Transportation Average (DJTA) and the 15 stocks of the Dow Jones Utility Average (DJUA). The paper is organized as follows: we will first introduce all the definitions of the used variables and the model that gives all the relations between them, then we will analyse real data and show that the model conclusion is applicable to real data.

2. Model

The daily returns of single stock (say α) are given by $r_\alpha(t) = \ln[S_\alpha(t)/S_\alpha(t-1)]$ where $S_\alpha(t)$ is the closing price of stock α at day t. Then, if one wants to extract volatility from data one can consider that $r_\alpha(t) = \sigma_\alpha(t)\omega_\alpha(t)$ where the $\omega(t)$ have vanishing averages and unitary variance. Volatility $\sigma_\alpha(t)$ can be eventually extracted considering the high frequency (intra-day) continuous trading, but the problem remains highly unsolved because of the overnight contribution to the return $r_\alpha(t)$ for which there is not continuous trading. Therefore, the best measure of (historical) daily volatility simply remains the absolute returns $|r_\alpha(t)|$.

If the aim is to measure the global volatility of a market, we will show that things can be different. One can address the problem considering volatility of a proper representative index. Nevertheless, if one considers a price-weighted index (as Nikkei 225) the main contributions will be artificially given by those stocks with a larger price. The problem is circumvented if one considers a capitalization-weighted index (as Hang Seng) or an equally-weighted index (as S&P 500 EWI). The daily return $r(t)$ of this last index is simply the plain average of the returns of its components. i.e.

$$r(t) = \frac{1}{N}\sum_{\alpha=1}^{N} r_\alpha(t) \tag{1}$$

where N is the number of stocks in the basket and $r_\alpha(t) = \ln(S_\alpha(t)/S_\alpha(t-1))$ and $S_\alpha(t)$ is the daily closing price of the stock α at day t. For the other two types of indexes the difference is that the average is weighed by price or by capitalization.

Again, the underlying index daily volatility $\sigma(t)$ is not directly observable from daily returns but it is indirectly defined by $r(t) = \sigma(t)\omega(t)$. Because of market efficiency, it can be assumed that the $\omega(t)$ are independent identically distributed random variables with vanishing average and unitary variance. One could argue that $\sigma(t)$ is, indeed, observable from high frequency data, but, as already mentioned, the problem of the overnight contribution to the daily returns remains.

Therefore, since daily market volatility is not objectively given by the index return (only the product $\sigma(t)\omega(t)$ is observable), its distribution depends on the model chosen for $\omega(t)$. Gaussianity is often assumed as in ARCH–GARCH modelling (the leptokurticity of the distribution of returns is, in this case, entirely charged to volatility). Nevertheless, one can do other choices for the distribution of $\omega(t)$ as, for example, the uniform distribution (between $-\sqrt{3}$ and $\sqrt{3}$ in order that the variance is unitary). We will show that the data analysed in this work show this second behaviour.

3. Data analysis and results

In this paper we consider $N = 65$ stocks of Dow Jones from 1973 to 2014 so that $1 \le t \le T \simeq 10\,000$. Dow Jones as an index is not equally weighted but we can construct ourselves an EW Dow Jones index for which the daily returns $r(t)$ are simply the plain average of returns of its components as in definition (1).

The absolute returns are then given by

$$|r(t)| = \sigma(t)|\omega(t)| = \frac{1}{N}\left|\sum_{\alpha=1}^{N} r_\alpha(t)\right| \tag{2}$$

which are the absolute returns of the associated equally-weighted index (for price-weighted and capitalization-weighted indexes the only difference is that some weights must be introduced).

The core of this paper is the definition of the volatility as

$$\sigma(t) = \frac{1}{\sqrt{3}N}\sum_{\alpha=1}^{N} |r_\alpha(t)| \tag{3}$$

so that

$$\omega(t) = \frac{r(t)}{\sigma(t)} \tag{4}$$

where $r(t)$ and $\sigma(t)$ are defined in Eqs. (1) and (3). Eq. (3) can be considered as the definition of Daily Market Volatility, next we will show all the properties of this definition of volatility.

Most of the models assume that $\sigma(t)$ and $|\omega(t+\tau)|$ are uncorrelated for any value of the lag τ (negative, positive or vanishing) or they assume (as ARCH–GARCH) a short range correlation (a correlation only for small values of $|\tau|$). Moreover $|r(t)|$ and $|\omega(t+\tau)|$ should be uncorrelated for any non vanishing τ as well as $|\omega(t)|$ and $|\omega(t+\tau)|$.

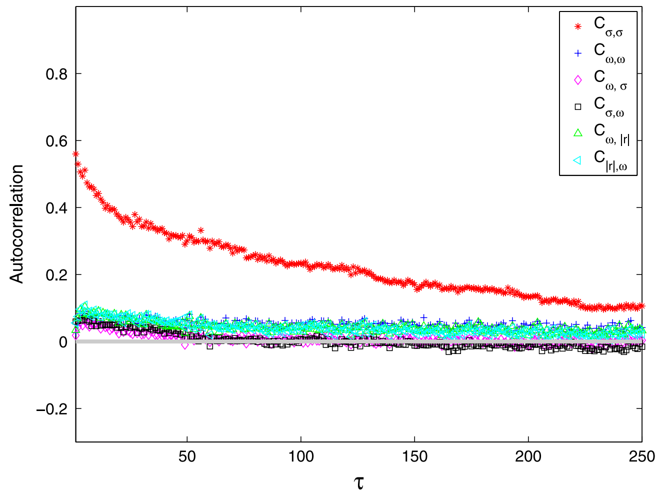

Fig. 1. Auto-correlation of σ, $|\omega|$ and all the cross-correlations between σ, ω and $|r|$.

Therefore, the first step is to show that all these properties hold according to our definition of volatility. After having computed the correlations $C_{\omega,\sigma} = C(\omega(t), \sigma(t+\tau))$, $C_{\sigma,\omega} = C(\sigma(t), \omega(t+\tau))$ and also $C_{\omega,\omega}$, $C_{\omega,|r|}$ and $C_{|r|,\omega}$, we have plotted them in Fig. 1. As it can be well appreciated the four above considered cross-correlations and the auto-correlation substantially vanish. This is better appreciated if compared with the volatility auto-correlation $C_{\sigma,\sigma}$ which is also plotted and which, on the contrary, exhibits a strong lag dependence and it is significantly positive for lags up to 250 working days.

Once it has been shown that the variables $\omega(t)$ are uncorrelated from each other, from the absolute returns (2) and from the volatilities (3), we need to show that they have vanishing expected value ($\langle\omega(t)\rangle = 0$) and unitary variance ($\langle\omega^2(t)\rangle = 1$). Indeed, we obtain a better result, in fact, the distribution of the $\omega(t)$ is uniform in the range $[-\sqrt{3}, \sqrt{3}]$ (which implies vanishing expected value and unitary variance but also implies $\langle|\omega(t)|\rangle = \sqrt{3}/2$). This result, which is our second step, can be appreciated in Fig. 2 where the empirical distribution or $|\omega|$, as defined by (4), is plotted.

We emphasize that this result comes directly form the data and the definition of $|\omega|$ and it is not assumed *a priori*. We still have a third step, to complete our argument. Having assumed uncorrelation of the $\omega(t)$ from the volatilities one forcefully has that the auto-correlation $C_{|r|,|r|}$ only differs for a positive multiplicative constant $0 < k < 1$ from the auto-correlation $C_{\sigma,\sigma}$ at any time lag $\tau \geq 1$. In fact, the auto-correlation of the absolute returns is

$$C_{|r|,|r|}(\tau) = \frac{\langle|r(t+\tau)|\,|r(t)|\rangle - \langle|r(t)|\rangle^2}{\langle r^2(t)\rangle - \langle|r(t)|\rangle^2} \tag{5}$$

then taking into account that $|r(t)| = \sigma(t)|\omega(t)|$ and assuming mutual uncorrelation between the $\sigma(t)$ and the $\omega(t)$, one has for any $\tau \geq 1$:

$$C_{|r|,|r|}(\tau) = k\, C_{\sigma,\sigma}(\tau) \tag{6}$$

where $C_{\sigma,\sigma}(\tau)$ is the volatility auto-correlation

$$C_{\sigma,\sigma}(\tau) = \frac{\langle\sigma(t+\tau)\sigma(t)\rangle - \langle\sigma(t)\rangle^2}{\langle\sigma^2(t)\rangle - \langle\sigma(t)\rangle^2} \tag{7}$$

and k is the constant

$$k = \frac{\langle\sigma^2(t)\rangle - \langle\sigma(t)\rangle^2}{3\langle\sigma^2(t)\rangle/4 - \langle\sigma(t)\rangle^2} \tag{8}$$

where we have used the uniform distribution value $\langle|\omega(t)|\rangle^2/\langle\omega^2(t)\rangle = 3/4$. We compute from sample $\langle\sigma^2(t)\rangle = 0.00008583$ and $\langle\sigma(t)\rangle = 0.008388$ so that $k = 1/2.85$.

Moreover, following the same steps, one can easily compute

$$C_{|r|,\sigma}(\tau) = C_{\sigma,|r|}(\tau) = \sqrt{k}\, C_{\sigma,\sigma}(\tau) \tag{9}$$

where k is the same value (8) previously computed ($k \simeq 1/2.85$) so that (6) and (9) are very strict requirements. It turns out that both relations (6) and (9) hold, in fact, the four correlations, rescaled by k (or by $\sqrt{k}$), are plotted in Fig. 3 where it can be appreciated that they are almost identical for all values of the lag τ.

Notice, that the coefficient k is not a fitting parameter, but it is independently derived by market data. The fact that after rescaling the four correlations coincide ultimately confirms that it is correct to write $r(t) = \sigma(t)\omega(t)$ where $\omega(t)$ and $\sigma(t)$ are the mutually uncorrelated variables we have defined.

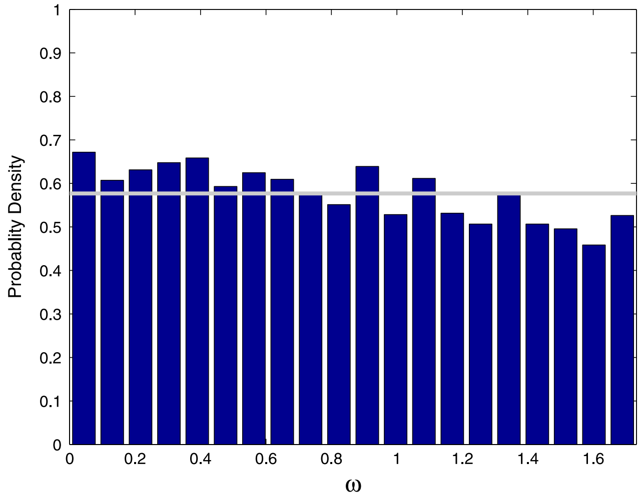

Fig. 2. Uniform (almost) probability density of the $|\omega|$. One has $\langle\omega^2\rangle = 0.924 \simeq 1$ and $\langle|\omega|\rangle = 0.823 \simeq \sqrt{3}/2 = 0.866$. The horizontal line represents the theoretical uniform distribution between 0 and $\sqrt{3}$.

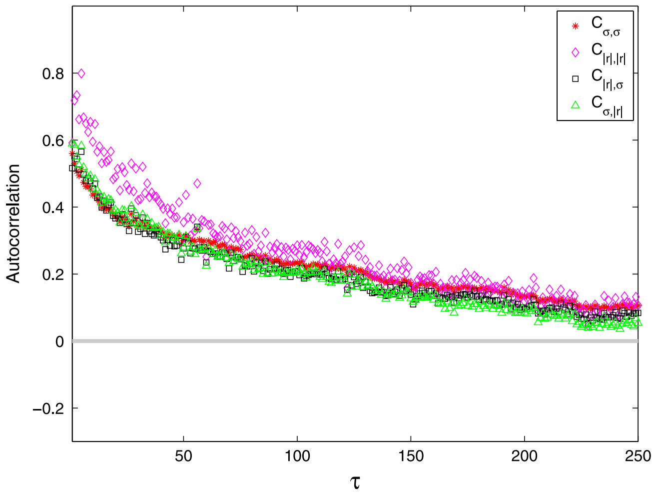

Fig. 3. Auto-correlations of σ and $|r|$ and their cross-correlations.

4. Conclusions

In sum, we have defined the volatilities $\sigma(t)$ and, consequently, the variables $\omega(t)$ so that they (a) are mutually uncorrelated, (b) the $\omega(t)$ are also uncorrelated from absolute returns, (c) the $\omega(t)$ are i.i.d. uniformly distributed with vanishing expected value and unitary variance, (d) the auto-correlation of the volatility exhibits a strong lag dependence and it is significantly positive for lags up to 250 working days, (e) the correct scaling of cross-correlations and auto-correlations involving the $r(t) = \sigma(t)\omega(t)$ and the $\sigma(t)$ holds. Therefore, we have given a definition of daily market volatility which is observable and keep all the statistical features expected for this variable.

We can conclude by saying that while for a single stock it is impossible to extract the volatility from return using $r(t) = \sigma(t)\omega(t)$, we have found a simple way to do that for an index using all absolute returns of the single stocks entering into the index definition.

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