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Random Motion of Light-Speed Particles

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Abstract
In 1956 Mark Kac proposed a process related to the telegrapher equation where the particle travels at constant speed (say the speed of light c) and randomly inverts its velocity. This process had important applications concerning the path-integral solution and the probabilistic interpretation of the 1+1 dimensions Dirac equation. The extension to 3+1 dimensions requires that the particle only moves at light-speed, which implies that velocity can be represented as a point on the surface of a sphere of radius c. The realizations of the process for the velocity only may connect these points, and, by strict analogy with the Kac model, it can be assumed that the velocity jumps from one value to another. In this paper we follow a new and different strategy assuming that the velocity performs continuous trajectories (velocity changes direction in a continuous way) which are the realization of a Wiener process on the surface. The processes which emerge transform one in the other by Lorentz boost. The associate Forward Kolmogorov Equation for the joint probability density of position and velocity, which is the (3+1) dimensional analogous of the telegrapher equation, is examined and a simplification is performed by means of variables separation.

Keywords Brownian motion · Wiener process · Relativity · Lorentz boost · Ito calculus

1 Introduction

In 1956 the Polish-American physicist and mathematician Mark Kac proposed a process where the particle travels in one space dimension at constant speed (left or right) and randomly inverts its velocity, and he proved that the associated probability density satisfies the telegrapher equation [1].

About 30 years after the Kac pioneering work, Gaveau et al. noticed that this process could be easily associated to the Dirac equation in 1+1 dimensions (one space dimension + time) by an analytical continuation of time. Then, using this equivalence, they were able to provide a probabilistic solution of the Dirac equation in terms of the Kac light-speed trajectories [2].

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Indeed, the process considered by Gaveau et al. is part of a larger class, in fact, by Lorentz boosts new processes can be obtained. The particles still move the speed of light c (a simple consequence of the fact that a light-speed particle in an inertial frame is also light-speed in any other inertial frame) but velocity inversions from right to left and from left to right occur at a different probability rate.

The class of these (1+1)-dimensional light-speed process can be further extended by considering inversion rates which not only depend on the sign of the velocity but also on position and time. This extension gave the possibility to reformulate the (1+1)-dimensional quantum mechanics of a relativistic particle in terms of stochastic processes [3].

In symbols, the position x satisfy the differential equation $dx(t) = cn(t)dt$ where $c(t) = cn(t)$ is the velocity, and $n(t) = \pm 1$ is a dichotomous variable. $n(t)$ changes sign according a probability rate $a(n, x, t)$ which, in general, depends on the dichotomous variable n, on the one-dimensional position x and on time t.

Given that $\rho(n, x, t)$ is the probability density that the particle is in x with velocity cn at time t, one has the Forward Kolmogorov Equation

$$\frac{\partial \rho(n,x,t)}{\partial t} = -\,cn\frac{\partial \rho(n,x,t)}{\partial x} + a(-n,x,t)\rho(-n,x,t) - a(n,x,t)\rho(n,x,t). \qquad (1)$$

Each process of the above class transforms in another of the same class by Lorentz boost [3]. In the original Kac model the simplest choice was made: constant rates, i.e., $a(n, x, t) = a(-n, x, t) = a$.

The extension to 3+1 dimensions requires that the particle only moves at light speed and velocity may only change direction. This implies that the velocity can be represented as a point on the surface of a sphere of radius c and the realizations of the process for the velocity only may connect points on this surface.

By strict analogy with the Kac model, it can be assumed that the velocity jumps from one point on the surface of the sphere to another, keeping in this way the speed constant. In symbols: the three-dimensional position $\mathbf{x}$ satisfy the differential equation $d\mathbf{x}(t) = c\mathbf{n}(t)dt$ where $\mathbf{c}(\mathbf{t}) = c\mathbf{n}(t)$ is the velocity and $\mathbf{n}(t)$ is a unitary vector which can be represented as a point on the surface of a sphere of unitary radius. $\mathbf{n}(t)$ jumps from $\mathbf{n}$ to $\mathbf{n}'$ according to a probability rate $a(\mathbf{n}, \mathbf{n}', \mathbf{x}, t)$ which, in general, depends on the starting vector $\mathbf{n}$, on the arrival vector $\mathbf{n}'$, on the three-dimensional position $\mathbf{x}$ and on time t.

Given that $\rho(\mathbf{n}, \mathbf{x}, t))$ is the probability density that the particle is in $\mathbf{x}$ with velocity $c\,\mathbf{n}$ at time t, one has the Forward Kolmogorov Equation

$$\begin{aligned}\frac{\partial \rho(\mathbf{n},\mathbf{x},t)}{\partial t} &= -\,c\mathbf{n}\nabla\rho(\mathbf{n},\mathbf{x},t) + \int dS(\mathbf{n}')\, a(\mathbf{n}',\mathbf{n},x,t)\rho(\mathbf{n}',\mathbf{x},t) \\ &\quad - \int dS(\mathbf{n}')\, a(\mathbf{n},\mathbf{n}',x,t)\rho(\mathbf{n},\mathbf{x},t), \end{aligned} \qquad (2)$$

where $dS(\mathbf{n})$ is the surface element in correspondence of $\mathbf{n}$ and integrals go over all the surface of the sphere of unitary radius. Again, each process of the above class transform in another of the same class by Lorentz boost. The simplest process has constant jumping rates, i.e., $a(\mathbf{n}, \mathbf{n}', x, t) = a$.

Remark that in this model the probability density has a singularity. Consider, for example, a process that starts in the origin $\mathbf{x}(0) = 0$ which has a constant jumping rate a, then the probability that $|\mathbf{x}(t)| = ct$ is e^{-at}. This implies a singularity of the probability density $\rho(\mathbf{n}, \mathbf{x}, t)$ in all, or some of the points were $|\mathbf{x}| = ct$.

2 The Model

In this paper we follow a different strategy which removes singularities. We assume that the velocity performs continuous trajectories (velocity changes direction in a continuous way) which are the realizations of uniform Wiener process on the surface of a sphere of radius c. In this way while the speed is always c, velocity direction changes following trajectories which are almost everywhere continuous but not differentiable, while the trajectories of the position are continuous and differentiable.

In symbols:

$$\begin{aligned} d\mathbf{x}(t) &= c\mathbf{n}(t)dt, \\ d\mathbf{n}(t) &= -\omega^2\mathbf{n}(t)\,dt + \omega\,d\mathbf{w}(t), \end{aligned} \tag{3}$$

where $d\mathbf{w}(t)$ is a two-dimensional Wiener increment perpendicular to $\mathbf{n}(t)$. The equation is written according to Ito, so that $d\mathbf{n}(t) = \mathbf{n}(t+dt) - \mathbf{n}(t)$ and $d\mathbf{w}(t) = \mathbf{w}(t+dt) - \mathbf{w}(t)$ is a two component standard Wiener increment on the plane perpendicular to $\mathbf{n}(t)$ such that $E[\,|d\mathbf{w}(t)|^2] = 2dt$.

The second of the above equations describes a uniform Wiener process on surface of a unitary sphere (so that $\mathbf{c(t)} = c\mathbf{n}(t)$ remains on the surface of the sphere of radius c). The Wiener process on a spherical surface was studied for the first time at least 70 years ago [4,5] and later continued to receive attention [6–11]. It is straightforward to prove from (3) that the speed of the particle remains constantly luminal *i.e,* $|\mathbf{c}(t)| = c$ at any time $t \geq 0$. In fact, according to Ito:

$$d|\mathbf{n}|^2 = -2\omega^2|\mathbf{n}|^2\,dt + 2\omega\mathbf{n}\cdot d\mathbf{w}(t) + \omega^2 E[\,|d\mathbf{w}(t)|^2] = -2\omega^2(|\mathbf{n}|^2 - 1)dt, \tag{4}$$

where the first equality is obtained by Ito calculus while for the second equality we have used the mutual orthogonality of $d\mathbf{w}(t)$ and $\mathbf{n}(t)$ and we have also used $E[\,|d\mathbf{w}(t)|^2] = 2dt$. The equality $d|\mathbf{n}|^2 = -2\omega^2(|\mathbf{n}|^2 - 1)dt$ implies that $\mathbf{n}(t)$ remains on the surface of a unitary sphere if it is there at initial time, consequently, the velocity $\mathbf{c}(t) = c\mathbf{n}(t)$ is a point on the surface of a sphere of radius c at any time.

This is the 'rest frame' process, we may also consider the general class of processes generated by Lorentz boosts. Assume, for example, that the 'rest frame' moves at constant velocity $\mathbf{v}$ with respect to another frame, using Ito calculus and Lorentz boost rules, after a long but straightforward calculation one gets that the process in the new frame reads

$$\begin{aligned} d\mathbf{x}(t) &= c\mathbf{n}(t)dt, \\ d\mathbf{n} &= -\frac{\omega^2}{\alpha^3}\left[1 - \frac{\mathbf{v}\cdot\mathbf{n}}{c}\right]^3 \mathbf{n}dt + \omega\left[\frac{1}{\alpha}\left(1 - \frac{\mathbf{v}\cdot\mathbf{n}}{c}\right)\right]^{\frac{3}{2}} d\mathbf{w}, \end{aligned} \tag{5}$$

where $\alpha = \left(1 - v^2/c^2\right)^{\frac{1}{2}}$. Notice that we have not renamed the variables. It is easy to verify that the speed remains unchanged, this is not astonishing since a particle moving at the speed of light also moves at the speed of light in any other inertial frame. The more the product $\mathbf{v}\cdot\mathbf{n}$ is large, the more the diffusion slows down favouring the motion in the $\mathbf{v}$ direction.

The above equation defines a class of light-speed processes which transform one in the other by Lorentz boost. The special 'rest frame' case (3) corresponds to $\mathbf{v} = 0$.

All processes (5) are (3+1)-dimensional versions of the Kac process, nevertheless, they can be also considered as peculiar Ornstein–Uhlenbeck processes constructed in such a way that the speed remains constant. Therefore, they also belong to the long tradition of relativistic diffusion process of this type [12–19]. The Forward Kolmogorov Equation for the joint probability density of position and velocity in the 'rest frame' (3) can be written as

$$\frac{\partial \rho(\mathbf{n}, \mathbf{x}, t)}{\partial t} = -c\mathbf{n}\nabla \rho(\mathbf{n}, \mathbf{x}, t) + \frac{\omega^2}{2}\Delta_c \rho(\mathbf{n}, \mathbf{x}, t), \tag{6}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, while Δ_c is the velocity Laplacian on the surface of the unitary sphere. In case, using spherical coordinates, one can write $\mathbf{n} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta)$ and

$$\Delta_c = \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The use of spherical coordinates for the velocity makes clear that an integration of the position coordinates leaves an equation for the density probability of the sole velocity, while the opposite is not true i.e., an integration of the velocity coordinates doesn't leave an equation for the density probability of the sole position. This fact can be also seen from the stochastic equation (3), where the second equation is autonomous with respect to the first, but not the opposite.

3 Special Variables

The question is if there is a way to separate the position variables in order to solve autonomously the associated equation. The answer is positive, but somehow hybrid variables have to be considered.

Consider two unitary vectors $\mathbf{n_2}(t)$ and $\mathbf{n_3}(t)$ which are both orthogonal to $\mathbf{n}(t))$ and which are mutually orthogonal, then one can make the choice $d\mathbf{w}(t) = \mathbf{n_2}dw_2(t) + \mathbf{n_3}dw_3(t)$, where $dw_1(t)$ and $dw_2(t)$ are two standard independent Wiener increments ($E[|dw_1|^2] = E[|dw_2|^2] = dt$, $E[dw_1 dw_2] = 0$). Consequently, the second equation in (3) rewrites as

$$d\mathbf{n} = -\omega^2\mathbf{n}\,dt + \omega\mathbf{n_2}dw_2 + \omega\mathbf{n_3}dw_3. \tag{7}$$

Then, assume the following stochastic equations for $\mathbf{n_2}(t)$ and $\mathbf{n_3}(t)$:

$$\begin{aligned} d\mathbf{n_2} &= -\frac{\omega^2}{2}\mathbf{n_2}\,dt - \omega\mathbf{n}dw_2, \\ d\mathbf{n_3} &= -\frac{\omega^2}{2}\mathbf{n_3}\,dt - \omega\mathbf{n}dw_3,\,. \end{aligned} \tag{8}$$

Assume that at a given time the three vectors are unitary and mutually orthogonal, then, according to equation (4), we have $d|\mathbf{n}|^2 = 0$. Moreover, using again Ito calculus we also obtain from (7) and (8):

$$\begin{aligned} d|\mathbf{n_2}|^2 &= -\omega^2(|\mathbf{n_2}|^2 - 1)dt, \\ d|\mathbf{n_3}|^2 &= -\omega^2(|\mathbf{n_3}|^2 - 1)dt, \\ d(\mathbf{n}\cdot\mathbf{n_2}) &= -\frac{5\omega^2}{2}(\mathbf{n}\cdot\mathbf{n_2})dt, \\ d(\mathbf{n}\cdot\mathbf{n_3}) &= -\frac{5\omega^2}{2}(\mathbf{n}\cdot\mathbf{n_3})dt, \\ d(\mathbf{n_2}\cdot\mathbf{n_3}) &= -\omega^2(\mathbf{n_2}\cdot\mathbf{n_3})dt, \end{aligned} \tag{9}$$

where we used orthogonality to set to zero all the random increments at the right side of the above equations (as in the second equality in (4)). Moreover, orthogonality and unitarity also imply that all the five deterministic increments at the right side in equations (9) vanish as well. Therefore, if $\mathbf{n}(t)$, $\mathbf{n_2}(t)$ and $\mathbf{n_3}(t)$ are initially a set of mutually orthogonal unitary vectors, they preserve this property at any time so that they can be used as a basis for the decomposition of the position vector $\mathbf{x}(t)$.

Let us define $\xi_1 = \mathbf{x} \cdot \mathbf{n}$, $\xi_2 = \mathbf{x} \cdot \mathbf{n_2}$ and $\xi_3 = \mathbf{x} \cdot \mathbf{n_3}$. Using the first equation in (3) and both the Eqs. (7) and (8), we derive by Ito calculus:

$$\begin{aligned} d\xi_1 &= c dt - \omega^2 \xi_1 dt + \omega \xi_2 dw_2 + \omega \xi_3 dw_3, \\ d\xi_2 &= -\frac{\omega^2}{2} \xi_2 dt - \omega \xi_1 dw_2, \\ d\xi_3 &= -\frac{\omega^2}{2} \xi_3 dt - \omega \xi_1 dw_3. \end{aligned} \tag{10}$$

The associated Forward Kolmogorov Equation for the probability density $\rho(\xi, \xi_2, \xi_3, t)$, restricted to these three variables, is

$$\frac{\partial \rho}{\partial t} = -\nabla_\xi (\mathbf{b}\rho) + \frac{\omega^2}{2} \sum_{i,j} \frac{\partial^2 D_{i,j} \rho}{\partial \xi_1 \partial \xi_j}, \tag{11}$$

where ∇_ξ is the gradient $(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3})$, $\mathbf{b}$ is the drift $(c - \omega^2 \xi_1, -\frac{\omega^2}{2}\xi_2, -\frac{\omega^2}{2}\xi_3)$ while D is the matrix

$$\begin{bmatrix} \xi_2^2 + \xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & \xi_1^2 & 0 \\ -\xi_1\xi_3 & 0 & \xi_1^2 \end{bmatrix}. \tag{12}$$

The fact that these three variables can be isolated does not mean that the complete probability density (position + velocity) only depends on them. Nevertheless, in case the process starts from the origin ($\mathbf{x}(0) = \boldsymbol{\xi}(0) = 0$) and the initial velocity distribution is uniform, it is easy to realize that the complete probability density only depends on these three variables at any time.

There are important averages of these variables (or combinations of them) that can be explicitely computed. Given that $|\mathbf{x}|^2 = (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2$, from Eq. (10) (but also directly from the first equation in (3)) we find $d|\mathbf{x}|^2 = 2c\xi_1 dt$. Taking the average of this equation as well the average of the first equation in (10), we obtain the following system

$$\begin{aligned} dE[\xi_1] &= c dt - \omega^2 E[\xi_1] dt, \\ dE[|\mathbf{x}|^2] &= 2c E[\xi_1]\, dt. \end{aligned} \tag{13}$$

These equations can be easily solved obtaining

$$\begin{aligned} E[\xi_1(t)] &= \frac{c}{\omega^2} - \frac{c}{\omega^2} e^{-\omega^2 t}, \\ E[|\mathbf{x}(t)|^2] &= \frac{2c^2}{\omega^2}\, t - \frac{2c^2}{\omega^4} \left(1 - e^{-\omega^2 t}\right), \end{aligned} \tag{14}$$

where we have assumed without loss of generality that the system is initially in the origin, i.e., $\mathbf{x}(0) = \boldsymbol{\xi}(0) = 0$.

The above averages imply, for large times, a diffusive behaviour with coefficient $\frac{c^2}{\omega^2}$, in this limit one has, in fact, $E[|\mathbf{x}(t)|^2] \sim \frac{2c^2}{\omega^2}\, t$. On the contrary, for short times $E[|\mathbf{x}(t)|^2] \sim c^2 t^2$ which means ballistic behaviour at the speed of light.

The short-term ballistic behaviour it is not completely unexpected. In fact, for a small time $\Delta t \ll 1/\omega^2$ the velocity of a particle remains almost constant. This can be understood from the second equation in (3) which, having defined $\Delta\mathbf{n} = \mathbf{n}(\Delta t) - \mathbf{n}(0)$, implies $|\Delta\mathbf{n}| \simeq |\omega^2\mathbf{n}\,\Delta t - \omega\,\mathbf{w}(\Delta t)| \leq \omega^2\Delta t + \omega|\mathbf{w}(\Delta t)|$. Since $|\mathbf{w}(\Delta t)|$ is of the order of $\sqrt{\Delta t}$ and given that $\Delta t \ll 1/\omega^2$, one has $|\Delta\mathbf{c}|/c = |\Delta\mathbf{n}| \ll 1$.

4 Conclusion

The class of processes we described here, which is new and hopefully interesting in its own, could probably be suitable for modeling the Brownian motion of massless particles.

A point which we would like to investigate is the possible connection between the Backward Kolmogorov Equation and some (3+1)-dimensional relativistic equations as Klein–Gordon and Dirac and, in case, to find their path-integral solution in terms of light-speed trajectories.

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