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Particles with constant speed and random velocity in 3+1 space-time: separation of the space variables

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Abstract

We consider a particle in 3+1 space-time dimensions which constantly moves at speed of light c, randomly changing its velocity which can be represented by a point on the surface of a sphere of radius c. The velocity performs an isotropic Wiener process on this surface so that the velocity trajectories are almost everywhere continuous although not differentiable, while the position trajectories are continuous and differentiable. Together with the process that describes the particle in the 'rest frame', where the diffusion of velocity on the surface of the sphere is isotropic, the entire family of anisotropic processes which result from Lorentz boosts is also described. The focus of this article is on stochastic evolution in space. We identify a reduced set of variables whose stochastic evolution is autonomous from the remaining variables, but, nevertheless, carry all the relevant information concerning the spatial properties of the process. The associated stochastic equations as well the Forward Kolmogorov equation are considerably simplified compared to those of the complete set of position and velocity variables.

Keywords: Wiener process, Itô calculus, relativity, Lorentz boost, forward Kolmogorov equation

1. Introduction

In this paper we will focus on the motion of a particle that constantly moves at the speed of light c while randomly changing its velocity.

In 1+1 space-time dimensions (one space dimension + time) the particle has only two alternatives: moving in one direction or the opposite with same speed, with the only option of randomly switching from one to the other of the two possible velocities. This stochastic process was considered in 1956 by the Polish physicist and mathematician Mark Kac who assumed that the probability rate of a jump is constant and independent of the direction of motion (velocity sign). Then, he proved that the associated probability density satisfies the telegrapher's Equation [1, 2]. In order to ensure relativistic invariance one has also to consider all processes which can be generated by Lorentz boost from the Kac process (let us call it the 'rest frame' process). As we will discuss in the next section, the rate of jumps of the process in this relativistically invariant class is, in general, asymmetric, *i.e.* it depends on the velocity sign.

In a recent research [3–5] we considered a family of processes which generalizes the Kac approach to the (3+1)-dimensional case (three space dimensions + time). We assumed that the particle only moves at the speed of light c which implies that velocity can be represented by a point on the surface of a sphere of radius c. We also assumed that the velocity performs an isotropic Wiener process on this surface. This second choice is not compulsory as processes with discontinuous velocity switches (jumps on the surface of the sphere) can also be considered, however it ensures that the velocity trajectories are almost everywhere continuous although not differentiable and that the position trajectories are continuous and differentiable. As we will see, this avoids singularities in the position probability density which, on the contrary, appear forcefully in the 1+1 space-time model with discontinuous velocity changes.

Since our particle lives in a relativistic world, together with the process describing the particle in the 'rest frame', where the diffusion of velocity on the sphere surface is isotropic, we also considered the whole family of processes which result from Lorentz boosts [3]. In the general case the velocity continues to be confined to the sphere surface (still a light-speed particle), but the diffusion is anisotropic.

The goal of this paper is to identify a reduced set of variables whose stochastic evolution is autonomous from the remaining variables, but, nevertheless, carry all the relevant information concerning the spatial properties of the processes. Both the associated stochastic equations and the Forward Kolmogorov equation (FKE) are considerably simplified compared to those of the complete set of position and velocity variables.

In the next section we revisit both the 1+1 an the 3+1 space-time families of process explicitly describing their relativistic invariance. In section 3 we focus on the 1+1 case, testing our strategy which, as a side result, also provides a new technique for the explicit solution of the telegrapher's equation. The solution is plotted at different times deploying its singular component and showing its convergence to the gaussian. In section 4 we replicate the strategy for the 3+1 case showing that it is possible togo from the FKE and the stochastic equations of the complete set of three position variables plus two velocity variables to the FKE and the stochastic equations of a set of only three autonomous space variables. In section 5 we further reduce to the FKE and the stochastic equations of just two variables which in the final version are the distance from the origin and the angle between position and velocity. The stochastic equations are then numerically solved as described in section 6. The probability density of the distance from the origin is plotted at different time showing the absence of singular components and the convergence to the normal behavior. Section 7 is devoted to the conclusions.

Before starting to describe our results it is worth highlighting that the study of relativistic stochastic processes in 3+1 dimensions has a long tradition and has produced an infinite number of results (see for example [5–15]), however, the ultra-relativistic model, with continuous velocity described in [3, 4] and in the present article, is totally new.

2. 1+1 and 3+1 space-time dimensions: stochastic equations and Lorentz invariance

Mark Kac considered a particle moving in one space dimension at constant speed and inverting velocity at random times, more precisely, he assumed the following stochastic equations:

$$\begin{aligned} x(t+\mathrm{d}t) &= x(t) + c\sigma(t)\,\mathrm{d}t, \\ \sigma(t+\mathrm{d}t) &= \begin{cases} \sigma(t) & \text{prob} = 1-a\mathrm{d}t \\ -\sigma(t) & \text{prob} = a\mathrm{d}t, \end{cases} \end{aligned} \tag{1}$$

where $\sigma(t) = \pm 1$ is a dichotomous variable and a is the constant jump rate. Therefore, the velocity is $c\sigma(t)$ and the speed is constant and equals c.

The above stochastic equations hold in the 'rest frame'. Assume that this 'rest frame' moves at constant velocity v with respect to a second inertial frame, since the probability rate of a jump behaves like the inverse of a time interval, it modifies according to

$$\begin{aligned} a &\to a_+(v) = a\gamma\left(1 - \frac{v}{c}\right), \\ a &\to a_-(v) = a\gamma\left(1 + \frac{v}{c}\right), \end{aligned} \tag{2}$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ and where $a_+(v)$ refers to jumps from positive to negative velocity and $a_-(v)$ to the reverse jumps. These two expression can be merged by defining

$$a(v,\sigma) = a\gamma\left(1 - \frac{v\sigma}{c}\right). \tag{3}$$

Then, equations (1) are replaced by

$$\begin{aligned} x(t+\mathrm{d}t) &= x(t) + c\sigma(t)\,\mathrm{d}t, \\ \sigma(t+\mathrm{d}t) &= \sigma(t+\mathrm{d}t) = \begin{cases} \sigma(t) & \text{prob} = 1-a(v,\sigma(t))\,\mathrm{d}t \\ -\sigma(t) & \text{prob} = a(v,\sigma(t))\,\mathrm{d}t. \end{cases} \end{aligned} \tag{4}$$

Given the anisotropy of the jump rates, the particle spends longer times with a velocity σ with same sign of v so that, as it can be easily verified, the long term average velocity of the particle equals v.

The stochastic equations (4) determine a relativistically invariant family of processes which transform one into each other via Lorentz boost.

The process (1) was used in later research [16, 17] to provide a probabilistic solution of the 1+1 dimensional Dirac equation (a relativistic analogous of the Feynman–Kac formula). Moreover, more general processes, where the probability rate of velocity inversion not only depends on the velocity v but also also on the position x, were used to construct the Nelson Stochastic Mechanics of a Dirac particle, always in 1+1 dimensions [18].

In three space dimensions, constant speed means that velocity is $\mathbf{c}(t) = c\mathbf{n}(t)$ where the unitary vector $\mathbf{n}(t)$ can be represented as a point on the surface of a sphere with unitary radius (or, which is the same, $\mathbf{c}(t)$ can be represented as a point on the surface of a sphere of radius c). A stochastic process for the velocity can be realized by considering random jumps on the surface, or by continuous trajectories on it. Our choice is the second, more precisely we assume that $\mathbf{n}(t)$ performs a isotropic Wiener process on the surface of a sphere of unitary radius. The equations governing this process (Itô notation) are:

$$\begin{aligned} \mathrm{d}\mathbf{x}(t) &= c\mathbf{n}(t)\,\mathrm{d}t, \\ \mathrm{d}\mathbf{n}(t) &= -\omega^2\mathbf{n}(t)\,\mathrm{d}t + \omega\mathrm{d}\mathbf{w}(t), \end{aligned} \tag{5}$$

where $\mathrm{d}\mathbf{x}(t) = \mathbf{x}(t+\mathrm{d}t) - \mathbf{x}(t)$ and, according to Itô, $\mathrm{d}\mathbf{n}(t) = \mathbf{n}(t+\mathrm{d}t) - \mathbf{n}(t)$, $\mathrm{d}\mathbf{w}(t) = \mathbf{w}(t+\mathrm{d}t) - \mathbf{w}(t)$. The differential $\mathrm{d}\mathbf{w}(t)$ is a two component standard Wiener increment on the plane perpendicular to $\mathbf{c}(t)$ (the plane tangent to the sphere surface). Standard means that the increment, being on a plane, satisfies $E[|\mathrm{d}\mathbf{w}(t)|^2] = 2\mathrm{d}t$. Notice that the diffusion coefficient ω has the dimension of the inverse of the square root of a time.

It is straightforward to verify that $|\mathbf{n}(t)| = 1$ at any time $t > 0$ if $|\mathbf{n}(0)| = 1$ (see [3]). Therefore, equation (5) describes a particle which has constant speed and whose velocity changes direction following continuous but not differentiable trajectories.

Let us remark once more that while a constant speed process in one space dimension can be only constructed by considering jumps between the two possible velocities (the 1D sphere of radius c only contains the two points $\pm c$), in three space dimensions the points of the surface can be connected both by jumps and by continuous paths as in present paper.

Models that consider velocity jumps on the surface of the sphere rather than continuous velocity trajectories have been previously considered as, for example, in [19] where the times between jumps are exponentially distributed.

More recently, in [20, 21], a similar model was considered, with random times between jumps following an Erlang distribution. The authors, who consider several higher-dimensional cases including the 3+1 space-time model, show that the resulting probability density for the position satisfy a telegraph-type equation of which they find the solution. Also in this case the main difference with our continuous velocity model is that the velocity makes jumps, a second difference is that the distribution of the initial velocity is assumed uniform on the surface of the sphere, while here we do not make this assumption.

The telegrapher's equation in 3+1 space-time dimensions was also studied and solved in [22, 23]. What is very interesting is that the authors provide an application to cosmic-ray transport by comparing the solution of the telegrapher's equation with an intensity profile obtained from a simulation of particles moving in a magnetic field with a turbulent and a homogeneous component. From the comparison it emerges that the telegrapher's equation solution is able to reproduce both qualitatively and quantitatively the simulated profile while the latter differs strongly from the solution of the heat equation associated to ordinary diffusion.

In the 'rest frame' our model assumes that the particle velocity performs a isotropic Wiener process on the surface of a sphere of radius c according to (5). Things change in a generic frame: assume that the 'rest frame' moves at constant velocity $\mathbf{v}$ (without rotating) with respect to a second inertial frame, then, equations (5) are replaced by

$$\begin{aligned} \mathrm{d}\mathbf{x}(t) &= c\mathbf{n}(t)\,\mathrm{d}t, \\ \mathrm{d}\mathbf{n}(t) &= -\omega^2\left[\gamma\left(1-\frac{\mathbf{v}\cdot\mathbf{n}(t)}{c}\right)\right]^3\mathbf{n}(t)\,\mathrm{d}t + \omega\left[\gamma\left(1-\frac{\mathbf{v}\cdot\mathbf{n}(t)}{c}\right)\right]^{\frac{3}{2}}\mathrm{d}\mathbf{w}(t), \end{aligned} \tag{6}$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ with $v = |\mathbf{v}|$. We laboriously but straightforwardly derived these equations in [3] using Lorentz boost properties and Itô calculus.

The second of the above equations, given that $\mathrm{d}\mathbf{w}(t)$ is a two-component Wiener increment tangent to the surface of the sphere, describes an anisotropic Wiener process on the surface of a sphere. It is simple to verify that, again, $|\mathbf{n}(t)| = 1$ at any time $t \geqslant 0$ if $|\mathbf{n}(0)| = 1$ (see

[3]). The particle spends longer times on the surface regions where $\mathbf{v}\cdot\mathbf{n}(t)$ is positive than in those where it is negative because the diffusion coefficient is smaller when the product $\mathbf{v}\cdot\mathbf{n}(t)$ is larger and *viceversa*. This intuitively explains why in the moving reference frame the large time average of the particle velocity equals $\mathbf{v}$, a fact which can be directly derived by Lorentz boost rules considering that in the 'rest frame' the long term average velocity vanishes.

The stochastic equations (6) define a relativistically invariant family of processes of parameter $\mathbf{v}$ in 3+1 space-time which transform one into another by Lorentz boost, just as the stochastic equations (4) define a relativistically invariant family of processes of parameter v in 1+1 space-time.

3. 1+1 space-time dimensions: a new approach to the telegrapher's equation

The probability density that the particle is at position x with velocity $c\sigma$ at time t is $\rho(x,\sigma,t)$. According to (1), the master equation for an infinitesimal time step dt is

$$\rho(x,\sigma,t+\mathrm{d}t) = (1-a\mathrm{d}t)\,\rho(x-c\sigma\mathrm{d}t,\sigma,t) + \rho(x,-\sigma,t)\,a\mathrm{d}t. \tag{7}$$

Neglecting terms of order $(\mathrm{d}t)^2$ or higher, one can write

$$\rho(x-c\sigma\mathrm{d}t,\sigma,t) = \rho(x,\sigma,t) - \frac{\partial\rho(x,\sigma,t)}{\partial x}c\sigma\mathrm{d}t, \tag{8}$$

which can be inserted into (7). After that, again neglecting the term of order $(\mathrm{d}t)^2$, the FKE of the rest frame process is obtained simply by dividing by dt. One finds:

$$\frac{\partial\rho(x,\sigma,t)}{\partial t} = -c\sigma\frac{\partial\rho(x,\sigma,t)}{\partial x} + a\rho(x,-\sigma,t) - a\rho(x,\sigma,t), \tag{9}$$

which is a first order equation for a probability density that depends on two variables (the position x and the dichotomous velocity $c\sigma$). Further details and a practical application for the probabilistic solution of the Dirac equation can be found in [16].

Furthermore, the probability density $\rho(x,t)$ that the particle is in x at time t, regardless of the sign of the velocity σ, is

$$\rho(x,t) = \rho(x,1,t) + \rho(x,-1,t). \tag{10}$$

Note that we leave the symbol ρ to also identify this probability density, in fact, given that the velocity sign σ does not appear as an argument, there is not ambiguity.

It can be easily shown by (9) that the density $\rho(x,t)$ satisfy the second order telegrapher's equation

$$\frac{\partial^2\rho(x,t)}{\partial t^2} + 2a\frac{\partial\rho(x,t)}{\partial t} - c^2\frac{\partial^2\rho(x,t)}{\partial x^2} = 0. \tag{11}$$

The initial condition is that the particle is at the origin, which means $\rho(x,0)=\delta(x)$, the support of this density is $x\in[-ct,ct]$. We may further assume equal probability $1/2$ for negative and positive velocity at time $t=0$. This second assumption can be traduced in the additional initial condition $\frac{\partial\rho(x,t)}{\partial t}\big|_{t=0}=0$, since equal probability implies $\rho(x,t)\simeq\frac{\delta(x+ct)+\delta(x-ct)}{2}$ for small times t (see also [1, 2]). Equivalently, one can say that the symmetric solution with initial condition $\delta(x)$ is the one corresponding to equal probability for $\sigma(0)=\pm1$.

The novelty we introduce now is not particularly relevant for this 1+1 dimensions case, but the idea can be transferred to the 3+1 dimension model where it turns out to be very useful. The point is that we are able to reformulate the differential problem as a single first order Equation for a single variable that replaces the other two alternatives: the first order equation (9) for two variables (position and velocity) or the second order telegrapher's equation (11) for a single variable (position).

We start defining

$$\xi(t) = \sigma(t)\, x(t)\,, \tag{12}$$

with the trivial equality $|\xi(t)| = |x(t)|$. For this variable we have the stochastic equation

$$\xi(t + \mathrm{d}t) = \begin{cases} \xi(t) + c\mathrm{d}t & \text{prob} = 1 - a\mathrm{d}t \\ -\xi(t) & \text{prob} = a\mathrm{d}t. \end{cases} \tag{13}$$

It should be noticed that this variable is autonomous at variance with $x(t)$ whose stochastic equation involves a second variable $\sigma(t)$.

Given that $\bar{\rho}(\xi, t)$ is the probability density that the particle is in ξ at time t (we use the different symbol $\bar{\rho}$ to identify this probability density in order to avoid confusion), one has the FKE

$$\frac{\partial \bar{\rho}(\xi, t)}{\partial t} = -c \frac{\partial \bar{\rho}(\xi, t)}{\partial \xi} + a\bar{\rho}(-\xi, t) - a\bar{\rho}(\xi, t)\,, \tag{14}$$

which is a first order equation for a single component density. The initial condition is that the particle is at the origin, which means $\bar{\rho}(\xi, 0) = \delta(\xi)$, the support is $\xi \in [-ct, ct]$. No other initial conditions are needed, since the equation is first order. The solution, as it can be directly checked (see the appendix), is

$$\bar{\rho}(\xi, t) = e^{-at}\delta(\xi - ct) + \frac{ae^{-at}}{2c}\left[I(z) + a\,\frac{ct + \xi}{cz}\,\frac{\partial I(z)}{\partial z}\right]\Theta(ct - |\xi|)\,, \tag{15}$$

where z is the dimensionless derived variable

$$z = \frac{a}{c}\sqrt{c^2 t^2 - \xi^2} \tag{16}$$

and $I(z)$ is the zero-order modified Bessel function

$$I(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}, \tag{17}$$

moreover, $\delta(\xi - ct)$ is a Dirac delta and $\Theta(ct - |\xi|)$ is the Heaviside step function which equals 1 when $\xi \in [-ct, ct]$ and equals 0 otherwise.

The solution consists in the sum of two parts: a regular one and the singular part $e^{-at}\delta(\xi - ct)$, which accounts for the probability e^{-at} that the particle has never jumped and, consequently, it has continued to move according to $\xi(t) = ct$.

The particle is at a distance $|x|$ from the origin when $\xi = |x|$ or $\xi = -|x|$, which means that the probability density for $|x|$ is

$$f(|x|, t) = \bar{\rho}(|x|, t) + \bar{\rho}(-|x|, t) = \bar{\rho}(x, t) + \bar{\rho}(-x, t)\,. \tag{18}$$

This equality holds independently of the initial velocity sign or distribution of it. Moreover, if the initial velocity distribution is symmetric (same probability for $\sigma(0) = \pm 1$), the following equality also holds

$$\rho(x, t) = \frac{f(|x|, t)}{2} = \frac{\bar{\rho}(x, t) + \bar{\rho}(-x, t)}{2}, \tag{19}$$

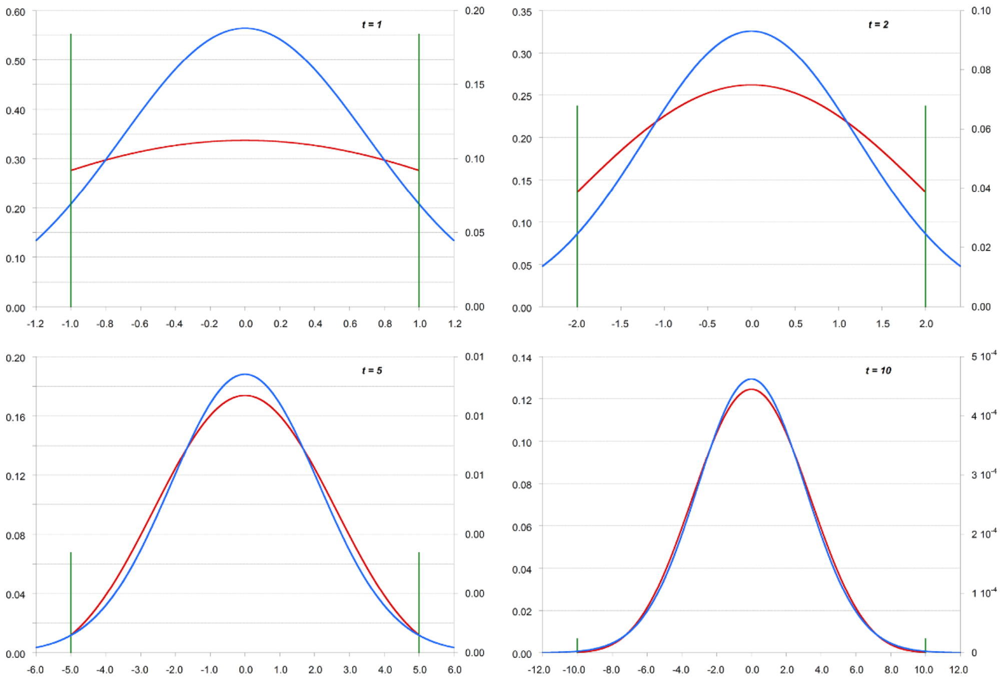

Figure 1. Symmetric probability density $\rho(x,t)$ at different times $t = 1, 2, 5, 10$ for the telegrapher's process plotted as a function of x. We have chosen $c = a = 1$. The density $\rho(x,t)$ consists of two parts: a regular part (red, the scale is at the left of the figures) and the singular one $\frac{e^{-at}}{2}[\delta(x-ct)+\delta(x+ct)]$ which accounts for the probability that the velocity made no jumps. This singular component corresponds to the lateral spikes (green, the scale at the right of the figures refers to the coefficient $\frac{e^{-at}}{2}$ of the Dirac's deltas). The symmetric probability density $\rho(x,t)$ implies $E[x(t)] = 0$ and, for large t, also $E[x^2(t)] \simeq \frac{c^2}{a}t - \frac{c^2}{2a^2} = t - 1/2$, therefore, in the same figures it is compared with a symmetric gaussian (blue) of variance $t - 1/2$ ($t = 1, 2, 5, 10$). The gradual convergence of $\rho(x,t)$ to the gaussian can be visually appreciated.

which corresponds to the symmetric solution $\rho(x,t)$ of (11). Given (15) and (19), the probability density $\rho(x,t)$ can be explicitly written as

$$\rho(x,t) = \frac{e^{-at}}{2}\left[\delta(x-ct)+\delta(x+ct)\right] + \frac{ae^{-at}}{2c}\left[I(z) + \frac{at}{z}\frac{\partial I(z)}{\partial z}\right]\Theta(ct-|x|), \tag{20}$$

where now $z = \frac{a}{c}\sqrt{c^2t^2 - x^2}$. Notice that the singular part, which accounts for the probability that the velocity made no jumps, is now $\frac{e^{-at}}{2}[\delta(x-ct)+\delta(x+ct)]$. The above probability density is represented at different times $t = 1, 2, 5, 10$ in figure 1, where we have chosen without loss of generality $c = 1$, $a = 1$.

Finally, using (1), it can be easily shown that

$$\begin{aligned} &E[x(t)] = 0, \\ &E\left[x^2(t)\right] = \frac{c^2}{a}t - \frac{c^2}{2a^2}\left(1 - e^{-2at}\right) \simeq \frac{c^2}{a}t - \frac{c^2}{2a^2}, \end{aligned} \tag{21}$$

where the equality at the first line holds for a uniform initial distribution of velocity (positive and negative initial velocity have probability 1/2), while the equality at the second line, given the initial position in the origin, always holds. Moreover, the approximate equality at the second line holds for large times ($t \gg 1/a$). The choice $c = 1$ and $a = 1$ for the parameters implies, for large times, $E[|\mathbf{x}(t)|^2] \simeq t - 1/2$, therefore, in figure 1 the probability density $\rho(x,t)$ is compared with a symmetric gaussian of variance $t - 1/2$. The comparison, which is done at different times $t = 1, 2, 5, 10$, visually shows the gradual convergence of $\rho(x,t)$ to the gaussian. tremo

4. 3+1 space-time dimensions: separation of space variables

According to the stochastic equations (5), we can easily write down the FKE for the the density $\rho(\mathbf{x},\mathbf{n},t)$ which will be the 3+1 dimensions analogous of the 1+1 dimensions FKE equation (9). At variance with the density $\rho(x,\sigma,t)$ in (9) whose argument consists in two variables, the density $\rho(\mathbf{x},\mathbf{n},t)$ has five variables as argument, three for locating the particle in space and two for individuating the velocity (a point on a sphere surface). Our goal is now to reduce the set of relevant autonomous variables according to the same line of reasoning that we followed in the previous section for the case of 1+1 space-time dimensions.

Let's start from a premise: the increment $\mathrm{d}\mathbf{w}(t)$ in (5) has to be a two component Wiener differential perpendicular to $\mathbf{n}(t)$ (which means tangent to the surface), nevertheless, the recipe for its construction is not univocal. In our research we have proposed (see [3, 4]).

$$\mathrm{d}\mathbf{w}(t) = \mathbf{n}_2(t)\,\mathrm{d}w_2(t) + \mathbf{n}_3(t)\,\mathrm{d}w_3(t), \tag{22}$$

where $\mathbf{n}_2(t)$ and $\mathbf{n}_3(t)$ are two unitary vectors perpendicular each other and also perpendicular to $\mathbf{n}(t)$ and where $\mathrm{d}w_1(t)$ and $\mathrm{d}w_2(t)$ are two scalar standard independent Wiener increments ($E[|\mathrm{d}w_1|^2] = E[|\mathrm{d}w_2|^2] = \mathrm{d}t$, $E[\mathrm{d}w_1] = E[\mathrm{d}w_2] = E[\mathrm{d}w_1\mathrm{d}w_2] = 0$).

It must be clear that the orientation of $\mathbf{n}_2(t)$ and $\mathbf{n}_3(t)$ can be arbitrarily chosen on the plane perpendicular to $\mathbf{n}(t)$. In [3] we made the simplest choice to achieve the goal of that article, which was to construct, by Itô calculus, the general family of processes generated by Lorentz boosts. In [4] we made a totally different choice which turns out to be also the most efficient for the present goal.

First of all let us rewrite the second equation in (5) by means of (22) as

$$\mathrm{d}\mathbf{n} = -\omega^2\mathbf{n}\,\mathrm{d}t + \omega\mathbf{n}_2\mathrm{d}w_2 + \omega\mathbf{n}_3\mathrm{d}w_3, \tag{23}$$

then, assume the following stochastic equations for $\mathbf{n}_2(t)$ and $\mathbf{n}_3(t)$:

$$\begin{aligned} \mathrm{d}\mathbf{n}_2 &= -\frac{\omega^2}{2}\mathbf{n}_2\,\mathrm{d}t - \omega\mathbf{n}\mathrm{d}w_2, \\ \mathrm{d}\mathbf{n}_3 &= -\frac{\omega^2}{2}\mathbf{n}_3\,\mathrm{d}t - \omega\mathbf{n}\mathrm{d}w_3. \end{aligned} \tag{24}$$

If $\mathbf{n}(t)$, $\mathbf{n}_2(t)$ and $\mathbf{n}_3(t)$ are initially a set of unitary mutually orthogonal vectors, they preserve this property at any time (see [4]) so that they can be used as a basis for the decomposition of the position vector $\mathbf{x}(t)$. This can be easily shown by direct Itô computing and intuitively understood noticing that $\omega\mathrm{d}w_2$ is the infinitesimal angle of a rigid rotation of the vectors $\mathbf{n}$ and $\mathbf{n}_2$ around $\mathbf{n}_3$ and $\omega\mathrm{d}w_3$ is the infinitesimal angle of a rigid rotation of the vectors $\mathbf{n}$ and $\mathbf{n}_3$ around $\mathbf{n}_2$.

Let us define $\xi_1 = \mathbf{x}\cdot\mathbf{n}$, $\xi_2 = \mathbf{x}\cdot\mathbf{n}_2$ and $\xi_3 = \mathbf{x}\cdot\mathbf{n}_3$ (the three components of $\mathbf{x}$ along the three orthogonal vectors $\mathbf{n}$, $\mathbf{n}_2$ and $\mathbf{n}_3$). Using the first equation in (5) and all the equations in (23) and (24), we derive

$$\begin{aligned} \mathrm{d}\xi_1 &= c\mathrm{d}t - \omega^2\xi_1\mathrm{d}t + \omega\xi_2\mathrm{d}w_2 + \omega\xi_3\mathrm{d}w_3, \\ \mathrm{d}\xi_2 &= -\frac{\omega^2}{2}\xi_2\mathrm{d}t - \omega\xi_1\mathrm{d}w_2, \\ \mathrm{d}\xi_3 &= -\frac{\omega^2}{2}\xi_3\mathrm{d}t - \omega\xi_1\mathrm{d}w_3. \end{aligned} \tag{25}$$

We have thus obtained an autonomous system of three equations for three variables, separating the space problem from the velocity problem, exactly as we did for the 1+1 space-time case in previous section. In other words, the above system is the 3+1 dimensions equivalent of equation (13).

The associated FKE for the probability density $\bar{\rho}(\xi_1,\xi_2,\xi_3,t)$, restricted to these three variables, is

$$\frac{\partial\bar{\rho}}{\partial t} = -\nabla_\xi(\mathbf{b}\bar{\rho}) + \frac{\omega^2}{2}\sum_{i,j}\frac{\partial^2 D_{i,j}\bar{\rho}}{\partial\xi_i\partial\xi_j}, \tag{26}$$

where ∇_ξ is the gradient $(\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2},\frac{\partial}{\partial\xi_3})$, $\mathbf{b}$ is the drift $(c-\omega^2\xi_1, -\frac{\omega^2}{2}\xi_2, -\frac{\omega^2}{2}\xi_3)$ and D is the matrix

$$\begin{bmatrix} \xi_2^2+\xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & \xi_1^2 & 0 \\ -\xi_1\xi_3 & 0 & \xi_1^2 \end{bmatrix}. \tag{27}$$

The initial condition for the density is $\bar{\rho}(\xi_1,\xi_2,\xi_3,0) = \delta(\xi_1)\,\delta(\xi_2)\,\delta(\xi_3)$ and the support is $|\boldsymbol{\xi}|\in[0,ct]$. This FKE is the exact analogous of the FKE (14) in previous section, that's also why we indicate the density $\bar{\rho}(\xi,t)$ in (14) and the present density $\bar{\rho}(\xi_1,\xi_2,\xi_3,t)$ with the same symbol $\bar{\rho}$.

The fact that these three variables can be isolated does not mean that the complete probability density (position + velocity) only depends on them. Nevertheless, in case the process starts from the origin ($\mathbf{x}(0)=0$) one has $\xi(0)=\xi_2(0)=\xi_3(0)=0$ and the initial velocity distribution is uniform, it is easy to realize that the complete probability density only depends on these three variables at any time.

From the solution of this equation, given that $|\mathbf{x}|^2 = \xi_1^2+\xi_2^2+\xi_3^2$, one has the probability density for the distance from the origin

$$f(|\mathbf{x}|,t) = \int\bar{\rho}(\xi_1,\xi_2,\xi_3,t)\,\delta\left(|\mathbf{x}| - \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{28}$$

which is the equivalent of (18) and which holds independently of the initial velocity distribution. The integration can be restricted to the sphere of radius ct. Moreover, given a uniform distribution of initial velocities (only in this very special case), one also gets the equivalent of (19)

$$\rho(\mathbf{x},t) = \frac{f(|\mathbf{x}|,t)}{4\pi|\mathbf{x}|^2}, \tag{29}$$

where to go from the density of $|\mathbf{x}|$ to the density of $\mathbf{x}$ we have divided by the surface $4\pi|\mathbf{x}|^2$ of the sphere of radius $|\mathbf{x}|$. Let us stress again that this last passage requires an homogeneous initial distribution of velocities which, in turn, implies that the probability density $\rho(\mathbf{x},t)$ only depends on $|\mathbf{x}|$.

All steps and results in this section follow exactly from the same lines of reasoning in previous one, the difference is that, unfortunately, we do not now have an explicit solution of (26) as we have for (14) but, thankfully, we can further simplify the problem.

5. 3+1 space-time dimensions: reduced set of space variables

Let us define, in order to simplify the notation, $\xi_1 = \xi$. From equations (25) and from the trivial equality $|\mathbf{x}|^2 = \xi^2 + \xi_2^2 + \xi_3^2$, or directly from the first equation in (5), we find $\mathrm{d}|\mathbf{x}|^2 = 2c\xi\,\mathrm{d}t$. Moreover, having defined,

$$\mathrm{d}w = \frac{\xi_2 \mathrm{d}w_2 + \xi_3 \mathrm{d}w_3}{\sqrt{\xi_2^2 + \xi_3^2}}, \tag{30}$$

the first equation in (25) becomes $\mathrm{d}\xi = c\mathrm{d}t - \omega^2\xi\,\mathrm{d}t + \omega\sqrt{\xi_2^2+\xi_3^2}\,\mathrm{d}w$. The Wiener increment $\mathrm{d}w$ is standard, in fact, it can be easily verified that $E[\mathrm{d}w] = 0$ and $E[(\mathrm{d}w)^2] = \mathrm{d}t$. Then, taking into account again that $\xi_2^2 + \xi_3^2 = |\mathbf{x}|^2 - \xi^2$, and also taking into account that $\mathrm{d}|\mathbf{x}|^2 = 2|\mathbf{x}|\mathrm{d}|\mathbf{x}|$ we get the pair of autonomous stochastic equations:

$$\begin{aligned} \mathrm{d}\xi &= c\mathrm{d}t - \omega^2\xi\,\mathrm{d}t + \omega\sqrt{|\mathbf{x}|^2 - \xi^2}\,\mathrm{d}w, \\ \mathrm{d}|\mathbf{x}| &= c\frac{\xi}{|\mathbf{x}|}\mathrm{d}t, \end{aligned} \tag{31}$$

which, without loss of generality, can be solved assuming that the particle is in the origin at initial time, *i.e.* $|\mathbf{x}(0)| = 0$, $\xi(0) = 0$. Notice that $\frac{\xi}{|\mathbf{x}|}$ never diverges since at any time $|\xi(t)| \leqslant |\mathbf{x}(t)|$. Also notice that this implies that at any time T (included initial time) for which $|\mathbf{x}(T)| = 0$ one also must have $\xi(T) = 0$. Moreover, at time T one also has $\frac{\xi(T)}{|\mathbf{x}(T)|} = 1$, this can be seen by continuity since the above stochastic equations impliy $\xi(T+\mathrm{d}t) = c\mathrm{d}t$ and $|\mathbf{x}(T+\mathrm{d}t)|^2 = 2c\int_T^{T+\mathrm{d}t}\xi(T+\mathrm{d}s)\mathrm{d}s = (c\mathrm{d}t)^2$.

We can write down the FKE associated to the stochastic equations (31) for the joint probability density for $|\mathbf{x}|$ and ξ. After renaming $|\mathbf{x}| = r$ one has

$$\frac{\partial\hat{\rho}(r,\xi,t)}{\partial t} = -c\frac{\partial}{\partial r}\left[\frac{\xi}{r}\hat{\rho}(r,\xi,t)\right] - \frac{\partial\left[\left(c-\omega^2\xi\right)\hat{\rho}(r,\xi,t)\right]}{\partial\xi} + \frac{\omega^2}{2}\frac{\partial^2\left[\left(r^2-\xi^2\right)\hat{\rho}(r,\xi,t)\right]}{\partial\xi^2}, \tag{32}$$

with initial condition $\hat{\rho}(r,\xi,0) = \delta(r)\delta(\xi)$. This density has support in the region $r \in [0, ct]$, $\xi \in [-r, r]$. In particular, when $r = 0$ one must have $\xi = 0$ and, for what we said before, $\frac{\xi}{r} = 1$. Notice that we ahve introduced a new symbol $\hat{\rho}$ for this density.

From the solution of this equation one has

$$f(|\mathbf{x}|,t) = \int_{-|\mathbf{x}|}^{|\mathbf{x}|}\hat{\rho}(|\mathbf{x}|,\xi,t)\,\mathrm{d}\xi \quad\rightarrow\quad \rho(\mathbf{x},t) = \frac{f(|\mathbf{x}|,t)}{4\pi\,|\mathbf{x}|^2}, \tag{33}$$

where $f(|\mathbf{x}|,t)$ is the same marginal probability density in (28) and $\rho(\mathbf{x},t)$ is the same probability density in (29). Recall that this $\rho(\mathbf{x},t)$ presupposes, unlike $f(|\mathbf{x}|,t)$, an initial homogeneous distribution of velocities which, in turn, implies that $\rho(\mathbf{x},t)$ only depends on $|\mathbf{x}|$.

It may be convenient to substitute the auxiliary variable ξ with another one, whose geometrical meaning is more transparent. Let us first define $\nu = \frac{\xi}{|\mathbf{x}|}$, by construction $\nu \in [-1,1]$. First of all we remark that the second equation in (31) can be rewritten as

$$\mathrm{d}|\mathbf{x}| = c\nu \mathrm{d}t. \tag{34}$$

Taking into account this result and $\xi\,\mathrm{d}\left(\frac{1}{|\mathbf{x}|}\right) = -\xi\left(\frac{\mathrm{d}|\mathbf{x}|}{|\mathbf{x}|^2}\right) = -\frac{\nu^2}{|\mathbf{x}|}c\mathrm{d}t$ one has from the first equation in (31)

$$\mathrm{d}\nu = \frac{c}{|\mathbf{x}|}\left(1-\nu^2\right)\mathrm{d}t - \omega^2\nu\mathrm{d}t + \omega\sqrt{1-\nu^2}\mathrm{d}w. \tag{35}$$

Equations (34) and (35) form an autonomous system. The initial conditions are $|\mathbf{x}(0)| = 0$ and $\nu(0) = 1$ (see the discussion below (31)).

It should be remarked that $\nu = \frac{\xi}{|\mathbf{x}|} = \frac{\mathbf{x}\cdot\mathbf{n}}{|\mathbf{x}|}$, therefore, it is the cosine of the angle between the position $\mathbf{x}$ and the velocity $\mathbf{c}$. It is straightforward to define $\nu = \cos\theta$, by construction $\theta \in [0,\pi]$, where $\theta = 0$ corresponds to $\nu = 1$ and $\theta = \pi$ to $\nu = -1$. Equation (35) rewrites as

$$\mathrm{d}\cos\theta = \frac{c}{|\mathbf{x}|}(\sin\theta)^2\,\mathrm{d}t - \omega^2\cos\theta\mathrm{d}t + \omega\sin\theta\mathrm{d}w, \tag{36}$$

which imply, as it can be easily checked following Itô, the first Equation of the system below

$$\begin{aligned}\mathrm{d}\theta &= -\frac{c}{|\mathbf{x}|}\sin\theta\mathrm{d}t + \frac{\omega^2}{2}\cot\theta\mathrm{d}t - \omega\mathrm{d}w,\\ \mathrm{d}|\mathbf{x}| &= c\cos\theta\mathrm{d}t,\end{aligned} \tag{37}$$

while the second equation is a trivial consequence of (34). The initial conditions are $\theta(0) = 0$ (which corresponds to $\nu(0) = 1$) and $|\mathbf{x}(0)| = 0$.

Having defined $r = |\mathbf{x}|$ one has the following FKE (we use again the symbol $\hat\rho$ for this density since there is not possible confusion with the density in (32) given that the argument is specified)

$$\frac{\partial\hat\rho(r,\theta,t)}{\partial t} = -c\cos\theta\frac{\partial\hat\rho(r,\theta,t)}{\partial r} + \frac{c}{r}\frac{\partial[\sin\theta\hat\rho(r,\theta,t)]}{\partial\theta} - \frac{\omega^2}{2}\frac{\partial[\cot\theta\hat\rho(r,\theta,t)]}{\partial\theta} + \frac{\omega^2}{2}\frac{\partial^2\hat\rho(r,\theta,t)}{\partial\theta^2}, \tag{38}$$

with initial condition $\hat\rho(r,\theta,0) = \delta(r)\delta(\theta)$. The support is $r\in[0,ct]$, $\theta\in[0,\pi]$. The last two terms of the expression at the right of the equality sign are associated to the isotropic diffusion on the surface of the sphere of unitary radius (latitude component), In fact, they can be rewritten in a compact form as

$$\frac{\omega^2}{2}\text{Ł}^*\hat\rho(r,\theta,t) = \frac{\omega^2}{2}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\hat\rho(r,\theta,t)}{\sin\theta}\right)\right], \tag{39}$$

where the operator Ł^* is the adjoint of Ł which is the latitude component of the Laplacian operator in spherical coordinates (unitary radius).

The $\hat{\rho}(r,\theta,t)$ is the probability density that the particle is at distance $r = |\mathbf{x}|$ from the origin and at latitude θ (the angle between $\mathbf{x}$ and $\mathbf{n}$). This density is independent of the velocity initial conditions, i.e. of the initial distribution of the velocity.

From the solution of this equation one has

$$f(|\mathbf{x}|,t) = \int_0^{\pi} \hat{\rho}(|\mathbf{x}|,\theta,t)\,\mathrm{d}\theta \quad \rightarrow \quad \rho(\mathbf{x},t) = \frac{f(|\mathbf{x}|,t)}{4\pi\,|\mathbf{x}|^2}, \tag{40}$$

where, again, the second equality holds only in case of a uniform distribution of initial velocities which, in turn, implies that the density $\rho(\mathbf{x},t)$ only depends on $|\mathbf{x}|$.

6. 3+1 space-time dimensions: asymptotic behavior, stochastic simulations and gaussian convergence

Again, we are unable to explicitly solve the FKE (32) or the FKE (38), all we can do is to obtain the probability densities $f(|\mathbf{x}|,t)$ as histograms generated by means of 200 000 simulated realizations of the process. For the simulations we used a time step $\mathrm{d}t = 0.0001$. The values of the parameters are chosen, without loss of generality, as $c = 1$ and $\omega = 1$.

The densities $f(|\mathbf{x}|,t)$ are plotted in figure 2 as a function of $|\mathbf{x}|$ for different times $t = 2,5,10,20$ (with $c = 1$, $\omega = 1$). It should be noticed that no singular component of the density is present at $|\mathbf{x}| = ct$ where, on the contrary, the probability density vanishes. In fact, at variance with the 1+1 space-time case, the probability that the velocity remains unchanged until time t equals zero. This would be different for a model with velocity modeled by a jump process on the sphere surface, in this case, in fact, there would always be a finite probability that the velocity remains unchanged until time t.

It can be easily shown, using (31) or directly (5), that

$$\begin{aligned} &E[\mathbf{x}(t)] = 0, \\ &E\left[|\mathbf{x}(t)|^2\right] = \frac{2c^2}{\omega^2}t - \frac{2c^2}{\omega^4}\left(1 - e^{-\omega^2 t}\right) \simeq \frac{2c^2}{\omega^2}t - \frac{2c^2}{\omega^4}, \end{aligned} \tag{41}$$

where for the first equality we took advantage of the assumption of uniform initial velocity distribution while the first equality at the second line, given the initial position in the origin, always holds. Moreover, the approximate equality at the second line holds for large times ($t \gg 1/\omega^2$).

The choice of parameters $c = 1$ and $\omega = 1$ in figure 2 implies, for large times, $E[|\mathbf{x}(t)|^2] \simeq 2(t-1)$ which means that the variance of each of the three components of the position is $\frac{2}{3}(t-1)$. Thus, te probability density $f(|\mathbf{x}|,t)$ can be compared with the corresponding density $f_G(|\mathbf{x}|,t)$ of a 3D gaussian variable (ordinary diffusion) where each of the three components has a vanishing mean value and a variance $s = \frac{2}{3}(t-1)$:

$$f_G(|\mathbf{x}|,t) = \left(\frac{2}{\pi s^3}\right)^{\frac{1}{2}} |\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{2s}}. \tag{42}$$

The probability density $f_G(|\mathbf{x}|,t)$ is also plotted in figure 2 with respect to $|x|$ for different times $t = 2,5,10,20$ where it is superposed to $f(|\mathbf{x}|,t)$. The convergence of $f(|\mathbf{x}|,t)$ to $f_G(|\mathbf{x}|,t)$ at increasing times can be visually appreciated [24].

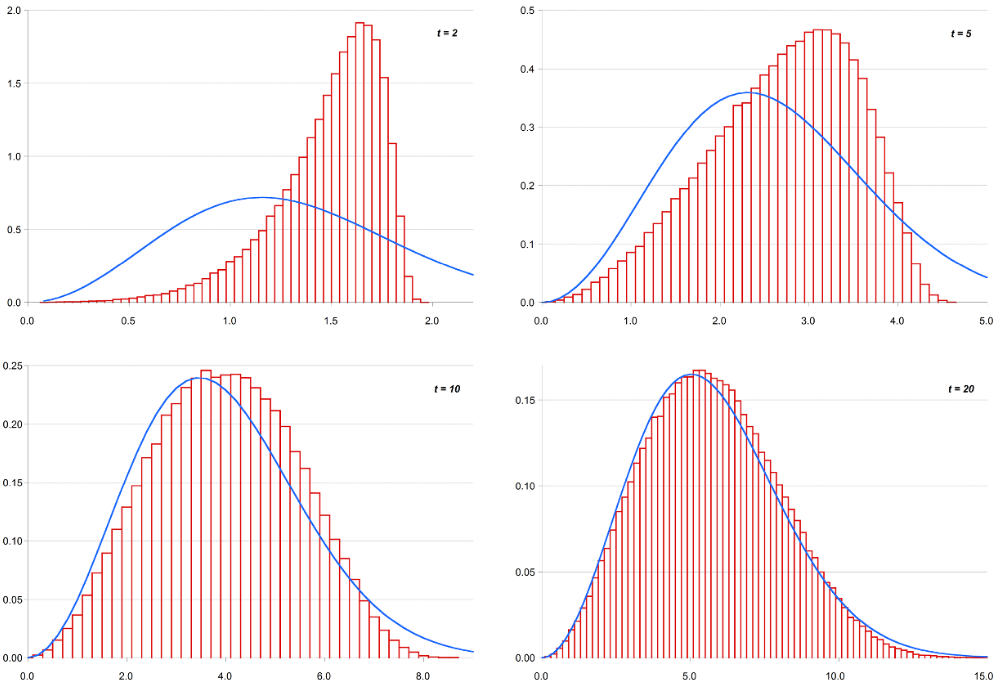

Figure 2. Probability density of the distance from origin $f(|\mathbf{x}|,t)$ at different times $t = 2, 5, 10, 20$ plotted (red histograms) as a function of $|x|$. We have chosen $c = \omega = 1$. No singular component is present at $|\mathbf{x}| = ct$ since the probability that the velocity remains unchanged until time t equals zero. The density $f(|\mathbf{x}|,t)$ implies $E[\mathbf{x}(t)] = 0$ and, for large t, also $E[|\mathbf{x}(t)|^2] \simeq \frac{2c^2}{\omega^2}t - \frac{2c^2}{\omega^4} = 2(t-1)$, therefore, in the same figures it is compared with the analogous density $f_G(|\mathbf{x}|,t) = \left(\frac{2}{\pi s^3}\right)^{\frac{1}{2}} |\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{2s}}$, $s = \frac{2}{3}(t-1)$ of a 3D gaussian variable with the same averages (blue lines). The convergence to the gaussian behavior at larger times can be visually appreciated as well the almost ballistic behavior of the trajectories for smaller times ($|\mathbf{x}|$ is smaller than ct but distributed close to it).

7. Conclusions

The 3+1 space-time dimensions process considered here extends the 1+1 space-time dimensions process considered by Mark Kac in 1956. As in the Kac model, the speed is constant, but the velocity follows a Wiener process. The speed, if interpreted as the speed of light, is always the maximum possible given the relativistic constraint and therefore it deploys the trajectories that better mimic the Brownian motion trajectories (whose speed is infinite).

In this paper we focus on the spatial aspects of the stochastic evolution. We identify a reduced set of variables whose stochastic evolution is autonomous from the remaining variables, but, nevertheless, carry all the relevant information concerning the spatial properties of the process. The advantage is that the associated stochastic equations as well the FKE are considerably simplified with respect to those governing the evolution of the complete set of position and velocity variables.

This approach turns out to be also useful for the classical Kac model where one reduces from two to a single variable and where the resulting simplified FKE can be also used as a tool for a new solution of the telegrapher's equation. Nevertheless, it is much more useful for the 3+1 model where the reduction is from five to a couple of variables.

The distribution of the distance from the origin is computed explicitly (1+1 space-time dimensions) and numerically (3+1 space-time dimensions) by means of 200 000 simulated realizations of the process. In the 3+1 space-time dimensions model no singular component of the density is present at $|\mathbf{x}| = ct$, where, on the contrary, the probability density vanishes. In fact, at variance with the 1+1 space-time case, the probability that the velocity remains unchanged until time t equals zero. This would be different for a 3+1 model with constant speed and velocity modeled by a jump process, in this case, in fact, there would always be a finite probability that the velocity remains unchanged.

Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: https://people.disim.univaq.it/~serva/research/dataset/dataset.html.

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Conflict of interest

The author has not received support from any organization for the submitted work.

Appendix

We show here that the probability density (15) satisfies the FKE (14) with initial value $\bar{\rho}(\xi,0) = \delta(\xi)$.

Let us rewrite (15) in the form

$$\bar{\rho}(\xi,t) = e^{-at}\delta(\xi - ct) + R(\xi,t)\,\Theta(ct - |\xi|)\,, \tag{43}$$

where

$$R(\xi,t) = \frac{ae^{-at}}{2c}\left[I(z) + a\,\frac{ct+\xi}{cz}\,\frac{\partial I(z)}{\partial z}\right] \tag{44}$$

and where

$$I(z) = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k} \quad \text{with} \quad z = \frac{a}{c}\sqrt{c^2t^2 - \xi^2}. \tag{45}$$

We preliminary remark that both $I(z)$ and $\frac{2}{z}\frac{\partial I(z)}{\partial z}$ equal 1 when $z=0$, which implies $R(-ct,t) = \frac{ae^{-at}}{2c}$ and $R(ct,t) = \frac{ae^{-at}}{2c}(1+at)$. We also remark that the Heaviside step function satisfies the equality $\Theta(ct - |\xi|) = \Theta(\xi + ct) - \Theta(\xi - ct)$ and that $(\frac{\partial}{\partial t} + c\frac{\partial}{\partial \xi})\Theta(\xi - ct) = 0$ while $(\frac{\partial}{\partial t} + c\frac{\partial}{\partial \xi})\Theta(\xi + ct) = 2c\delta(\xi + ct)$. Therefore,

$$R(\xi,t)\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)\Theta(ct-|\xi|)=R(\xi,t)\,2c\delta(\xi+ct)$$
$$=R(-ct,t)\,2c\delta(\xi+ct)=ae^{-at}\delta(\xi+ct)\,, \tag{46}$$

moreover, given that $(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi})\delta(\xi-ct)=0$, we also find

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)e^{-at}\delta(\xi-ct)=-ae^{-at}\delta(\xi-ct)\,. \tag{47}$$

By these preliminary calculations we have

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)\bar{\rho}(\xi,t)=-ae^{-at}\delta(\xi-ct)+ae^{-at}\delta(\xi+ct)$$
$$+\Theta(ct-|\xi|)\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)R(\xi,t)\,, \tag{48}$$

so that, the FKE (14) is satisfied if the above expression equals

$$-a\bar{\rho}(\xi,t)+a\bar{\rho}(-\xi,t)=-ae^{-at}\delta(\xi-ct)+ae^{-at}\delta(\xi+ct)$$
$$+a\Theta(ct-|\xi|)\left[R(-\xi,t)-R(\xi,t)\right]. \tag{49}$$

At this point all that remains is to prove that

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)R(\xi,t)=a\left[R(-\xi,t)-R(\xi,t)\right]. \tag{50}$$

From (44) one has

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)R(\xi,t)=-aR(\xi,t)+\frac{ae^{-at}}{2c}\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)\left[I(z)+a\frac{ct+\xi}{cz}\frac{\partial I(z)}{\partial z}\right], \tag{51}$$

than, given that

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)I(z)=\left(\frac{\partial z}{\partial t}+c\frac{\partial z}{\partial \xi}\right)\frac{\partial I(z)}{\partial z}=a^2\frac{ct-\xi}{cz}\frac{\partial I(z)}{\partial z} \tag{52}$$

and that

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)\left[\frac{ct+\xi}{cz}\frac{\partial I(z)}{\partial z}\right]=\frac{2}{z}\frac{\partial I(z)}{\partial z}+\frac{ct+\xi}{c}\left(\frac{\partial z}{\partial t}+c\frac{\partial z}{\partial \xi}\right)\frac{\partial}{\partial z}\left[\frac{1}{z}\frac{\partial I(z)}{\partial z}\right]$$
$$=\frac{2}{z}\frac{\partial I(z)}{\partial z}+z\frac{\partial}{\partial z}\left[\frac{1}{z}\frac{\partial I(z)}{\partial z}\right]=I(z)\,, \tag{53}$$

one obtains

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial \xi}\right)R(\xi,t)=-aR(\xi,t)+\frac{a^2e^{-at}}{2c}\left[I(z)+2\frac{ct-\xi}{cz}\frac{\partial I(z)}{\partial z}\right]$$
$$=-aR(\xi,t)+aR(-\xi,t)\,, \tag{54}$$

which completes the proof.

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