Strong relaxations and cutting planes for the Stable Set Problem

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The stable set problem (SSP)

$G = (V, E)$ simple undirected graph

A vertex set $S \subseteq V$ is called *stable* if the vertices in S are pairwise non-adjacent

stable set problem: determine a stable set of *maximum cardinality* (*weight*, if a weight vector $w \in \mathbb{Q}_+^{|V|}$ is given)

- equivalent to computing the *max-clique* in the complement $\bar{G}$ of G
- NP-hard in the strong sense (and even hard to approximate Hastad 99)
- many applications, including coding theory, combinatorial auctions, forest planning, databases, air traffic management, telecom, scheduling.

Linear and semidefinite programming

- "Unstructured stable set problems appear to be very difficult integer programming problems for LP-based branch-and-bound algorithms" (Nemhauser '92)
- In fact, graphs with $\simeq 400$ vertices can be very hard to solve to optimality even for state-of-the-art MIP solvers (compare to the TSP!).
- One major reason is that constructing strong linear relaxations is not straightforward

- Semidefinite programming has been giving strong relaxations to several CO problems, including the stable set
- Although tractable in theory, SDP algorithms often show numerical problems and embedding them into branch-and-bound is not straighforward (e.g. slow reoptimization)

Outline

- Linear programming relaxations
- Semidefinite programming: The Lovász theta relaxation
- Computational comparison
- An ellipsoidal relaxation

Linear programming formulations

Edge formulation

$$\alpha_w(G) = \max \quad \sum_{i \in V} w_i x_i$$
$$\text{s.t.} \quad x_i + x_j \leq 1 \quad \forall \{i,j\} \in E \qquad (1)$$
$$x_i \in \{0,1\} \quad \forall i \in V$$

Fractional Stable Set Polytope
$\mathrm{FRAC}(G) = \{x \in \mathbb{R}^{|V|} : x_i \geq 0, \forall i \in V$ and (1) hold $\}$

Stable Set Polytope
$\mathrm{STAB}(G)$ denotes the convex hull of the incidence vectors of stable sets in G.

Clique inequalities (Padberg'73)

For any maximal clique $C \subseteq V$ the inequality

$$\sum_{i \in C} x_i \leq 1 \qquad (2)$$

induces a facet of $\mathrm{STAB}(G)$

Denote by Ω the set of all maximal cliques of G and let

$$\mathrm{QSTAB(G)} = \{x \in \mathbb{R}^{|V|} : (2) \text{ holds } \forall C \in \Omega, x_i \geq 0, \forall i \in V\}$$

Clearly, we have:

$$\boxed{\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G) \subseteq \mathrm{FRAC}(G)}$$

Example

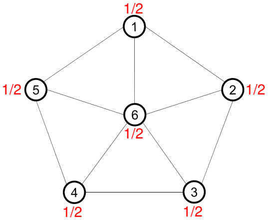
$$\max x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

FRAC(G)

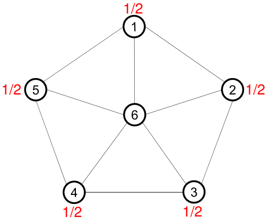
$$x_i + x_j \leq 1, \{i, j\} \in E$$
$$x_i \geq 0, i \in V$$

QSTAB(G)

$$x_1 + x_2 + x_6 \leq 1$$
$$x_1 + x_5 + x_6 \leq 1$$
$$x_2 + x_3 + x_6 \leq 1$$
$$x_3 + x_4 + x_6 \leq 1$$
$$x_4 + x_5 + x_6 \leq 1$$
$$x_i \geq 0, i \in V$$



$$UB_{\text{FRAC(G)}} = 3$$



$$UB_{\text{QSTAB(G)}} = 2.5$$

Separation of clique inequalities

- $|\Omega|$ is exponential in general, so we need a *separation oracle* to optimize over $\mathrm{QSTAB(G)}$
- the separation problem is strongly NP-hard but greedy-like heuristics (variants of Hoffman & Padberg 93) are effective
- experiment:
  - run a cutting plane algorithm embedding the separation heuristic and stop when it fails (violation tolerance $1E^{-9}$):

    $\mathcal{C}$ collection of cliques generated,
    $A_\mathcal{C}$ incidence matrix of $\mathcal{C}$ versus V

    $\mathcal{Q}(\mathcal{C}) = \{A_\mathcal{C} x \leq \mathbf{1}, x \geq 0\}$ clique relaxation

    $UB_\mathcal{C} = \{\max w^T x : \mathcal{Q}(\mathcal{C})\}$ upper bound on $\alpha(G)$

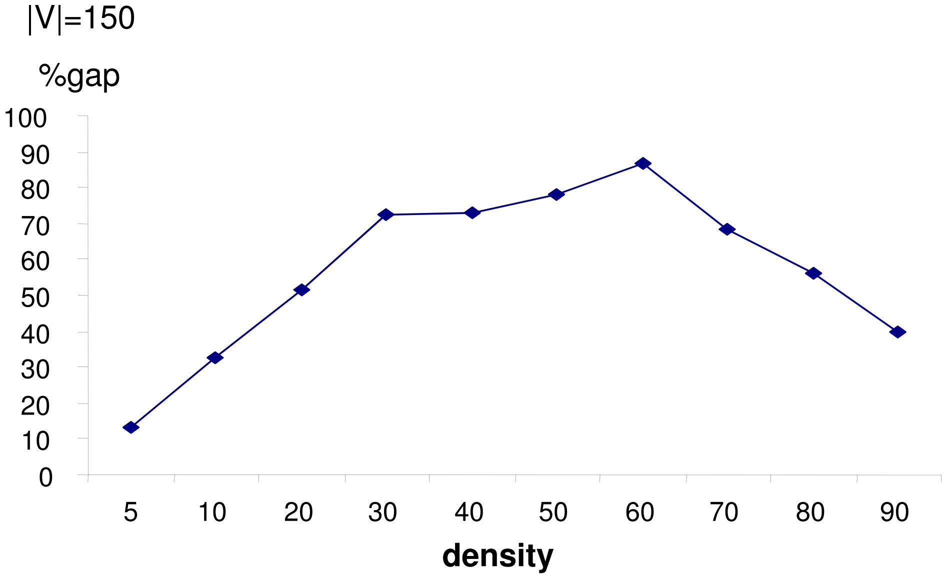
  - go on with an exact separation algorithm and stop when it fails (violation tolerance $1E^{-9}$) $\Rightarrow UB_{QSTAB}$

Heuristic vs. exact separation

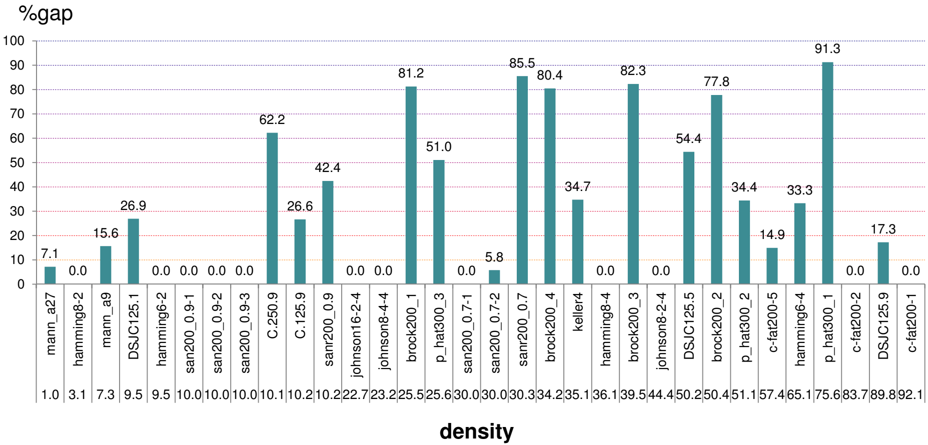
Graph	$\|V\|$	Density	$\alpha(G)$	$UB_{\mathcal{C}}$	$\% \frac{UB_{\mathcal{C}}-\alpha}{\alpha}$	# cuts	UB QSTAB	% Gap closed	# add. cuts
brock200_1	200	0.25	21	38.03	81.1	2,001	38.02	0.1	8
brock200_2	200	0.5	12	21.27	77.3	3,975	21.13	1.6	111
brock200_3	200	0.39	15	27.35	82.3	2,861	27.23	1.0	64
brock200_4	200	0.34	17	30.70	80.6	2,529	30.63	0.5	35
C.125.9	125	0.1	34	43.07	26.7	486	43.07	0.0	0
C.250.9	250	0.1	44	71.38	62.23	1,722	-	-	-
c-fat200-5	200	0.57	58	66.67	14.9	7,561	66.67	0.0	0
DSJC125.1	125	0.09	34	43.14	26.9	460	43.14	0.0	0
DSJC125.5	125	0.5	10	15.46	54.6	1,522	15.37	1.6	42
DSJC125.9	125	0.9	4	4.66	16.6	2,904	4.58	12.2	236
mann_a27	378	0.01	126	135.00	7.1	468	135.00	0.0	0
mann_a45	1035	0	345	360.00	4.3	1,320	360.00	0.0	0
hamming6-4	64	0.65	4	5.33	33.4	170	5.33	0.0	0
keller4	171	0.35	11	14.83	34.8	942	14.83	0.0	11
p_hat300_1	300	0.76	8	15.30	91.25	7,197	-	-	-
p_hat300_2	300	0.51	25	33.59	34.36	3,280	-	-	-
p_hat300_3	300	0.26	36	54.37	51.0	3,968	54.31	0.3	20
san200_0.7-2	200	0.3	18	19.04	6.5	1,489	18.65	37.5	26
sanr200_0.7	200	0.3	18	33.40	85.5	2,350	33.34	0.4	16
sanr200_0.9	200	0.1	42	59.82	42.4	1,170	59.82	0.0	0

Optimizing over $\mathcal{Q}(\mathcal{C})$ is "close" to doing it over $\mathrm{QSTAB(G)}$

Integrality gap of $\mathcal{Q}(C)$: uniform random graphs

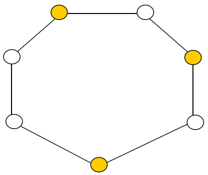


Integrality gap of $\mathcal{Q}(C)$: DIMACS graphs

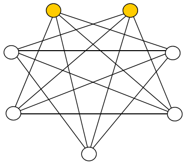


(some) Other linear inequalities

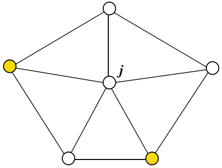
Odd hole: $\sum_{i \in H} x_i \leq \left\lfloor \frac{|H|}{2} \right\rfloor$
$H \subseteq V$ inducing a chordless cycle of odd cardinality.



Odd antihole: $\sum_{i \in A} x_i \leq 2$
$A \subseteq V$ inducing an odd antihole (i.e., the complement of an odd cordless cycle).



Odd wheel: $\sum_{i \in H} x_i + \left\lfloor \frac{|H|}{2} \right\rfloor x_j \leq \left\lfloor \frac{|H|}{2} \right\rfloor$
$H \subseteq V$ inducing an odd hole and $j \notin H$ is adjacent to all vertices in H.



On the first Chvàtal closure of $\mathrm{QSTAB}(G)$

Given our "nice" clique relaxation
$\mathcal{Q}(\mathcal{C}) = \{A_{\mathcal{C}} x \leq \mathbf{1}, x \geq 0\}$

look at its first Chvátal closure:

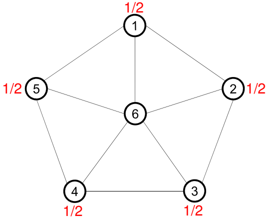
$$\mathcal{Q}^1(\mathcal{C}) = \{A_{\mathcal{C}} x \leq \mathbf{1}, \lfloor u^T A_{\mathcal{C}} \rfloor x \leq \lfloor u^T \mathbf{1} \rfloor, x \geq 0, u \in \mathbb{R}^m_+\}$$

it is a polytope, thanks to a fundamental result of Chvátal.

- it includes odd-hole, odd-antihole (Holm, Torres & Wagler 10) and wheel inequalities (Cheng & Cunningham 97)
- We optimize over $\mathcal{Q}^1(\mathcal{C})$ by an exact oracle based on a MIP (Fischetti and Lodi 07)
- the upper bound $UB_{\mathcal{Q}^1(\mathcal{C})}$ approximates the one from the first Chvátal closure of $\mathrm{QSTAB}(G)$

Example (continued)

$$\max \; x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$
$$x_1 + x_2 + x_6 \leq 1$$
$$x_1 + x_5 + x_6 \leq 1$$
$$x_2 + x_3 + x_6 \leq 1$$
$$x_3 + x_4 + x_6 \leq 1$$
$$x_4 + x_5 + x_6 \leq 1$$
$$x_i \geq 0, i = 1, \ldots, 6$$



the Chvátal-Gomory cut

$$x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 \leq 2$$

,

obtained by multipliers $u^T = (1/2, \ldots, 1/2)$, cuts-off the fractional point and closes the gap

How strong is it in practice?

Graph	$\alpha(G)$	$UB_{\mathcal{Q}(\mathcal{C})}$	$UB_{\mathcal{Q}^1(\mathcal{C})}$	$\frac{UB_{\mathcal{Q}(\mathcal{C})} - UB_{\mathcal{Q}^1(\mathcal{C})}}{UB_{\mathcal{Q}(\mathcal{C})} - \alpha(G)}\%$
brock200_1	21	38.02	37.83	1.12
brock200_2	12	21.21	21.12	0.98
brock200_3	15	27.3	27.22	0.65
brock200_4	17	30.66	30.54	0.88
brock400_1	27	63.96	63.92	0.11
brock400_2	29	64.39	64.34	0.14
brock400_3	31	64.18	64.13	0.15
brock400_4	33	64.21	64.16	0.16
C125.9	34	43.05	42.59	5.08
C250.9	44	71.39	70.99	1.46
c-fat200-5	58	66.67	65.76	10.5
DSJC125.1	34	43.16	42.64	5.68
DSJC125.5	10	15.39	15.25	2.6
mann_a9	16	18	17	50
mann_a27	126	135	134.15	9.44
hamming6-4	4	5.33	5.23	7.52
keller4	11	14.82	14.76	1.57
p_hat300-1	8	15.26	15.24	0.28
p_hat300-2	25	33.59	33.54	0.58
p_hat300-3	36	54.33	54.18	0.82
san200_0.7-2	18	20.36	20.28	3.39
san200_0.9-3	42	45.13	44	36.1
sanr200_0.7	18	33.34	33.17	1.11
sanr200_0.9	42	59.82	59.39	2.41
sanr400_0.5	13	41.29	41.26	0.11
sanr400_0.7	21	57.02	56.96	0.17

Rank inequalities and liftings

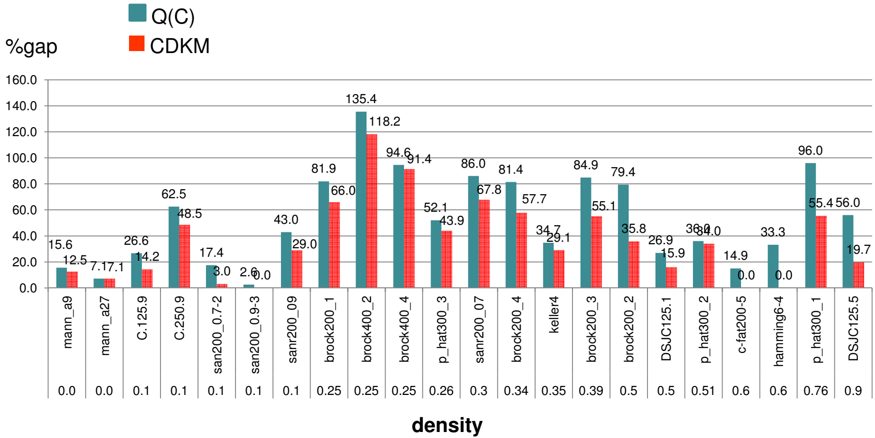
$$\sum_{i \in W} x_i \leq \alpha(G[W]), \qquad W \subseteq V$$

- template: clique, (lifted) odd-hole Nemhauser & Sigismondi 92
- non-template: *project-and-lift* separation heuristics Rossi & S. 01
- rank inequalities + local cuts + mod$-k$ cuts Rebennack et al. 11

CDKM non-rank lifted inequalities Corrêa, Delle Donne, Koch, Marenco 14, 15

- exact separation Coniglio & Gualandi 13

Which progress?



- non-rank inequalities obtained by advanced lifting close a significant portion of the gap on specific instances
- their contribution is not conclusive in general

Take away I: the LP picture

- clique inequalities are "plug-and-play" facet-inducing inequalities but often leave large integrality gaps
- optimizing over their first Chvàtal closure (including several combinatorial inequalities) does not give any substantial progress
- unstructured rank inequalities and their lifted versions yield a progress but are not conclusive in general

Semidefinite programming: the Lovász theta relaxation

Quadratic formulation

non-convex quadratically-constrained formulation:

$$\begin{array}{rll} \max & \sum_{i \in V} x_i & \\ \text{s.t.} & x_i^2 - x_i = 0 & (i \in V) \\ & x_i x_j = 0 & (\{i,j\} \in E). \end{array}$$

To linearize, associate a variable X_{ij} to product $x_i x_j$ and look at the *matrix variable* $X = xx^T$, along with the augmented matrix

$$Y := \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

Y is real, symmetric and *positive semidefinite*

The Lovász theta relaxation

$$Y := \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

This leads to the SDP relaxation:

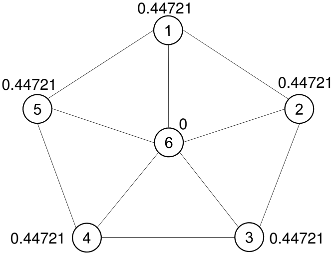
$$\begin{aligned} \theta(G) = \max \quad & \textstyle\sum_{i \in V} x_i \\ \text{s.t.} \quad & x = \operatorname{diag}(X) && (3) \\ & X_{ij} = 0 \qquad (\{i,j\} \in E) && (4) \\ & Y \in \mathcal{S}^+_{|V|+1} && (5) \end{aligned}$$

where $\mathcal{S}^+_{|V|+1}$ is the cone of symmetric psd matrices of order $|V| + 1$.

$\theta(G)$ can be computed in polynomial time to arbitrary precision

Example

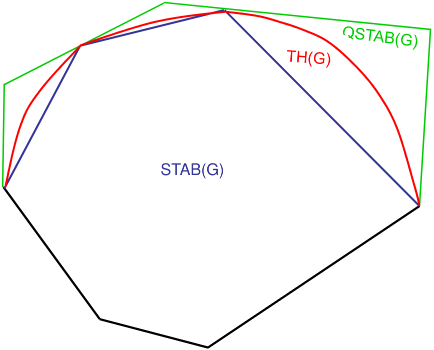
$$Y = \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_1 & X_{11} & 0 & X_{13} & X_{14} & 0 & 0 \\ x_2 & 0 & X_{22} & 0 & X_{24} & X_{25} & 0 \\ x_3 & X_{31} & 0 & X_{33} & 0 & X_{35} & 0 \\ x_4 & X_{41} & X_{42} & 0 & X_{44} & 0 & 0 \\ x_5 & 0 & X_{52} & X_{53} & 0 & X_{55} & 0 \\ x_6 & 0 & 0 & 0 & 0 & 0 & X_{66} \end{pmatrix}$$



$$\theta(G) = 2.2362 = \sqrt{5}$$

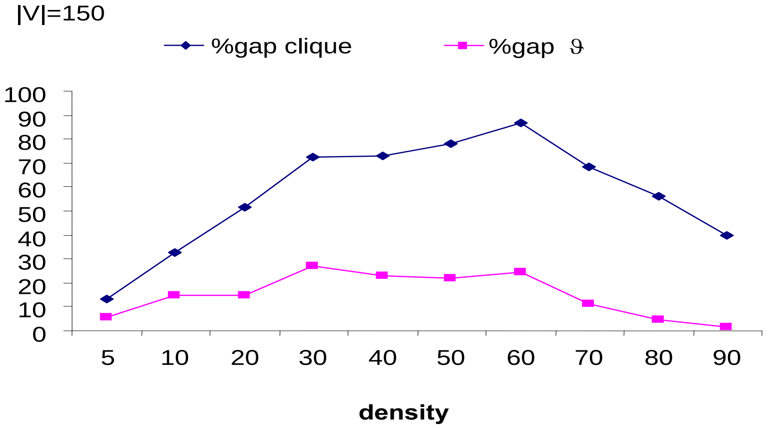
The *theta body*

$TH(G) = \{x \in \mathbb{R}^n : \exists X \in \mathbb{R}^{n \times n} : (3)(4)(5) \text{ holds}\}$
projection of the feasible region onto the x-subspace

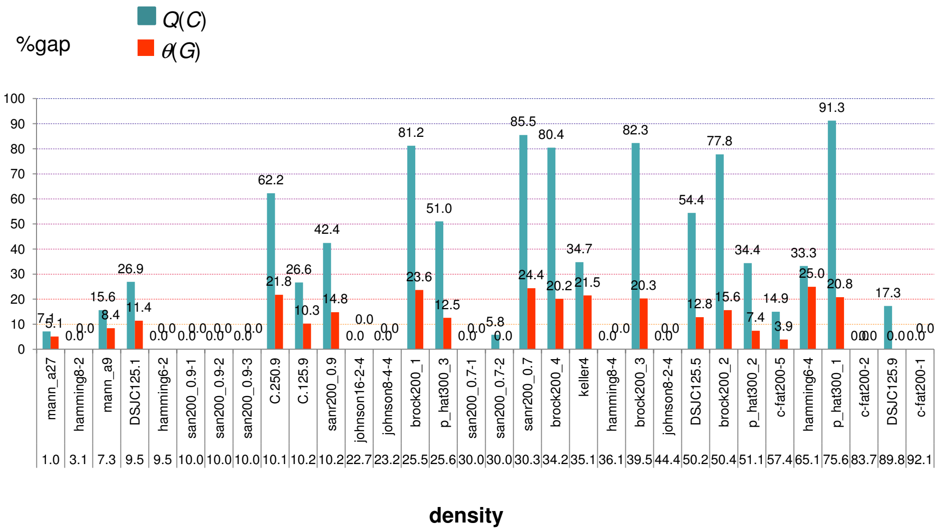


- $TH(G)$ convex but not polyhedral in general
- $\mathrm{STAB}(G) \subseteq TH(G) \subseteq \mathrm{QSTAB}(G)$ [equality iff G is perfect] (Grötschel, Lovász, Schrijver '88)

$UB_{Q(\mathcal{C})}$ vs. $\theta(G)$



$UB_{Q(\mathcal{C})}$ vs. $\theta(G)$



Take away II: the SDP picture

- the θ relaxation can be further strengthened by adding linear inequalities
- unfortunately this leads to computational challenges Burer & Vandenbussche 06, Gruber & Rendl 03, Dukanovich & Rendl 07, Locatelli 15
- the θ relaxation provides a nice compromise between tightness and computational tractability
- thanks to tailored solvers (Povh, Rendl and Wiegele 06; Malik Povh, Rendl and Wiegele 07) using $\theta(G)$ in branch-and-bound looks now viable (Wilson 09)
- however, LP solvers are by far more efficient and suitable for branch-and-bound

An ellipsoidal relaxation

or...how to "capture" the θ bound by linear programming!

Go quadratic

$$\begin{array}{rll} \max & \sum_{i\in V} x_i & \\ \text{s.t.} & x_i^2 = x_i & (i \in V) \\ & x_i x_j = 0 & (\{i,j\} \in E). \end{array}$$

for any $\lambda \in \mathbb{R}^{|V|},\ \mu \in \mathbb{R}^{|E|}$ the quadratic inequality

$$\sum_{i\in V} \lambda_i (x_i^2 - x_i) + \sum_{\{i,j\}\in E} \mu_{ij} x_i x_j \le 0$$

is satisfied by the incidence vectors of all stable sets in G.

in compact form: $\quad x^T(\text{Diag}(\lambda) + M(\mu))x \le \lambda^T x$

$M(\mu)$ symmetric with $M_{ij} = \mu_{ij}/2$ if $\{i,j\} \in E$ and zero otherwise.

$TH(G)$ by ellipsoids

If the matrix $\text{Diag}(\lambda) + M(\mu)$ is psd, the set

$$E(\lambda, \mu) = \{x \in \mathbb{R}^n : \ x^T(\text{Diag}(\lambda) + M(\mu))x \leq \lambda^T x\}$$

defines an ellipsoid that contains $\text{STAB}(G)$

Lemma (Fujie & Tamura, 2002)

For any graph G, we have:

$$TH(G) = \bigcap_{\lambda,\mu:\text{Diag}(\lambda)+M(\mu)\succeq 0} E(\lambda, \mu)$$

does not give a straightforward algorithm to optimize over $TH(G)$, but there is one special ellipsoid showing a very nice property

The dual SDP

recall the Lovász theta relaxation:

$$
\begin{aligned}
\theta(G) = \max \quad & tr(X) & \\
\mathbf{t}] \quad & Y_{00} = 1 & (6)\\
\lambda] \quad & Y_{0i} = X_{ii} \quad (i \in V) & (7)\\
\mu] \quad & X_{ij} = 0 \quad (\{i,j\} \in E) & (8)\\
& Y \in \mathcal{S}^+_{|V|+1} &
\end{aligned}
$$

and look at its dual:

$$
\begin{aligned}
& t^* = \min \ t \\
\text{s.t.} \quad & \begin{pmatrix} t & -(e+\lambda)^T/2 \\ -(e+\lambda)/2 & \mathrm{Diag}(\lambda) + M(\mu) \end{pmatrix} \succeq 0 \qquad (9)\\
& \lambda \in \mathbb{R}^n, \ \mu \in \mathbb{R}^m, \ t \in \mathbb{R},
\end{aligned}
$$

A nice ellipsoid

Theorem (GLRS, 2011)

Let $\lambda^* \in \mathbb{R}^{|V|}$, $\mu^* \in \mathbb{R}^{|E|}$ be optimal dual vectors. Then

$$\theta(G) = t^* = \max\left\{\sum_{i\in V} x_i : x \in E(\lambda^*, \mu^*)\right\}.$$

that is, if we optimize over $E^* = E(\lambda^*, \mu^*)$ we obtain the same value $\theta(G)$ as if we would optimize over $\mathrm{TH}(G)$.

we refer to $E(\lambda^*, \mu^*)$ as the optimal ellipsoid

Proof (sketch)

Fact 1 An optimal dual solution (λ^*, μ^*, t^*) exists, strong duality holds, and $t^* = \theta(G)$ (easy to show)

Fact 2 look at the quadratic formulation

$$\begin{array}{rll} \max & \sum_{i \in V} x_i & \\ \text{s.t.} & x_i^2 = x_i & (i \in V) \\ & x_i x_j = 0 & (\{i,j\} \in E). \end{array}$$

dualize all the contraints by multipliers $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^{|E|}$ and look at the Lagrangian dual

$$\min_{\lambda \in \mathbb{R}^{|V|},\, \mu \in \mathbb{R}^{|E|}} \max_{x \in \mathbb{R}^{|V|}} \{\sum_{i \in V} x_i - x^T(\mathrm{Diag}(\lambda) + M(\mu))x + \lambda \cdot x\}$$

Proof (sketch)

Lemma[Shor '87]
The dual SDP yields the same lower bound as the Lagrangian dual.

Moreover, (λ^*, μ^*) is optimal to the Lagrangian dual **iff** there exists an optimal solution (λ^*, μ^*, t^*) to the dual SDP. Therefore

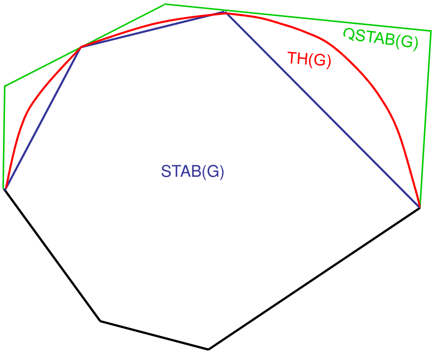
$$t^* = \max_{x \in \mathbb{R}^n} \left\{ \sum_{i \in V} x_i - x^T (\mathsf{Diag}(\lambda^*) + M(\mu^*))x + \lambda^* \cdot x \right\}$$

Fact 3 the matrix $\mathsf{Diag}(\lambda^*) + M(\mu^*)$ is psd (by dual feasibility) and, therefore, the objective is concave. As a consequence (easy to show)

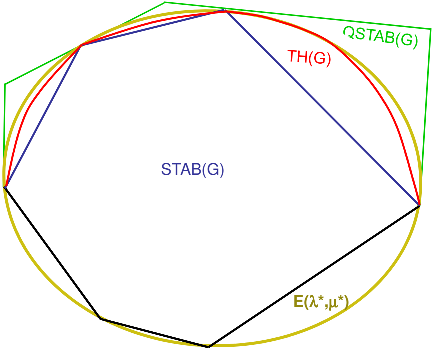
$$\theta(G) = t^* = \max \left\{ \sum_{i \in V} x_i : x \in E(\lambda^*, \mu^*) \right\}.$$

□

The optimal ellipsoid



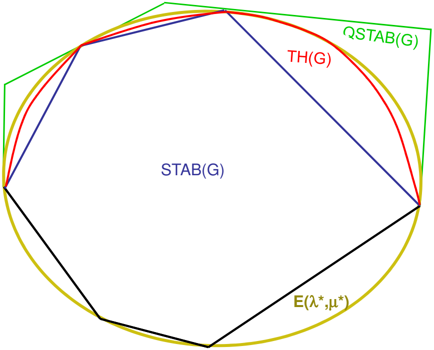
The optimal ellipsoid



$$TH(G) \subseteq E(\lambda^*, \mu^*)$$

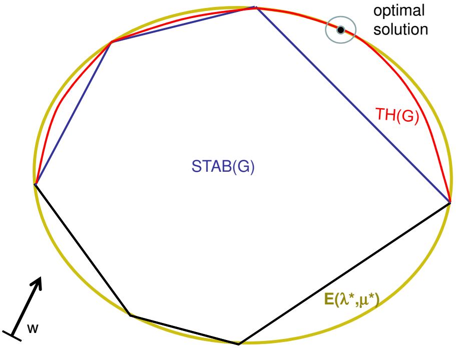
$$E(\lambda^*, \mu^*) \not\subseteq \mathrm{QSTAB}(G),\ \mathrm{QSTAB}(G) \not\subseteq E(\lambda^*, \mu^*)$$

The optimal ellipsoid



the incidence vectors of stable sets lie on the boundary

The optimal ellipsoid



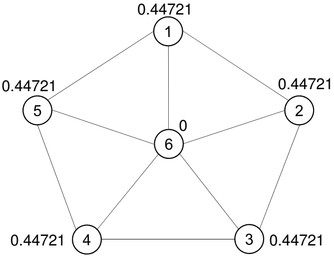
$$\theta(G) = \max\{\sum_{i \in V} w_i x_i : x \in TH(G)\} =$$
$$= \max\{\sum_{i \in V} w_i x_i : x \in E(\lambda^*, \mu^*)\}$$

Example

5-wheel: optimal ellipsoid

$$0.3727 \sum_{i=1}^{5} x_i^2 + 0.3884 x_6^2 + 0.4606 \sum_{ij \in E(C)} x_i x_j + 0.3404 \sum_{j \in C} x_6 x_j +$$

$$-0.3727 \sum_{i=1}^{5} x_i - 0.3884 x_6 \leq 0$$



optimal value $= 2.2362 = \sqrt{5}$

How to use the optimal ellipsoid algorithmically?

1. use the formulation with a *single convex quadratic constraint*:

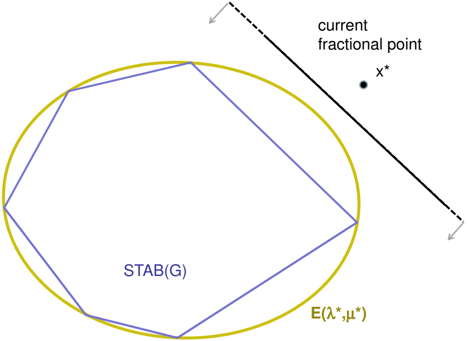
$$\begin{array}{lll} \max & \sum_{i \in V} x_i & \\ \text{s.t.} & \sum_{i \in C} x_i \leq 1 & (\forall C \in \mathcal{C}) \\ & x \in E(\lambda^*, \mu^*) \cap \mathbb{R}^n_+ & \end{array}$$

   where $\mathcal{C}$ is a collection of maximal cliques in G.

   rather compact but reoptimizing after branching is slow with interior point methods

2. construct a polyhedral outer-approximation via a set of linear constraints. We use the Kelley cutting plane method

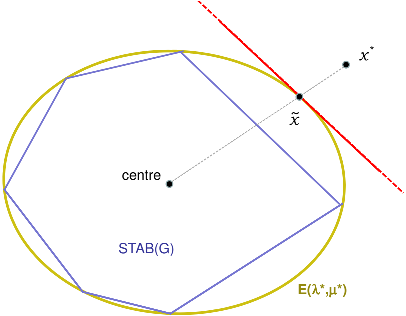
Separating hyperplane



any hyperplane separating x^* from $E(\lambda^*, \mu^*)$ defines a valid inequality for $\mathrm{STAB(G)}$

Separation algorithm

the center $\hat{x}$ exists as the ellipsoid is invariably non-degenerate in practice



1. Find point $\tilde{x} = \epsilon x^* + (1-\epsilon)\hat{x}$ on the boundary. (by solving a quadratic equation in the single real variable ϵ.)
2. Generate a cut $a \cdot x \leq b$ that defines a tangent hyperplane to $E(\lambda^*, \mu^*)$ at $\tilde{x}$ (just compute the gradient!)
3. scale by a constant factor and round down $\lfloor a \rfloor \cdot x \leq \lfloor b \rfloor$

Example (continued)

- Solve the LP with all cliques and get the otimal fractional point $x^* = (0.5, \dots, 0.5, 0)$
- compute the tangent hyperlane:

$$0.372900182 \sum_{i=1}^{5} x_i + 0.372833116 x_6 \leq 0.833912948$$

- scale and round down the cut

$$3729 \sum_{i=1}^{5} x_i + 3728 x_6 \leq 8339$$

- add the cut and solve LP: optimal value $= \sqrt{5} = \theta(G)$

Cut-and-branch

- Preprocessing: compute a near optimal ellipsoid $E(\lambda, \mu)$

  any dual feasible λ, μ implies $\text{Diag}(\lambda) + M(\mu) \succeq 0$ (and $E(\lambda, \mu)$ is an ellipsoid)

  we compute it by a tailored subgradient method (Giandomenico, Letchford, Rossi, S. '13); it was crucial to tackle large graphs at the price of some approximation

- Initial formulation: collection of clique inequalities

- After generating a Kelley cut the clique cutting plane is executed

Experiments

Graphs: DIMACS challenge benchmark graphs with $|V| \leq 700$ (very easy excluded) (http://www.dimacs.rutgers.edu/pub/challenge/graph)

Computer: 2 Intel Xeon 5150 processors clocked at 2.66 GHz with 8GB RAM

MIP/QCP solver: Gurobi 5.6.2

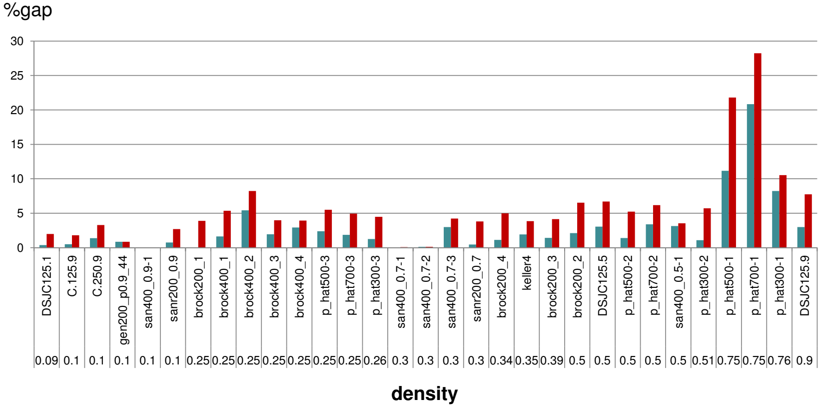
Questions:

Can the linear outer approximation achieve upper bounds close to $\theta(G)$?

Is such a description accessible at reasonable computational cost?

How do the Kelley cuts behave in a branch-and-cut?

Upper bounds: gap to $\theta(G)$



- gap from the ellipsoid is due to approximation of dual multipliers
- gap from the LP description includes also the convergence of the cutting plane

CPU times

Graph	Compute Ellipsoid Subgradient time	Opt. over ellipsoid Barrier time	time	LP approx. Kelley cuts	Clique
brock200_1	8.3	1.2	9.88	40	1,996
brock200_2	6.23	0.91	5.8	5	4,050
brock200_3	19.47	1.46	14.37	30	3,437
brock200_4	8.49	1.54	20.12	30	5,550
brock400_1	74.41	40.28	91.91	40	11,432
brock400_2	45.35	38.81	120.07	40	5,877
brock400_3	34.45	40.82	84.11	40	5,440
brock400_4	80.71	40.25	88.68	40	5,829
C.125.9	3.45	0.08	0.6	50	38
C.250.9	45.36	1.07	5.02	50	325
DSJC125.1	6.4	0.09	0.44	50	19
DSJC125.5	4.24	0.19	2.27	5	1,916
DSJC125.9	3.17	0.03	2.16	5	1,073
gen200_p0.9_44	20.48	0.36	0.07	2	25
keller4	3.99	0.2	0.64	10	839
p_hat300-1	29.92	0.87	4.23	5	8,424
p_hat300-2	39.97	2.77	3.35	5	1,837
p_hat300-3	85.42	6	25.94	40	2,537
p_hat500-1	166.87	6.84	40.46	5	23,585
p_hat500-2	241.38	24.59	93.69	5	12,185
p_hat500-3	211.57	88.85	168.52	50	6,082
p_hat700-1	372.69	28.42	215.79	5	43,855
p_hat700-2	479.25	123.83	230.92	10	18,215
p_hat700-3	1,427.96	602.93	**291.83**	50	25,834
san400_0.5-1	12.75	2.73	23.14	1	6,774
san400_0.7-1	16.85	21.88	**4.76**	2	757
san400_0.7-2	20.82	19.44	17.84	10	2,723
san400_0.7-3	758.07	9.72	3.04	1	996
san400_0.9-1	4.29	16.49	**0.52**	1	53
sanr200_0.7	12.23	1.32	6.02	30	1,640
sanr200_0.9	28.35	0.5	1.42	30	125

Cut-and-branch results

	Gurobi clique formulation		Gurobi MIQCP formulation		Branch-and-cut		
Graph	Subproblems	Time	Subproblems	Time	Subproblems	Time	**Speed-up**
brock200_1	283,763	429.23	88,176	419.81	135,791	251.701	1.71
brock200_2	5,600	33.76	4,469	105.20	1,764	33.08	1.02
brock200_3	18,890	48.84	9,794	172.28	15,306	82.506	—
brock200_4	35,940	74.64	15,619	179.35	15,619	101.36	—
brock400_4	3,926,519	51,644.22	***	***	1,676,237	46,730.35	1.11
C.125.9	3,940	4.98	2,790	15.29	3,636	7.47	—
C.250.9	53,842,560	65,064.41	***	***	29,525,099	52,996.94	1.23
DSJC125.1	3,599	3.52	2,983	21.18	3,940	9.23	—
DSJC125.5	440	4.22	258	8.28	342	7.59	—
DSJC125.9	0	0.37	0	3.49	0	5.54	—
gen200_p0.9_44	1,132	6.02	627	23.14	48	22.45	—
keller4	4,484	9.05	6,009	13.72	3,745	23.35	—
p_hat300-1	5,818	179.23	5,135	134.94	5,675	133.8	1.34
p_hat300-2	6,123	286.60	3,689	356.51	4,933	164.08	1.75
p_hat300-3	1,498,311	7,650.19	225,345	4,491.96	558,861	3,305.63	2.31
p_hat500-1	67,086	1,418.18	***	***	59,089	699.94	2.03
p_hat500-2	268,865	5,266.73	***	***	173,621	4,510.07	1.17
p_hat700-1	168,289	7,118.61	***	***	147,494	3,561.526	2.00
san400_0.5-1	29	14.77	31	26.17	0	39.64	—
san400_0.7-1	0	11.69	0	21.42	0	24.48	—
san400_0.7-2	501	55.15	0	37.26	1	41.33	—
san400_0.7-3	0	2.72	0	764.53	0	763.65	—
san400_0.9-1	0	0.98	0	5.48	0	5.33	—
sanr200_0.7	118,261	242.11	44,989	241.63	73,687	136.54	1.77
sanr200_0.9	1,171,075	921.88	243,331	1,454.59	302,031	303.15	3.04

Concluding remarks

- One can use an approximate solution to the dual SDP to construct an ellipsoid that wraps reasonably tightly around $\mathrm{STAB}(G)$
- this ellipsoid can be used to construct quite strong cutting planes in the original (linear) space
- branch-and-cut: re-optimise the dual SDP solution (and therefore the corresponding ellipsoid) after branching.
- current research: extend the approach to stronger SDP relaxations
- apply the method to other CO problems