Il metodo del simplesso

- implementazione matriciale
- implementazione "tableau"

rif. Fi 3.2;

Test di ottimalità

Test_Opt

Input: $\mathbf{B}^{-1}$

Output: $\bar{\mathbf{c}}$, $opt \in \{true, false\}$

INIT. $opt := false$

calcola $\mathbf{u}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$

$\bar{\mathbf{c}}_F^T := \mathbf{c}_F^T - \mathbf{u}^T F$

TEST **if** $\bar{\mathbf{c}}_F \geq \mathbf{0}$ **then** $opt := true$

Test di illimitatezza

Test_Illim

Input:	$\mathbf{B}^{-1}$, indice $h \notin \{B(1), \ldots, B(m)\}$
Output:	$\bar{\mathbf{A}}_h = \mathbf{B}^{-1}\mathbf{A}_h$, $illim \in \{true, false\}$

Init.	$illim := false$
	calcola $\bar{\mathbf{A}}_h = \mathbf{B}^{-1}\mathbf{A}_h$
Test	**if** $\bar{a}_{ih} \leq 0, i \in \{1, \ldots, m\}$ **then** $illim := true$

Metodo del Simplesso (forma matriciale)

Input:	$\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{B} = [\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}]$ (base ammissibile iniziale)
Output:	$\mathbf{x}$ sol. ottima **OR** $illim = true$

INIT. $illim := false$, $opt := false$

MAIN LOOP **while** ($illim = false$ **and** $opt = false$)

calcola $\mathbf{B}^{-1}$

$\mathbf{Test\_Opt}(\mathbf{B}^{-1}) \rightarrow \bar{\mathbf{c}}, opt$

if ($opt = true$) **then** **return** $\begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_F \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$

VAR. IN **else** scegli $h \notin \{B(1), \ldots, B(m)\}$ con $\bar{c}_h < 0$

$\mathbf{Test\_Illim}(\mathbf{B}^{-1}, h) \rightarrow \bar{\mathbf{A}}_h, illim$

if ($illim = true$) **then** "STOP: prob. illimitato"

else

VAR. OUT calcola $t := \arg\min_{i \in \{1,\ldots,m\}} \{\bar{b}_i / \bar{a}_{ih} : \bar{a}_{ih} > 0\}$

UPDATE $\mathbf{B}$ $B(t) := h$

END LOOP **end_while**

Esempio

$$\min -3x_1 + 2x_2 + 4x_3$$

s.t.

$$-x_1 - x_2 + 2x_3 + x_4 = 1$$

$$x_1 - 2x_2 + x_3 + x_5 = -1$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

base iniziale: $B(1) = 2, B(2) = 3$

$$\mathbf{B} = [\mathbf{A}_2, \mathbf{A}_3] = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

sba iniziale:

$$\begin{bmatrix} \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Esempio

Iter 1.

$$\mathbf{B}^{-1} = \left[\begin{array}{cc} 1/3 & -2/3 \\ 2/3 & -1/3 \end{array}\right]$$

Test_Opt

$$\mathbf{u}^T = [2 \quad 4] \left[\begin{array}{cc} 1/3 & -2/3 \\ 2/3 & -1/3 \end{array}\right] = [10/3 \quad -8/3]$$

$$\bar{c}_1 = -3 - [10/3 \quad -8/3] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3$$

$$\bar{c}_4 = 0 - [10/3 \quad -8/3] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -10/3$$

$$\bar{c}_5 = 0 - [10/3 \quad -8/3] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 8/3$$

return $opt = false$

Esempio

Sceglie var. entrante x_4

Test_Illim

$\bar{\mathbf{b}} = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$\bar{\mathbf{A}}_4 = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

return $illim = false$

sceglie var. uscente $x_{B(t)}$

$\begin{cases} \frac{\bar{b}_1}{\bar{a}_{14}} = 3 \\ \frac{\bar{b}_2}{\bar{a}_{24}} = 3/2 = \theta \end{cases} \quad \Longrightarrow \; t = 2, x_{B(2)} = x_3$ var. uscente

update $B(2) = 4$, $\mathbf{B} = [\mathbf{A}_2, \mathbf{A}_4] = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$

Esempio

Iter 2.

$$\mathbf{B}^{-1} = \left[\begin{array}{cc} 0 & -1/2 \\ 1 & -1/2 \end{array} \right]$$

Test_Opt

$$\mathbf{u}^T = [2 \quad 0] \left[\begin{array}{cc} 0 & -1/2 \\ 1 & -1/2 \end{array} \right] = [0 \quad -1]$$

$$\bar{c}_1 = -3 - [0 \quad -1] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2$$

$$\bar{c}_3 = 4 - [0 \quad -1] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5$$

$$\bar{c}_5 = 0 - [0 \quad -1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

return $opt = false$

Esempio

Sceglie var. entrante x_1

Test_Illim

$$\bar{\mathbf{A}}_1 = \left[\begin{array}{cc} 0 & -1/2 \\ 1 & -1/2 \end{array} \right] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix}$$

$illim = true \implies$ **STOP: problema illimitato**

Esempio

$$\begin{aligned}
\min -10x_1 - 12x_2 - 12x_3 & \\
\text{s.t.} & \\
x_1 + 2x_2 + 2x_3 &\leq 20 \\
2x_1 + x_2 + 2x_3 &\leq 20 \\
2x_1 + 2x_2 + x_3 &\leq 20 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

Esempio

Trasformiamo il problema in forma standard:

$$\begin{aligned} \min -10x_1 - 12x_2 - 12x_3 \\ \text{s.t.} \\ x_1 + 2x_2 + 2x_3 + x_4 = 20 \\ 2x_1 + x_2 + 2x_3 + x_5 = 20 \\ 2x_1 + 2x_2 + x_3 + x_6 = 20 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Otteniamo una base $\mathbf{B} = [\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6] = \mathbf{I}$ composta dalle colonne associate alle variabili slack. Assumiamo $\mathbf{B}$ come base iniziale

Esempio

Iter 1.

$$\mathbf{B}^{-1} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Test_Opt

$$\mathbf{u}^T = [0 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 0]$$

quindi, $\bar{\mathbf{c}}_F^T := \mathbf{c}_F^T - \mathbf{u}^T F = \mathbf{c}_F^T$

$$\bar{c}_1 = -10,\ \bar{c}_2 = -12,\ \bar{c}_3 = -12$$

return $opt = false$

Esempio

Sceglie var. entrante x_2

Test_Illim

$$\bar{\mathbf{A}}_2 = \mathbf{A}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \implies \text{return} \quad illim = false$$

sceglie var. uscente $x_{B(t)}$

$$\bar{\mathbf{b}} = \mathbf{b} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}, \quad t = \arg\min\{20/2^*, 20/2, 20/1\} = 1$$

$x_{B(1)} = x_4$ var. uscente

update $B(1) = 2$, $\mathbf{B} = [\mathbf{A}_2, \mathbf{A}_5, \mathbf{A}_6] = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Esempio

Iter 2. $\mathbf{B}^{-1} \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Test_Opt

$$\mathbf{u}^T = [-12 \quad 0 \quad 0] \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [-6 \quad 0 \quad 0]$$

$$\bar{c}_1 = -10 - [-6 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -4$$

$$\bar{c}_3 = -12 - [-6 \quad 0 \quad 0] \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\bar{c}_4 = 0 - [-6 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 6$$

return $opt = false$

Esempio

Sceglie var. entrante x_1

Test_Illim

$$\bar{\mathbf{A}}_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \end{bmatrix}$$

return illim $=$ false

sceglie var. uscente $x_{B(t)}$

$$\bar{\mathbf{b}} = \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}, \quad t = \arg\min\{10/(1/2), 10/(3/2), 0/1\} = 3$$

$x_{B(3)} = x_6$ var. uscente

update $B(3) = 1$, $\mathbf{B} = [\mathbf{A}_2, \mathbf{A}_5, \mathbf{A}_1] = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$

Esempio

Iter 3. $\mathbf{B}^{-1} \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & -3/2 \\ -1 & 0 & 1 \end{bmatrix}$

Test_Opt

$$\mathbf{u}^T = [-12 \quad 0 \quad -10] \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & -3/2 \\ -1 & 0 & 1 \end{bmatrix} = [-2 \quad 0 \quad -4]$$

$$\bar{c}_3 = -12 - [-2 \quad 0 \quad -4] \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = -4$$

$$\bar{c}_4 = 0 - [-2 \quad 0 \quad -4] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2$$

$$\bar{c}_6 = 0 - [-2 \quad 0 \quad -4] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4$$

return $opt = false$

Esempio

Sceglie var. entrante x_3

Test_Illim

$$\bar{\mathbf{A}}_3 = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 1 & -3/2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/2 \\ -1 \end{bmatrix}$$

return illim = false

sceglie var. uscente $x_{B(t)}$

$$\bar{\mathbf{b}} = \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}, \quad t = \arg\min\{20/3, 4^*, \} = 2$$

$x_{B(2)} = x_5$ var. uscente

update $B(2) = 3$, $\mathbf{B} = [\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_1] = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

Esempio

Iter 4. $\mathbf{B}^{-1}\begin{bmatrix} 1 & -3/2 & 1 \\ 1 & 1 & -3/2 \\ -3/2 & 1 & 1 \end{bmatrix}$

Test_Opt

$$\mathbf{u}^T = [-12 \quad -12 \quad -10]\begin{bmatrix} 1 & -3/2 & 1 \\ 1 & 1 & -3/2 \\ -3/2 & 1 & 1 \end{bmatrix} = [-9 \quad -4 \quad -4]$$

$$\bar{c}_4 = 0 - [-9 \quad -4 \quad -4]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 9$$

$$\bar{c}_5 = 0 - [-9 \quad -4 \quad -4]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 4$$

$$\bar{c}_6 = 0 - [-9 \quad -4 \quad -4]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4$$

return $opt = true$ - **STOP**

Esempio

Soluzione ottima

$$\begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 1 \\ 1 & 1 & -3/2 \\ -3/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

Questioni da definire

- correttezza e convergenza del Metodo del Simplesso
- individuazione della base iniziale
- implementazioni efficienti

Il *tableau* del Simplesso

memorizziamo il problema in un *tableau*:

$\mathbf{c}_B^T$	$\mathbf{c}_F^T$	0
$\mathbf{B}$	$\mathbf{F}$	$\mathbf{b}$

e ricostruiamo la rappresentazione rispetto alla base:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{F}\mathbf{x}_F$$
$$z = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_F^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{F})\mathbf{x}_F$$

Il *tableau* del Simplesso

premoltiplichiamo per $\mathbf{B}^{-1}$

$\mathbf{c}_B^T$	$\mathbf{c}_F^T$	0
$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{F}$	$\mathbf{B}^{-1}\mathbf{b}$

sottraiamo alla riga 0 la matrice premoltiplicata per $\mathbf{c}_B^T$

$\mathbf{c}_B^T - \mathbf{c}_B^T$	$\mathbf{c}_F^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{F}$	$-\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{F}$	$\mathbf{B}^{-1}\mathbf{b}$

Il *tableau* del Simplesso

otteniamo quindi:

	$\mathbf{0}^T$	$\mathbf{c}_F^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{F} = \bar{\mathbf{c}}_F$	$-\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$
$x_{B(1)}$ $\vdots$ $x_{B(m)}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{F} = \bar{\mathbf{F}}$	$\mathbf{B}^{-1}\mathbf{b} = \bar{\mathbf{b}}$

tableau in forma canonica rispetto alla base $\mathbf{B}$

Implementazione Tableau

Input:	$\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{B} = [\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}]$
Output:	$\mathbf{x}$ sol. ottima **OR** $illim = true$

INIT.	$illim := false$, $opt := false$
	costruisce il tableau iniziale in forma canonica
MAIN LOOP	**while** ($illim = false$ **and** $opt = false$)
ROW 0	**if** ($\bar{c}_j \geq 0, j \in F$) **then** **return** $\begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_F \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{b}} \\ \mathbf{0} \end{bmatrix}$
VAR. IN	**else** scegli $h \notin \{B(1), \ldots, B(m)\}$ con $\bar{c}_h < 0$
COL h	**if** ($\bar{a}_{ih} \leq 0, i = 1, \ldots, m$) **then** "STOP: prob. illimitato"
	else
VAR. OUT	calcola $t := \arg\min_{i \in \{1,\ldots,m\}} \{\bar{b}_i / \bar{a}_{ih} : \bar{a}_{ih} > 0\}$
UPDATE $\mathbf{B}$	$PIVOT(t, h)$
END LOOP	**end_while**

$PIVOT(t, h)$

È la procedura che ci consente di aggiornare la base $\mathbf{B}$ facendo entrare x_h ed uscire $x_{B(t)}$.
Consiste nel ricavare la variabile x_h dalla riga t e nel sostituire la sua espressione nelle altre righe.
Equivale alla seguente sequenza di operazioni sul tableau:

- [riga 0] $:=$ [riga 0] $- \frac{\bar{c}_h}{a_{th}} \cdot$ [riga t]
- [riga i] $:=$ [riga i] $- \frac{\bar{a}_{ih}}{a_{th}} \cdot$ [riga t]
- [riga t] $:=$ [riga t]$/\bar{a}_{th}$ (normalizza la riga t)

$\bar{a}_{th}$ è detto elemento di pivot

Esempio

$$\min -2x_1 - 5x_2 - x_3$$

s.t.

$$x_1 + 3x_2 + x_4 = 4$$

$$5x_2 + x_3 + x_5 = 5$$

$$2x_1 + 4x_2 + x_3 + x_6 = 6$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

- 2	- 5	- 1	0	0	0	0	
1	3	0	1	0	0	4	x_4
0	5	1	0	1	0	5	x_5
2	4	1	0	0	1	6	x_6

tableau già in forma canonica

Esempio

iter 1

- 2	- 5	- 1	0	0	0	0	
1	3	0	1	0	0	4	x_4
0	**5**	1	0	1	0	5	x_5
2	4	1	0	0	1	6	x_6

var entrante x_2

$t = \arg\min\{4/3, 1, 6/4\} = 2$

$\Longrightarrow$ var uscente x_5

$PIVOT(t = 2, h = 2)$:

- [riga 0] $:=$ [riga 0] $- (-5/5) \cdot$ [riga 2]
- [riga 1] $:=$ [riga 1] $- 3/5 \cdot$ [riga 2]
- [riga 3] $:=$ [riga 3] $- 4/5 \cdot$ [riga 2]
- [riga 2] $:=$ [riga 2]$/5$ (normalizza la riga $t = 2$)

Esempio

iter 2

- 2	0	0	0	1	0	5
1	0	-3/5	1	-3/5	0	1	x_4
0	1	1/5	0	1/5	0	1	x_2
2	0	1/5	0	-4/5	1	2	x_6

var entrante x_1
$t = \arg\min\{1, 2/2\} = 3$
$\Longrightarrow$ var uscente x_6

$PIVOT(t = 3, h = 1)$:

- [riga 0] $:=$ [riga 0] $- (-2/2) \cdot$ [riga 3]
- [riga 1] $:=$ [riga 1] $- 1/2 \cdot$ [riga 3]
- [riga 2] $:=$ [riga 3] $- 0/2 \cdot$ [riga 3]
- [riga 3] $:=$ [riga 3]$/2$ (normalizza la riga $t = 3$)

Esempio

iter 3

0	0	1/5	0	1/5	1	7	
0	0	-7/10	1	-1/5	-1/2	0	x_4
0	1	1/5	0	1/5	0	1	x_2
1	0	1/10	0	-2/5	1/2	1	x_1

$$\bar{\mathbf{c}} \geq \mathbf{0} \implies \mathbf{x}^T = \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

soluzione ottima di valore -7