Part II
Strong lift-and-project cutting planes

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The Lovász and Schrijver $\mathrm{M}(K, K)$ Operator

Let K be a given linear system in $0-1$ variables.

- for any pair of inequalities $\alpha x - \beta \geq 0$ and $\alpha' x - \beta' \geq 0$, compute the 'product inequality':

$$\begin{pmatrix} -\beta & \alpha^T \end{pmatrix} Y \begin{pmatrix} -\beta' \\ \alpha' \end{pmatrix} \geq 0.$$

- products $x_i x_j$, for all $1 \leq i < j \leq n$ are replaced with variables x_{ij}
- terms x_i^2, for $1 \leq i \leq n$, are replaced with x_i (valid when x_i is binary.)

This yields an extended formulation which is provably stronger than the original (Lovász and Schrijver '91).

The $\mathrm{M}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$ relaxation

Recall that Ω is the collection of all (maximal) cliques

$$QSTAB(G) := \begin{cases} 1 - \sum_{i \in C} x_i \geq 0 & (C \in \Omega) \\ x_i \geq 0 & (i \in V) \end{cases}$$

$\mathrm{M}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$:

$$CVIs: \quad -x_i + \sum_{j \in C : \{i,j\} \in \bar{E}} x_{ij} \leq 0 \qquad (C \in \Omega, i \in V \setminus C)$$

$$CPIs: \quad \begin{array}{ll} \sum_{i \in C} x_i \leq 1 & (C \in \Omega) \\ \sum_{i \in C \cup C'} x_i - \sum_{\{i,j\} \in \bar{E}(C:C')} x_{ij} \leq 1 & (C, C' \in \Omega) \\ x_{ij} = 0 & (\{i,j\} \in E) \\ x_{ij} \geq 0 & (\{i,j\} \in \bar{E}) \\ x_i \geq 0 & (i \in V) \end{array}$$

On the strength of $\mathrm{N}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$

- **non-compact**: $n|\Omega|$ CVIs and $|\Omega|(|\Omega| - 1)/2$ CPIs
- separation of CVIs and CPIs NP-hard in the strong sense (Giandomenico '06)
- projection $\mathrm{N}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$ onto the x-space:
  - stronger than the Sherali-Adams relaxation:

    $$\mathrm{N}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G)) \subseteq \mathrm{N}(\mathrm{QSTAB}(G)) \subseteq \mathrm{N}(\mathrm{FRAC}(G))$$

  - neither contains nor is contained in $TH(G)$ or $N_+(\mathrm{FRAC(G)})$

Theorem (Giandomenico, Letchford, Rossi, S '09)

$\mathrm{N}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$ *implies all web and antiweb inequalities, together with various lifted versions*

On the strength of $\mathrm{N}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$

Relaxation	implied inequalities
$TH(G)$	clique [Grotschel, Lovász and Schrijver '88]
$\mathrm{N}_+(FRAC(G))$	clique, odd-cycle, odd-antihole and odd-wheel [Lovász and Schrijver '91] web [Giandomenico and Letchford '06]
$\mathrm{N}(QSTAB(G), QSTAB(G))$	web and antiweb and their lifted versions $\Rightarrow$ edge, clique, odd-hole, odd-antihole [Giandomenico, Letchford, Rossi, S. '09]

Benders reformulation $\Rightarrow$ $\mathrm{N(QSTAB, QSTAB)}$

	$\max \mathbf{1}^T x + \mathbf{0}^T y$		$\max \mathbf{1}^T x + \eta$
	s. t.		s. t.
cliques	$Ax \leq 1$	$\rightarrow$	$Ax \leq 1$
CVIs + CPIs	$Bx + Dy \leq d$		$v^T(d - Bx) \geq \eta,\ v \in \mathrm{EXT(Q)}$
			$v^T(d - Bx) \geq 0,\ v \in \mathrm{RAY(Q)}$
	$x \in \mathbb{R}^n_+, y \in \mathbb{R}^p_+$		$x \in \mathbb{R}^n_+,\ \eta \in \mathbb{R}$

where $\mathrm{EXT(Q)}$ and $\mathrm{RAY(Q)}$ contain resp. the extreme points and extreme rays of the slave polyhedron $Q := \{v \in \mathbb{R}^p : v^T D \geq \eta, v \geq \mathbf{0}\}$

- in our case, $\eta = 0$ and $Q = \{v \in \mathbb{R}^p : v^T D \geq 0, v \geq \mathbf{0}\}$ is a polyhedral cone
- $\mathrm{Ext}(Q)$ contains only the zero vector $\Rightarrow$ the "optimality cuts" disappear

Benders reformulation $\Rightarrow \mathrm{N}(\mathrm{QSTAB}, \mathrm{QSTAB})$

	$\max \mathbf{1}^T x$ s. t.		$\max \mathbf{1}^T x$ s. t.
cliques	$Ax \leq 1$	$\rightarrow$	$Ax \leq 1$
CVIs + CPIs	$Bx + Dy \leq d$		$v^T(d - Bx) \geq 0,\ v \in \mathrm{RAY}$ $\mathrm{RAY} :=$ {extreme rays of $v^T D \geq 0, v \geq 0$}
	$x \in \mathbb{R}^n_+, y \in \mathbb{R}^p_+$		$x \in \mathbb{R}^n_+$

Clever selection of Benders cuts accomplished by a *Cut Generating Linear Program* (CGLP):

$$\min v^T(d - Bx^*)$$
$$v^T D \geq \mathbf{0}$$
$$\sum_{i=1,\ldots,p} v_i = \mathbf{1}$$
$$v \geq \mathbf{0}$$

Benders reformulation $\Rightarrow$ $\mathrm{N(QSTAB, QSTAB)}$

	$\max \mathbf{1}^T x$ s. t.		$\max \mathbf{1}^T x$ s. t.
cliques	$Ax \leq 1$	$\rightarrow$	$Ax \leq 1$
CVIs + CPIs	$Bx + Dy \leq d$		$v^T(d - Bx) \geq 0,\ v \in \mathrm{RAY}$ $\mathrm{RAY} :=$
	$x \in \mathbb{R}^n_+, y \in \mathbb{R}^p_+$		$\{$extreme rays of $v^T D \geq 0, v \geq 0\}$ $x \in \mathbb{R}^n_+$

Clever selection of Benders cuts accomplished by a *Cut Generating Linear Program* (CGLP):

$$\min v^T(d - Bx^*)$$
$$v^T D \geq \mathbf{0}$$
$$\textstyle\sum_{i=1,\ldots,p} v_i = \mathbf{1}$$
$$v \geq \mathbf{0}$$

Cut Generating LP

i vertex, H, K cliques.

	$\cdots u_{Ki} \cdots$	$\cdots\cdots v_{HK} \cdots\cdots$	
	$\cdots x_i^* \cdots$	$\cdots (1 - x^*(H \cup K)) \cdots$	
	$\cdots\cdots\cdots$	$\cdots\cdots\cdots$	$\geq \quad 0$
	$\vdots$	$\vdots$	$\vdots$
$(i,j) \in \bar{E}$	$a_{(i,j),(K,i)}$	$\cdots b_{(i,j)(H,K)} \cdots$	$\geq \quad 0$
	$\vdots$	$\vdots$	$\vdots$
	$\cdots\cdots\cdots$	$\cdots\cdots\cdots$	$\geq \quad 0$
	$\cdots 1 \cdots$	$\cdots 1 \cdots$	$= \quad 1$

$a_{(i,j),(K,i)} = 1$ if $j \in K$ and 0 otherwise;
$b_{(i,j)(H,K)} = -1$ if $i \in H$ and $j \in K$ and 0 otherwise

Cut Generating LP: example

5-hole, $x^* = (0.5, \ldots, 0.5)$

obj	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$	$\{3,4\}$	$\{3,4\}$	$\{3,4\}$	$\{4,5\}$	$\{4,5\}$	$\{4,5\}$	$\{1,5\}$	$\{1,5\}$	$\{1,5\}$
	3	4	5	1	4	5	1	2	5	1	2	3	2	3	4
13	1	0	0	1	0	0	1	0	0	0	0	0	0	1	0
14	0	1	0	0	0	0	1	0	0	1	0	0	0	0	1
24	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0
25	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0
35	0	0	0	0	0	1	0	0	1	0	0	1	0	1	0
η	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

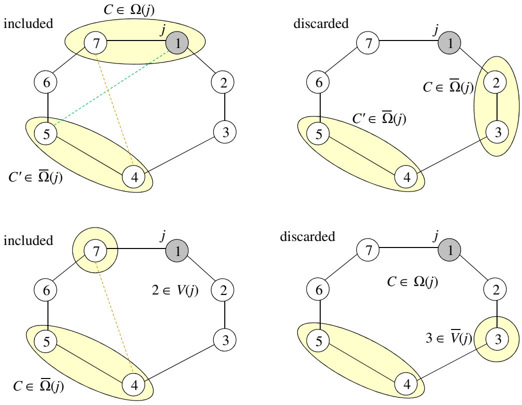
obj	1.5	2	2	1.5	1.5	2	2	1.5	2	1.5
	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$	$\{3,4\}$	$\{3,4\}$	$\{4,5\}$
	$\{2,3\}$	$\{3,4\}$	$\{4,5\}$	$\{1,5\}$	$\{3,4\}$	$\{4,5\}$	$\{1,5\}$	$\{4,5\}$	$\{1,5\}$	$\{1,5\}$
13	-1	-1	0	0	0	0	-1	0	-1	0
14	0	-1	-1	0	0	0	0	0	-1	-1
24	0	-1	-1	0	-1	-1	0	0	0	0
25	0	0	-1	-1	0	-1	-1	0	0	0
35	0	0	0	0	0	-1	-1	-1	-1	0
η	1	1	1	1	1	1	1	1	1	1

Cutting plane performance

- sparse graphs are tractable up to ~ 150 vertices (cut generation is too slow for $|V| = 200$)
- some cuts are sparse and "clean", but some other are quite dense
- sometimes cuts are helpful in branch-and-cut, but often confound it
- different normalization constraints may change things, but do not seen resolutive

Relaxing $\mathrm{M}(K,K)$

- pick a vertex j and only the CPIs containing variables x_{jk}, for $\{j,k\} \in \bar{E}$ (11 out of 21 CPIs are discarded)
- include the CVIs containing variables x_{hk} covered by the chosen CPIs (12 out of 35 discarded)



The $\mathrm{M}_j(K, K)$ relaxation

- $V(j)$ denotes the neighborhood of $j \in V$, $\bar{V}(j) = V \setminus V(j)$
- $\Omega(j)$ set of all maximal cliques containing j; $\bar{\Omega}(j) = \Omega \setminus \Omega(j)$

$$
\begin{aligned}
\max \sum_{i \in V} x_i & \\
\text{s.t.} & \\
\sum_{i \in C \cup C'} x_i - \sum_{\{i,k\} \in \bar{E}(C:C')} x_{ik} \leq 1 & \qquad (C \in \Omega(j), C' \in \bar{\Omega}(j)) \\
-x_i + \sum_{k \in C : \{i,k\} \in \bar{E}} x_{ik} \leq 0 & \qquad (C \in \bar{\Omega}(j), i \in \{j\} \cup V(j)) \\
-x_i + \sum_{k \in C : \{i,k\} \in \bar{E}} x_{ik} \leq 0 & \qquad (C \in \Omega(j), i \in \bar{V}(j) \cup V(\bar{V}(j))) \\
x_{ik} = 0 & \qquad (\{i,k\} \in E) \\
x_{ik} \geq 0 & \qquad (\{i,k\} \in \bar{E}) \\
x_i \geq 0 & \qquad (i \in V)
\end{aligned}
$$

On the strength of the closure $\cap_{j\in V} \mathrm{N}_j(K, K)$

Relaxation	implied inequalities
$TH(G)$	clique [Grotschel, Lovász and Schrijver '88]
$\mathrm{N}_+(FRAC(G))$	clique, odd-cycle, odd-antihole and odd-wheel [Lovász and Schrijver '91] web [Giandomenico and Letchford '06]
$\mathrm{N}(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$	web and antiweb and their lifted versions $\Rightarrow$ edge, clique, odd-hole, odd-antihole [Giandomenico, Letchford, Rossi, S. '07]
$\cap_{j\in V} \mathrm{N}_j(\mathrm{QSTAB}(G), \mathrm{QSTAB}(G))$	odd-hole and antiweb and their lifted versions $\Rightarrow$ edge, clique, odd-antihole a large class of web [Giandomenico, Rossi, S. '10]

Implementation

- (aggressive) clique separation heuristic $\to \Omega$
- $\Omega(j)$ filtered by slack: prefer cliques tight to the current fractional point
- **Cut generation (one round)** for each $j \in V$:

  STEP 1. build $\Omega(j), \bar{\Omega}(j) \subseteq \Omega$, build CGLP
  STEP 2. Solve CGLP; add a (eventually) violated Benders cut
  STEP 3. Run the clique separation heuristic;
  add violated clique cuts (and cliques to Ω)

- implemented in the IBM Cplex 11.2 framework; all default
- CGLPs solved by Cplex **hybaropt** option
- 2 Intel Xeon 5150 processors clock 2.66 GHz, 4GB of RAM

Upper bounds comparison

Graph	$\alpha(G)$	UB_{clq}	$\theta(G)$	$UB_{\mathcal{C}(N)}$	BV	Time $UB_{\mathcal{C}(N)}$	Time BV
brock200_1	21	38.06	27.50	33.59	27.98	373	28,590
brock200_2	12	21.33	14.22	18.27	17.08	190	67,302
brock200_3	15	27.34	18.82	23.55	20.79	338	51,665
brock200_4	17	30.67	21.29	26.77	22.80	196	43,433
C.125.9	34	43.06	37.89	37.81	*	391	*
C.250.9	44	71.38	56.24	63.95	*	8,908	*
c-fat200-5	58	66.67	60.34	58	58.17	45	44,483
DSJC125.1	34	43.15	38.39	38.22	*	297	*
DSJC125.5	10	15.44	11.47	13.21	*	27	*
mann_a9	16	18.5	17.47	17.11	17.17	0.26	50
mann_a27	126	135	132.76	132.44	*	120	*
mann_a45	345	360	356.04	355.86	*	1,062	*
hamming6-4	4	5.33	5.33	4.64	4.54	5	1,416
keller4	11	14.82	14.01	14.29	15.41	9,586	19,319
p_hat300-1	8	15.3	10.10	13.45	18.66	767	322,287
p_hat300-2	25	33.59	27.00	30.73	30.10	2,207	244,428
p_hat300-3	36	54.36	41.16	49.79	43.32	2,419	101,995
san200_0.7-2	18	19.04	18	18	20.01	151	37,102
sanr200_0.7	18	33.39	23.8	29.45	24.97	762	36,576
sanr200_0.9	42	59.82	49.3	54.52	49.31	949	9,428

$\cap_j N_j$ **outperforms** $\theta(G)$

Upper bounds comparison

Sparse random graphs ($\leq 5\%$), generation parameters as in Gruber & Rendl '03

Graph	$\|V\|$	$\|E\|$	$\alpha(G)$	UB_{clq}	$\theta(G)$	$UB_{\mathcal{C}(N)}$
150.4	150	459	58	67.50	62.40	60.80
150.5	150	556	55	62.00	58.01	56.19
170.3	170	451	70	79.50	73.51	70.00
200.2	200	420	93	97.50	94.77	93.00
200.3	200	603	80	89.00	83.63	81.13
300.2	300	905	121	142.00	128.10	124.62
350.2	350	1,206	132	156.00	141.94	139.25
400.1	400	816	187	199.00	191.42	187.19

Branch-and-cut-results

	CPLEX default		Clique-B&C		$\mathcal{C}(N)$-B&C	
Graph	Time	Subprob.	Time	Subprob.	Time	Subprob.
brock200_1	1,691.88	303,352	1,153.54	119,613	1,096.41	99,078
brock200_2	118.70	9,808	91.42	2,498	91.25	2,340
brock200_3	202.18	19,461	233.10	8,239	197.00	5,983
brock200_4	484.30	71,580	435.08	22,488	384.14	19,368
C.125.9	6.27	3,458	3.51	3,291	7.54	2,783
C.250.9	+++	+++	+++	+++	+++	+++
c-fat200-5	12.11	47	7.04	47	48.73	0
DSJC125.1	7.07	4,887	3.16	3,805	5.76	3,456
DSJC125.5	11.00	1,626	7.93	412	9.24	305
mann_a9	**0.02**	3	0.02	7	0.28	1
mann_a27	1.87	1,888	1.57	6,961	2.19	6,185
mann_a45	109.45	67,300	61.09	55,044	35.47	20,214
hamming6-4	0.11	6	0.09	5	0.40	1
keller4	23.40	5,814	27.86	2,713	38.99	2,602
p_hat300-1	136.23	4549	245.10	**2736**	259.12	2736
p_hat300-2	210.02	5,773	94.31	1,345	92.53	1,258
p_hat300-3	+++	+++	4,040.84	79,090	3,509.32	74,319
san200_0.7-2	6.34	517	4.07	170	12.21	0
sanr200_0.7	1,008.83	161,673	689.58	34,250	675.25	30,885
sanr200_0.9	2,795.50	853,581	1,357.64	395,538	1,039.78	278,983