Navier-Stokes equations in a moving domain

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Abstract

After recalling how to construct the abstract formulation of the Navier-Stokes equations on a fixed domain it is shown how to generalize them to the case of a moving domain. The traditional distinction between Eulerian and Lagrangian descriptions, unsatisfactory in this context, is replaced by the socalled ALE approach. This point of view, which historically has diverse origins, is cast into the context of a *working principle*, suitable also to model fluid wall interaction in a deformable tube.

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The *placement* is a function assigning a place to every element p of a body[1] $\mathcal{B}$, at a time t

$$\chi : \mathcal{B} \times \mathbb{R} \rightarrow \mathcal{E} \tag{1}$$

The *material velocity* is the field

$$v_m : \mathcal{B} \times \mathbb{R} \rightarrow \mathcal{V} \tag{2}$$

such that

$$v_m(p,t) := \lim_{\tau \rightarrow 0} \frac{1}{\tau}\big(\chi(p,t+\tau) - \chi(p,t)\big) = \dot{\chi}(p,t) \tag{3}$$

In a fixed domain $\mathcal{D} \in \mathcal{E}$ the *spatial velocity* is the field

$$v : \mathcal{D} \times \mathbb{R} \rightarrow \mathcal{V} \tag{4}$$

whose value at a place x and at time t is the velocity of the particle p occupying that place at that time

$$v\big(\chi(p,t),t\big) = v_m(p,t) \tag{5}$$

3 Acceleration fields

The *material acceleration* is the field

$$a_m : \mathcal{B} \times \mathbb{R} \rightarrow \mathcal{V} \tag{6}$$

such that

$$a_m(p,t) := \lim_{\tau \rightarrow 0} \frac{1}{\tau}\big(v_m(p,t+\tau) - v_m(p,t)\big) = \ddot{\chi}(p,t) \tag{7}$$

[1]We can think of a body as the fluid in a tank which is going to flow through a pipe and is in so large a quantity that the flow will last as long as we like. Before the flow starts we put a label on every particle so we can trace his path, or we can recognize which particle crosses a marked place where a probe detects its velocity and acceleration.

The *spatial acceleration* is the field

$$a : \mathcal{D} \times \mathbb{R} \to \mathcal{V} \tag{8}$$

whose value at a place x and at a time t is the acceleration of the particle p occupying that place at that time

$$a\big(\chi(p,t),t\big) = a_m(p,t) \tag{9}$$

By putting together the above definitions we can express the spatial acceleration field in terms of the spatial velocity field

$$\begin{aligned}
a(\chi(p,t),t) &= a_m(p,t) && (10)\\
&= \lim_{\tau\to 0} \frac{1}{\tau}\big(v_m(p,t+\tau) - v_m(p,t)\big) && (11)\\
&= \lim_{\tau\to 0} \frac{1}{\tau}\Big(v\big(\chi(p,t+\tau),t+\tau\big) - v(\chi(p,t),t)\Big) && (12)
\end{aligned}$$

By using the trick of adding the null expression

$$-\,v\big(\chi(p,t+\tau),t\big) + v\big(\chi(p,t+\tau),t\big) \tag{13}$$

and rearranging the terms between brackets, we get

$$\begin{aligned}
a\big(\chi(p,t),t\big) &= a_m(p,t) && (14)\\
&= \lim_{\tau\to 0} \frac{1}{\tau}\Big(v\big(\chi(p,t+\tau),t\big) - v\big(\chi(p,t),t\big)\Big) && (15)\\
&\quad + \lim_{\tau\to 0} \frac{1}{\tau}\Big(v\big(\chi(p,t+\tau),t+\tau\big) - v\big(\chi(p,t+\tau),t\big)\Big) && (16)\\
&= \nabla v\big(\chi(p,t),t\big)\,\dot{\chi}(p,t) + v'\big(\chi(p,t),t\big) && (17)\\
&= \nabla v\big(\chi(p,t),t\big)\,v_m(p,t) + v'\big(\chi(p,t),t\big) && (18)\\
&= \nabla v\big(\chi(p,t),t\big)\,v\big(\chi(p,t),t\big) + v'\big(\chi(p,t),t\big) && (19)
\end{aligned}$$

By replacing $\chi(p,t)$ with x, we arrive at

$$a(x,t) = \nabla v(x,t)\,v(x,t) + v'(x,t) \tag{20}$$

In short

$$a = (\nabla v)\,v + v' \tag{21}$$

Since, by definition of divergence of a tensor field, the following identity holds

$$\operatorname{div}(v \otimes v) = (\nabla v)v + v\,\operatorname{div} v \tag{22}$$

the acceleration field can also be given the expression

$$a = \operatorname{div}\big(v \otimes v\big) - v\,\operatorname{div} v + v' \tag{23}$$

4 Incompressibility

The placement gradient

$$F(p,t) := \nabla\chi(p,t) \tag{24}$$

is the tensor transforming a body line element ℓ, a vector tangent to a curve $\hat{p}(h)$ in $\mathcal{B}$ at $p = \hat{p}(0)$, into a vector e tangent to the corresponding curve $\chi(\hat{p}(h),t)$ in $\mathcal{E}$.

$$e := F(p,t)\,\ell = \lim_{h\to 0}\frac{1}{h}\Big(\chi\big(\hat{p}(h),t\big) - \chi\big(\hat{p}(0),t\big)\Big) \tag{25}$$

$$\ell := \lim_{h\to 0}\frac{1}{h}\big(\hat{p}(h) - \hat{p}(0)\big) \tag{26}$$

Differentiating with respect to time we obtain

$$\dot{F}(p,t)\,\ell = \lim_{h\to 0}\frac{1}{h}\Big(\dot{\chi}\big(\hat{p}(h),t\big) - \dot{\chi}\big(\hat{p}(0),t\big)\Big) \tag{27}$$

$$= \lim_{h\to 0}\frac{1}{h}\Big(v_m\big(\hat{p}(h),t\big) - v_m\big(\hat{p}(0),t\big)\Big) = \nabla v_m(p,t)\,\ell \tag{28}$$

On the other hand

$$\dot{F}(p,t)\,\ell = \lim_{h\to 0}\frac{1}{h}\Big(v_m\big(\hat{p}(h),t\big) - v_m\big(\hat{p}(0),t\big)\Big) \tag{29}$$

$$= \lim_{h\to 0}\frac{1}{h}\Big(v\big(\chi(\hat{p}(h),t),t\big) - v\big(\chi(\hat{p}(0),t),t\big)\Big) \tag{30}$$

$$= \nabla v(p,t)\,e = \nabla v(p,t)F(p,t)\ell \tag{31}$$

So we can conclude that

$$\dot{F} = \nabla v_m = \nabla v\,F \tag{32}$$

We define the volume of a part of $\mathcal{B}$ as the volume of the region of $\mathcal{E}$ occupied by it. Denoting the volume of a parallelepiped in $\mathcal{E}$ by

$$\mathrm{vol}\,(F\,\ell_1, F\,\ell_2, F\,\ell_3) \tag{33}$$

its time derivative is

$$\frac{d}{dt}\,\mathrm{vol}\,(F\,\ell_1, F\,\ell_2, F\,\ell_3) \tag{34}$$

$$= \mathrm{vol}\,(\dot{F}\,\ell_1, F\,\ell_2, F\,\ell_3) + \mathrm{vol}\,(F\,\ell_1, \dot{F}\,\ell_2, F\,\ell_3) + \mathrm{vol}\,(F\,\ell_1, F\,\ell_2, \dot{F}\,\ell_3) \tag{35}$$

$$= \mathrm{vol}\,(\nabla v\,F\,\ell_1, F\,\ell_2, F\,\ell_3) + \mathrm{vol}\,(F\,\ell_1, \nabla v\,F\,\ell_2, F\,\ell_3) + \mathrm{vol}\,(F\,\ell_1, F\,\ell_2, \nabla v\,F\,\ell_3) \tag{36}$$

$$= \mathrm{tr}\,\nabla v\;\;\mathrm{vol}\,(F\,\ell_1, F\,\ell_2, F\,\ell_3) = \mathrm{div}\,v\;\;\mathrm{vol}\,(F\,\ell_1, F\,\ell_2, F\,\ell_3) \tag{37}$$

by definition of tr and div. That is why the condition that the volume stays constant in time is

$$\mathrm{div}\,v = 0 \tag{38}$$

5 Balance

Balance principle: for any test velocity field w

$$\int_{\mathcal{D}} b \cdot w + \int_{\partial\mathcal{D}} t \cdot w - \int_{\mathcal{D}} T \cdot \nabla w = 0 \tag{39}$$

By definition of divergence of a tensor field

$$\operatorname{div} T \cdot w = \operatorname{div}(T^{\mathsf{T}} w) - T \cdot \nabla w \tag{40}$$

$$\int_{\mathcal{D}} b \cdot w + \int_{\partial\mathcal{D}} t \cdot w + \int_{\mathcal{D}} \operatorname{div} T \cdot w - \int_{\mathcal{D}} \operatorname{div}(T^{\mathsf{T}} w) = 0 \tag{41}$$

By the divergence theorem

$$\int_{\mathcal{D}} \operatorname{div}(T^{\mathsf{T}} w) = \int_{\partial\mathcal{D}} (T^{\mathsf{T}} w) \cdot n = \int_{\partial\mathcal{D}} Tn \cdot w \tag{42}$$

$$\int_{\mathcal{D}} (b + \operatorname{div} T) \cdot w + \int_{\partial\mathcal{D}} (t - Tn) \cdot w = 0 \tag{43}$$

Balance equations in $\mathcal{D}$ and on its boundary $\partial\mathcal{D}$

$$\operatorname{div} T + b = 0 \tag{44}$$
$$t = Tn \tag{45}$$

6 Material response

The response function for an incompressible linearly viscous fluid has the expression (see [1])

$$T = -pI + 2\mu \operatorname{sym} \nabla v \tag{46}$$

7 Navier Stokes equations

By substituting the stress with the response function we get

$$\operatorname{div} T = -\nabla p + \mu \left(\operatorname{div} \nabla v + \operatorname{div} \nabla v^{\mathsf{T}} \right) \tag{47}$$

Because of the following properties

$$\operatorname{div} \nabla v = \Delta v \tag{48}$$
$$\operatorname{div} \nabla v^{\mathsf{T}} = \nabla(\operatorname{tr} \nabla v) = \nabla \operatorname{div} v = 0 \tag{49}$$

the previous expression becomes

$$\operatorname{div} T = -\nabla p + \mu \Delta v \tag{50}$$

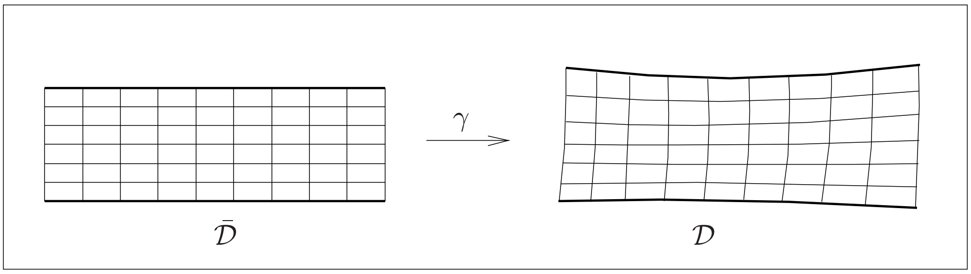

Figure 1: Grid deformation

As the only force distribution with respect to the volume is the inertial force density

$$b = -\rho a = -\rho \left(\nabla v\, v + v'\right) \tag{51}$$

the balance equation turns into

$$-\nabla p + \mu \Delta v - \rho \left(\nabla v\, v + v'\right) = 0 \tag{52}$$

8 Moving grid

By *grid deformation* we mean a regular function

$$\gamma : \bar{\mathcal{D}} \times \mathbb{R} \to \mathcal{E} \tag{53}$$

For a fixed time t

$$\gamma(\cdot, t) : \bar{\mathcal{D}} \to \mathcal{D} \tag{54}$$

Pull-back to $\bar{\mathcal{D}}$ of the velocity field v in $\mathcal{D}$

$$\mathrm{v}(\bar{x}, t) = v(\gamma(\bar{x}, t), t) \tag{55}$$

A vector tanget to a curve $\bar{c}(h)$ in $\bar{\mathcal{D}}$

$$\bar{e} := \lim_{h\to 0} \frac{1}{h}\big(\bar{c}(h) - \bar{c}(0)\big) \tag{56}$$

Grid deformation gradient

$$\Gamma := \nabla \gamma \tag{57}$$

The vector tanget to the curve $\gamma(\bar{c}(h), t)$ in $\mathcal{D}$

$$e = \Gamma(\bar{x}, t)\bar{e} = \lim_{h\to 0} \frac{1}{h}\Big(\gamma\big(\bar{c}(h), t\big) - \gamma\big(\bar{c}(0), t\big)\Big) \tag{58}$$

Pull-back to $\bar{\mathcal{D}}$ of the velocity field gradient ∇v in $\mathcal{D}$

$$\nabla \mathrm{v}(\bar{x}, t)\bar{e} = \lim_{h\to 0} \frac{1}{h}\Big(\mathrm{v}\big(\bar{c}(h), t\big) - \mathrm{v}\big(\bar{c}(0), t\big)\Big) \tag{59}$$

$$= \lim_{h\to 0} \frac{1}{h}\Big(v\big(\gamma(\bar{c}(h), t), t\big) - v\big(\gamma(\bar{c}(0), t), t\big)\Big) \tag{60}$$

$$= \nabla v\big(\gamma(\bar{x}, t), t\big)e = \nabla v\big(\gamma(\bar{x}, t), t\big)\Gamma(\bar{x}, t)\bar{e} \tag{61}$$

In short
$$\nabla \mathrm{v} = \nabla v \, \Gamma \tag{62}$$

The derivative with respect to time of the value of the velocity field at a place x transforms as follows
$$v'(x,t) = \lim_{\tau \to 0} \frac{1}{\tau}\big(v(x, t+\tau) - v(x,t)\big) \tag{63}$$
$$v'\big(\gamma(\bar{x},t),t\big) = \lim_{\tau \to 0} \frac{1}{\tau}\Big(v\big(\gamma(\bar{x},t), t+\tau\big) - v\big(\gamma(\bar{x},t),t\big)\Big) \tag{64}$$

By using the trick of adding the null expression
$$-\, v\big(\gamma(\bar{x}, t+\tau), t+\tau\big) + v\big(\gamma(\bar{x}, t+\tau), t+\tau\big) \tag{65}$$

and rearranging the terms between brackets, we get
$$v'\big(\gamma(\bar{x},t),t\big) = -\lim_{\tau \to 0} \frac{1}{\tau}\Big(v\big(\gamma(\bar{x},t+\tau), t+\tau\big) - v\big(\gamma(\bar{x},t),t+\tau\big)\Big) \tag{66}$$
$$+ \lim_{\tau \to 0} \frac{1}{\tau}\Big(v\big(\gamma(\bar{x},t+\tau), t+\tau\big) - v\big(\gamma(\bar{x},t),t\big)\Big) \tag{67}$$
$$= -\nabla v(\bar{x},t)\, \dot{\gamma}(\bar{x},t) + \dot{\mathrm{v}}(\bar{x},t) \tag{68}$$

Pull-back to $\bar{\mathcal{D}}$ of the acceleration field a in $\mathcal{D}$
$$\mathrm{a}(\bar{x},t) = a\big(\gamma(\bar{x},t),t\big) \tag{69}$$

Substituting the expression for $a(x,t)$ we get
$$\mathrm{a}(\bar{x},t) = \nabla v\big(\gamma(\bar{x},t),t\big)\, v\big(\gamma(\bar{x},t),t\big) + v'\big(\gamma(\bar{x},t),t\big) \tag{70}$$
$$= \nabla \mathrm{v}(\bar{x},t)\, \Gamma(\bar{x},t)^{-1}\, \mathrm{v}(\bar{x},t) - \nabla \mathrm{v}(\bar{x},t)\, \Gamma(\bar{x},t)^{-1}\, \dot{\gamma}(\bar{x},t) + \dot{\mathrm{v}}(\bar{x},t) \tag{71}$$
$$= \nabla \mathrm{v}(\bar{x},t)\, \Gamma(\bar{x},t)^{-1}\big(\,\mathrm{v}(\bar{x},t) - \dot{\gamma}(\bar{x},t)\big) + \dot{\mathrm{v}}(\bar{x},t) \tag{72}$$

In short
$$\mathrm{a} = \nabla \mathrm{v}\, \Gamma^{-1}(\mathrm{v} - \dot{\gamma}) + \dot{\mathrm{v}} \tag{73}$$

The incompressibility condition becomes
$$\operatorname{div} v = \operatorname{tr} \nabla v = \operatorname{tr} \nabla \mathrm{v} \Gamma^{-1} = \nabla \mathrm{v}^{\mathsf{T}} \cdot \Gamma^{-1} = 0 \tag{74}$$

As an alternative description of the grid deformation we can define the *grid displacement field* u, such that
$$u(\bar{x},t) = \gamma(\bar{x},t) - \bar{x} \tag{75}$$

whose gradient turns out to be
$$\nabla u = \Gamma - I \tag{76}$$

9 Volume and area changing

While $\det\Gamma$, by definition, is the ratio between the volume of a parallelepiped deformed by Γ and its former volume, the ratio between the area of a parallelogram tangent to $\partial\bar{\mathcal{D}}$ deformed by Γ and its former area, is given by (see sect. A)

$$\det\Gamma\|\Gamma^{-\mathsf{T}}m\| \tag{77}$$

where m is the unit external normal to the closed surface in $\bar{\mathcal{D}}$. The unit normal to the corresponding closed surface in $\mathcal{D}$ is

$$n = \frac{\Gamma^{-\mathsf{T}}m}{\|\Gamma^{-\mathsf{T}}m\|} \tag{78}$$

The expressions above give us a good reason to define the *cofactor* of Γ

$$\Gamma^{\star} := \det\Gamma\,\Gamma^{-\mathsf{T}} \tag{79}$$

It would be nice to have a volume preserving grid deformation. In such a case

$$\det\Gamma = 1 \tag{80}$$

In general we assume

$$\det\Gamma > 0 \tag{81}$$

10 Balance on a moving grid

Balance principle: for any test velocity field w

$$\int_{\mathcal{D}} b\cdot w + \int_{\partial\mathcal{D}} t\cdot w - \int_{\mathcal{D}} T\cdot\nabla w = 0 \tag{82}$$

In order to change the domain of integration, from the moving domain $\mathcal{D}$ to the fixed domain $\bar{\mathcal{D}}$, we have to define the pull-back of the test velocity field

$$\mathrm{w}(\bar{x},t) = w(\gamma(\bar{x},t),t) \tag{83}$$

whose gradient will be

$$\nabla\mathrm{w} = \nabla w\,\Gamma \tag{84}$$

From the balance above we get

$$\int_{\bar{\mathcal{D}}} b\cdot\mathrm{w}\det\Gamma + \int_{\partial\bar{\mathcal{D}}} t\cdot\mathrm{w}\det\Gamma\|\Gamma^{-\mathsf{T}}m\| - \int_{\bar{\mathcal{D}}} T\cdot\nabla\mathrm{w}\,\Gamma^{-1}\det\Gamma = 0 \tag{85}$$

Rearranging the integrands

$$\int_{\bar{\mathcal{D}}}(b\,\det\Gamma)\cdot\mathrm{w} + \int_{\partial\bar{\mathcal{D}}}\Big(t\,\det\Gamma\|\Gamma^{-\mathsf{T}}m\|\Big)\cdot\mathrm{w} - \int_{\bar{\mathcal{D}}}\Big(T\,\Gamma^{-\mathsf{T}}\det\Gamma\Big)\cdot\nabla\mathrm{w} = 0 \tag{86}$$

Using the definition of cofactor

$$\int_{\bar{\mathcal{D}}} (b \det \Gamma) \cdot \mathrm{w} + \int_{\partial \bar{\mathcal{D}}} \|\Gamma^\star m\| \, t \cdot \mathrm{w} - \int_{\bar{\mathcal{D}}} (T\,\Gamma^\star) \cdot \nabla \mathrm{w} = 0 \tag{87}$$

By definition of divergence of a tensor field

$$\operatorname{div}(T\,\Gamma^\star) \cdot \mathrm{w} = \operatorname{div}((T\,\Gamma^\star)^{\mathsf{T}} \mathrm{w}) - (T\,\Gamma^\star) \cdot \nabla \mathrm{w} \tag{88}$$

It is worth noting that here the divergence is for tensor and vector fields in $\bar{\mathcal{D}}$.

$$\int_{\bar{\mathcal{D}}} (b \det \Gamma) \cdot \mathrm{w} + \int_{\partial \bar{\mathcal{D}}} \|\Gamma^\star m\| \, t \cdot \mathrm{w} + \int_{\bar{\mathcal{D}}} \operatorname{div}(T\,\Gamma^\star) \cdot \mathrm{w} - \int_{\bar{\mathcal{D}}} \operatorname{div}((T\,\Gamma^\star)^{\mathsf{T}} \mathrm{w}) = 0 \tag{89}$$

By the divergence theorem

$$\int_{\bar{\mathcal{D}}} \operatorname{div}((T\,\Gamma^\star)^{\mathsf{T}} \mathrm{w}) = \int_{\partial \bar{\mathcal{D}}} \left((T\,\Gamma^\star)^{\mathsf{T}} \mathrm{w}\right) \cdot m = \int_{\partial \bar{\mathcal{D}}} (T\,\Gamma^\star m) \cdot \mathrm{w} \tag{90}$$

The balance equation above turns into

$$\int_{\bar{\mathcal{D}}} (b \det \Gamma + \operatorname{div}(T\,\Gamma^\star)) \cdot \mathrm{w} + \int_{\partial \bar{\mathcal{D}}} (\|\Gamma^\star m\| \, t - T\,\Gamma^\star m) \cdot \mathrm{w} = 0 \tag{91}$$

The usual localization argument gives us the balance equations in $\bar{\mathcal{D}}$ and on its boundary $\partial \bar{\mathcal{D}}$

$$\operatorname{div}(T\,\Gamma^\star) + b \det \Gamma = 0 \tag{92}$$

$$\|\Gamma^\star m\| \, t = T\,\Gamma^\star m \tag{93}$$

By setting

$$\breve{b} := b \det \Gamma \tag{94}$$

$$\breve{T} := T\,\Gamma^\star \tag{95}$$

$$\breve{t} := \|\Gamma^\star m\| \, t \tag{96}$$

the balance equations become

$$\operatorname{div} \breve{T} + \breve{b} = 0 \tag{97}$$

$$\breve{t} = \breve{T} m \tag{98}$$

11 Navier Stokes equations on a moving grid

The material response is

$$T = -pI + 2\mu \operatorname{sym} \nabla v = -pI + 2\mu \operatorname{sym}(\nabla \mathrm{v}\, \Gamma^{-1}) \tag{99}$$

Thus

$$\breve{T} = T\,\Gamma^\star = -p\,\Gamma^\star + 2\mu\ \mathrm{sym}\,(\nabla \mathrm{v}\,\Gamma^{-1})\,\Gamma^\star \tag{100}$$

The inertial forces are

$$b = -\rho\,\mathrm{a} = -\rho\left(\nabla \mathrm{v}\,\Gamma^{-1}(\mathrm{v} - \dot{\gamma}) + \dot{\mathrm{v}}\right) \tag{101}$$

Thus

$$\breve{b} = -\rho\,\mathrm{a}\ \det\Gamma = -\rho\left(\nabla \mathrm{v}\,\Gamma^{-1}(\mathrm{v} - \dot{\gamma}) + \dot{\mathrm{v}}\right)\det\Gamma \tag{102}$$

Putting all terms together we get

$$-\,\mathrm{div}\,(p\,\Gamma^\star) + 2\mu\ \mathrm{div}\,(\mathrm{sym}\,(\nabla \mathrm{v}\,\Gamma^{-1})\,\Gamma^\star) - \rho\left(\nabla \mathrm{v}\,\Gamma^{-1}(\mathrm{v} - \dot{\gamma}) + \dot{\mathrm{v}}\right)\det\Gamma = 0 \tag{103}$$

The first term expands into

$$\mathrm{div}\,(p\,\Gamma^\star) = p\ \mathrm{div}\,\Gamma^\star + \Gamma^\star\,\nabla p \tag{104}$$

By the property

$$\mathrm{div}\,\Gamma^\star = 0 \tag{105}$$

we get the new form of the Navier-Stokes equation

$$-\,\Gamma^\star\,\nabla p + 2\mu\ \mathrm{div}\,(\mathrm{sym}\,(\nabla \mathrm{v}\,\Gamma^{-1})\,\Gamma^\star) - \rho\left(\nabla \mathrm{v}\,\Gamma^{-1}(\mathrm{v} - \dot{\gamma}) + \dot{\mathrm{v}}\right)\det\Gamma = 0 \tag{106}$$

The incompressibility condition is

$$\mathrm{div}\,v = \mathrm{tr}\,(\nabla \mathrm{v}\Gamma^{-1}) = \nabla \mathrm{v}^{\mathsf{T}} \cdot \Gamma^{-1} = 0 \tag{107}$$

Differently from the case of non moving grid it is not possible here to use the incompressibility condition for simplifying the balance equation. Nevertheless the gradient of the espression on the left, multiplied by μ, can be subtracted from the Navier-Stokes equation just to recover the former case and hoping for computational simplifications.

12 The boundary as a membrane

Balance principle: for any test velocity field w

$$\int_{\partial\mathcal{D}} q \cdot w + \int_{\partial\partial\mathcal{D}} s \cdot w - \int_{\partial\mathcal{D}} S \cdot \nabla w = 0 \tag{108}$$

$$\int_{\partial\bar{\mathcal{D}}} q \cdot w\,\|\Gamma^\star m\| + \int_{\partial\partial\bar{\mathcal{D}}} s \cdot w\,\beta - \int_{\partial\bar{\mathcal{D}}} S \cdot \nabla w\,\Gamma^{-1}\|\Gamma^\star m\| = 0 \tag{109}$$

Balance equations in $\partial\bar{\mathcal{D}}$

$$\mathrm{div}_s\,(\|\Gamma^\star m\|\,\Gamma^{-\mathsf{T}}\,S) + q\,\|\Gamma^\star m\| = 0 \tag{110}$$

where

$$q\,\|\Gamma^\star m\| = (-\rho_w\,\mathrm{a} - Tn)\|\Gamma^\star m\| = -\rho_w\|\Gamma^\star m\|\,\mathrm{a} - T\,\Gamma^\star m \tag{111}$$

13 Fluid-wall interaction

14 Linearization

Let us assume that the flow is made up of a wave superposed to a stationary flow. Let δ be a wave control parameter: we can think of it as a scaling factor of the amplitude of the pressure wave. It is instrumental in the linearization process making it a well defined procedure. [After completion of the linearization process we can remove the parameter δ just by assigning it a unit value, if we like.]

$$\Gamma = I + \delta\,\tilde{\Gamma} + o(\delta) \tag{112}$$

$$\Gamma^{-1} = I - \delta\,\tilde{\Gamma} + o(\delta) \tag{113}$$

$$\det\Gamma = 1 + \delta\ \mathrm{tr}\,\tilde{\Gamma} + o(\delta) \tag{114}$$

$$\Gamma^{\star} = \det\Gamma\,\Gamma^{-\mathsf{T}} + o(\delta) \tag{115}$$

$$= (I - \delta\,\tilde{\Gamma}^{\mathsf{T}})(1 + \delta\ \mathrm{tr}\,\tilde{\Gamma}) + o(\delta) \tag{116}$$

$$= I - \delta\,\tilde{\Gamma}^{\mathsf{T}} + \delta\ \mathrm{tr}\,\tilde{\Gamma}\,I + o(\delta) \tag{117}$$

$$\mathrm{sym}\,(\nabla \mathrm{v}\,\Gamma^{-1})\Gamma^{\star} = \mathrm{sym}\,\nabla\bar{\mathrm{v}} + \delta \ldots \tag{118}$$

A Area changing

Let vol be any volume form in $\mathcal{E}$. An area form on $\partial\bar{\mathcal{D}}$ can be defined in the following way. Let m be the unit external normal to $\partial\bar{\mathcal{D}}$ at a position $\bar{x}$. Any couple of independent tangent vectors a_1, a_2 at $\bar{x}$ defines a parallelogram on the tangent plane whose area can be defined as

$$\mathrm{area}\,(a_1, a_2) = \mathrm{vol}\,(a_1, a_2, m) \tag{119}$$

In the grid deformation γ that couple of tangent vectors turns into a couple of vectors $\Gamma a_1, \Gamma a_2$, tangent to $\partial\mathcal{D}$ at $\gamma(\bar{x})$. Denoting by n any unit normal (one of the two possible choices) to $\partial\mathcal{D}$ at $\gamma(\bar{x})$, the area of the corresponding parallelogram is

$$\mathrm{area}\,(\Gamma a_1, \Gamma a_2) = \mathrm{vol}\,(\Gamma a_1, \Gamma a_2, n) \tag{120}$$

$$= \mathrm{vol}\,\big(\Gamma a_1, \Gamma a_2, \Gamma(\Gamma^{-1} n)\big) \tag{121}$$

$$= \det\Gamma\,\mathrm{vol}\,\big(a_1, a_2, (\Gamma^{-1} n)\big) \tag{122}$$

The vector $(\Gamma^{-1} n)$ can be decomposed into its orthogonal projection on span$\{m\}$

$$(m \otimes m)\Gamma^{-1} n = (\Gamma^{-1} n \cdot m) m \tag{123}$$

and its complement on span$\{a_1, a_2\}$. As for any vector $a \in \mathrm{span}\{a_1, a_2\}$

$$\mathrm{vol}\,(a_1, a_2, a) = 0 \tag{124}$$

the expression for the area above becomes

$$\begin{aligned}\operatorname{area}(\Gamma a_1, \Gamma a_2) &= \det\Gamma \operatorname{vol}\big(a_1, a_2, (\Gamma^{-1} n \cdot m) m\big) && (125)\\ &= \det\Gamma \operatorname{vol}\big(a_1, a_2, m\big)(\Gamma^{-1} n \cdot m) && (126)\\ &= \det\Gamma \operatorname{area}(a_1, a_2)(n \cdot \Gamma^{-\mathsf{T}} m) && (127)\end{aligned}$$

Note that the vector $\Gamma^{-\mathsf{T}} m$ is orthogonal to $\operatorname{span}\{\Gamma a_1, \Gamma a_2\}$, the tangent space to $\partial\mathcal{D}$ at $\gamma(\bar{x})$. For

$$\begin{aligned}\Gamma^{-\mathsf{T}} m \cdot \Gamma a_1 &= m \cdot \Gamma^{-1}\Gamma a_1 = m \cdot a_1 = 0 && (128)\\ \Gamma^{-\mathsf{T}} m \cdot \Gamma a_2 &= m \cdot \Gamma^{-1}\Gamma a_2 = m \cdot a_2 = 0 && (129)\end{aligned}$$

Hence

$$\begin{aligned}\Gamma^{-\mathsf{T}} m &= \alpha\, n && (130)\\ \Gamma^{-\mathsf{T}} m \cdot n &= \alpha && (131)\end{aligned}$$

The condition that the area does not change sign implies $\alpha > 0$. So the right choice for n is

$$n := \frac{\Gamma^{-\mathsf{T}} m}{\|\Gamma^{-\mathsf{T}} m\|} = \frac{\Gamma^{\star} m}{\|\Gamma^{\star} m\|} \tag{132}$$

We finally obtain

$$\frac{\operatorname{area}(\Gamma a_1, \Gamma a_2)}{\operatorname{area}(a_1, a_2)} = \det\Gamma \|\Gamma^{-\mathsf{T}} m\| = \|\Gamma^{\star} m\| \tag{133}$$

B Cofactor divergence

For any uniform vector field e in $\bar{\mathcal{D}}$

$$\operatorname{div}\Gamma^{\star} \cdot e = \operatorname{div}(\Gamma^{\star\mathsf{T}} e) \tag{134}$$

Integrating over $\bar{\mathcal{D}}$ and applying the divergence theorem we get

$$\begin{aligned}\int_{\bar{\mathcal{D}}} \operatorname{div}(\Gamma^{\star\mathsf{T}} e) &= \int_{\partial\bar{\mathcal{D}}} \Gamma^{\star\mathsf{T}} e \cdot m = \int_{\partial\bar{\mathcal{D}}} \Gamma^{\star} m \cdot e && (135)\\ &= \int_{\partial\bar{\mathcal{D}}} n \|\Gamma^{\star} m\| \cdot e = \int_{\partial\mathcal{D}} n \cdot e = \int_{\mathcal{D}} \operatorname{div} e = 0 && (136)\end{aligned}$$

This applies to every part of $\bar{\mathcal{D}}$ so, through an assumption of regularity, we come to the conclusion

$$\operatorname{div}\Gamma^{\star} = 0 \tag{137}$$

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