Blood flow through a curved artery

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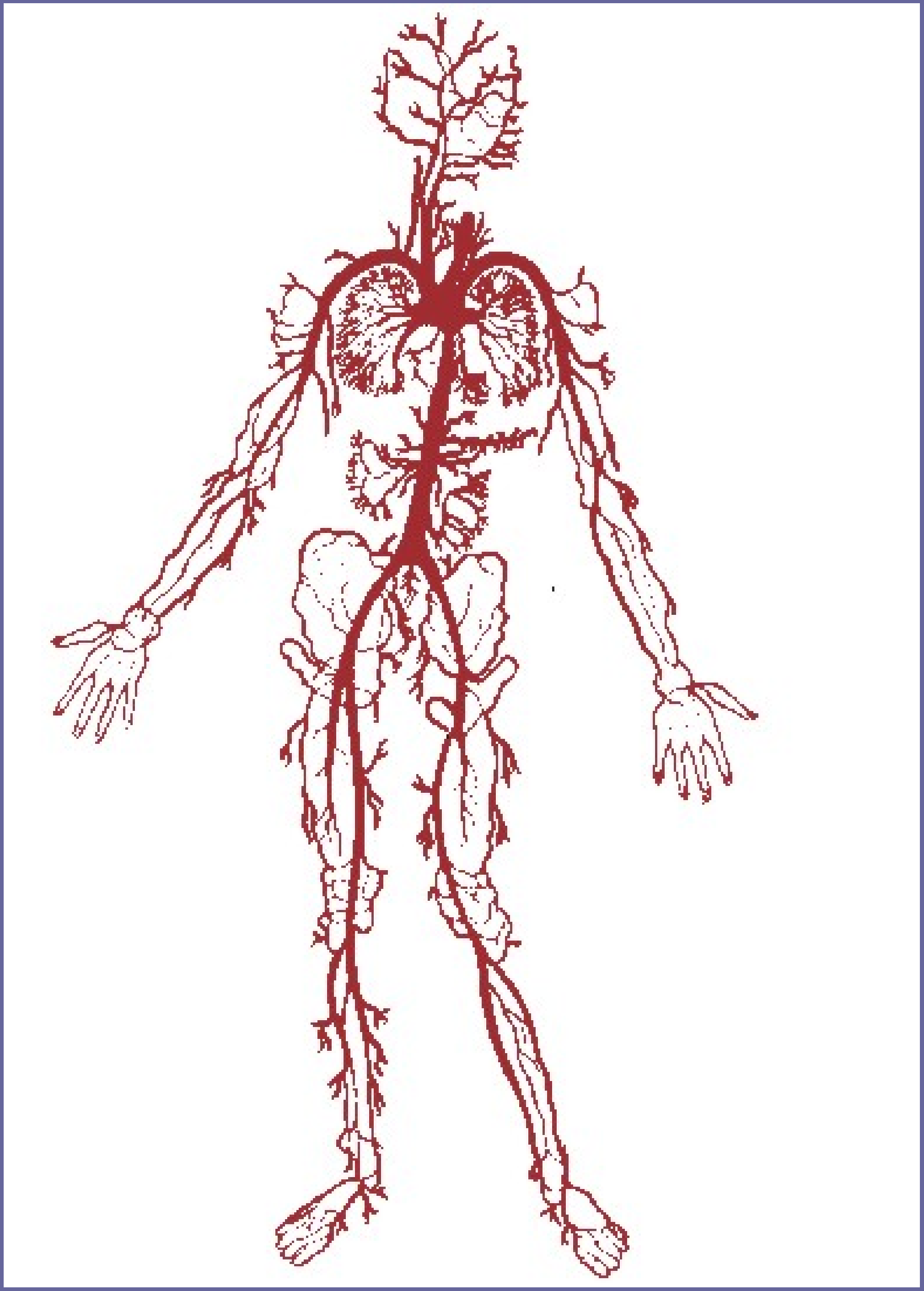


Circulatory system

K.B. Chandran, W.M. Swanson, D.N. Ghista, H.W.Vayo, Oscillatory Flow in Thin-Walled Curved Elastic Tubes, *Annals of Biomedical Engineering* 2, 392–412 (1974)

Project

- Mechanical model
- Problem formulation
- Solution procedure
- Results (velocity, pressure, tangential stress)

starting from scratch and self contained

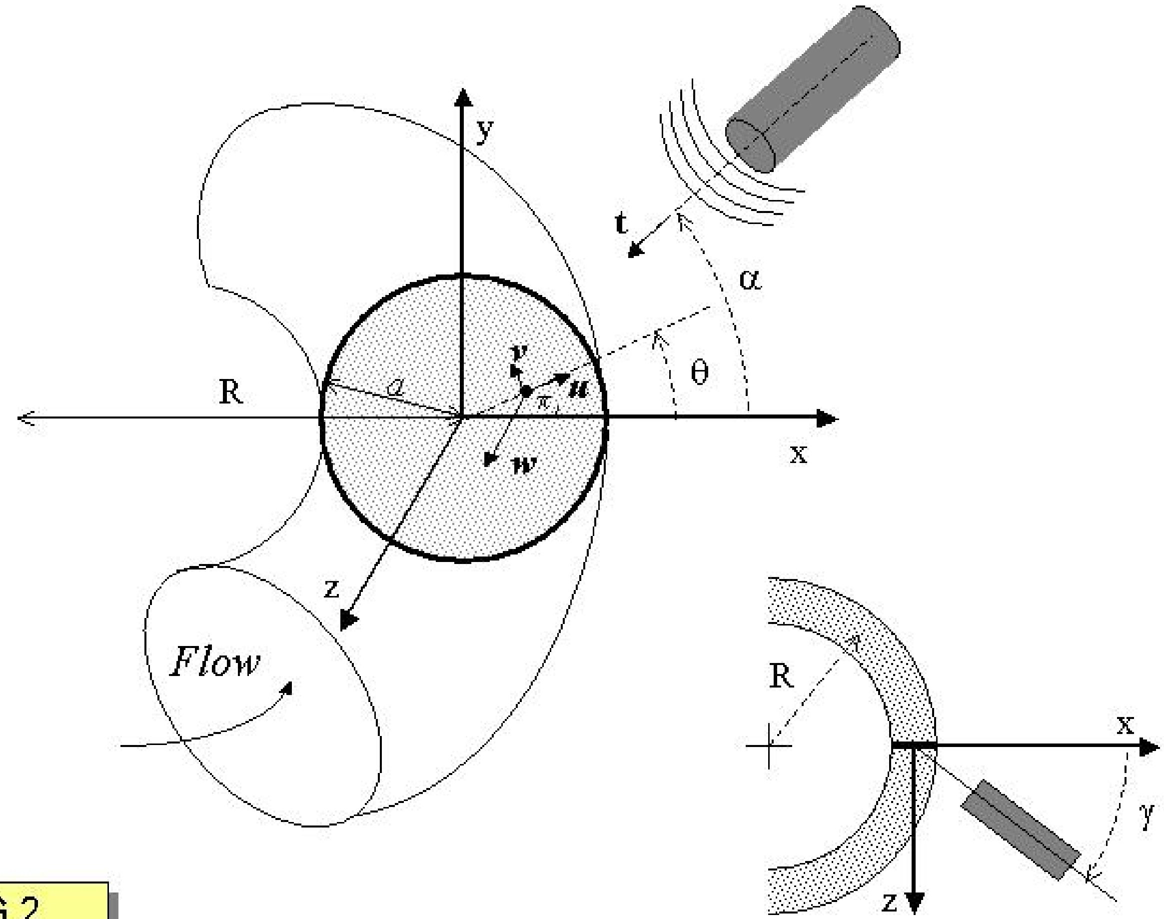


FIG 2

Laser Doppler ultrasound measurement (C. Guiot, Univ. Torino)

Motivations

- To provide velocity patterns for flux measurement
- To get some *ansatz* for one dimensional modelling

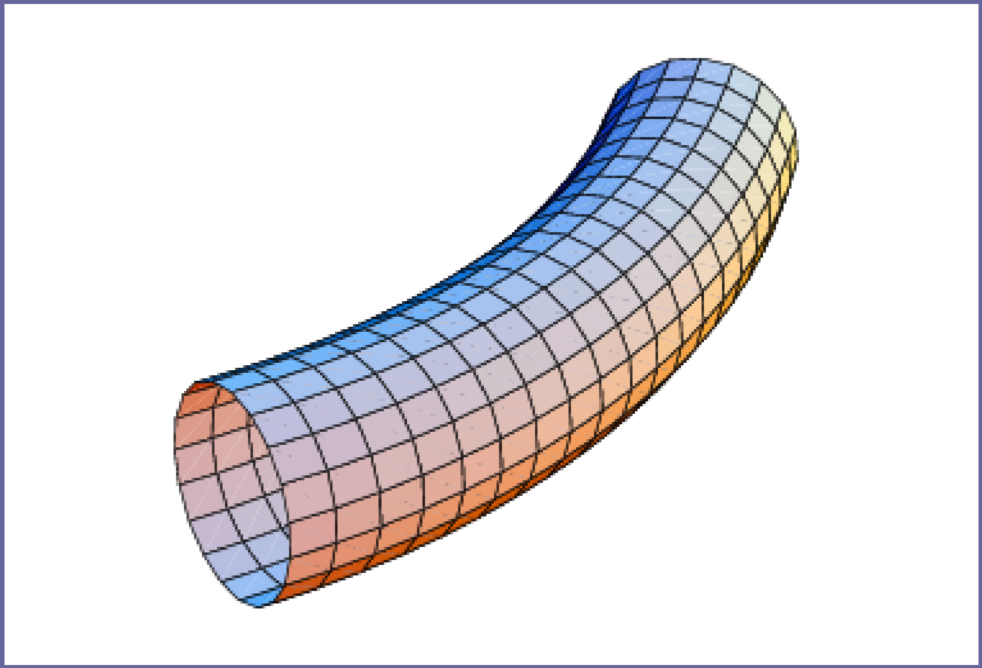
Basic references

axis	rigid wall	elastic wall	flow
cur	Dean (1927)		steady
str	Womersley (1955)	Morgan, Kiely (1953) Womersley (1955, 1957) Atabek (1968)	unsteady
cur	Lyne (1970) Smith (1975) Mullin, Greated (1980)	Chandran et al. (1974)	unsteady

Review paper: Berger, Talbot, Yao (1983).

Outline

- 3D Navier-Stokes equations (blood)
- 2D membrane model (wall)
- Linearization
- Wave propagating over a Poiseuille flow
- Curvature as a small perturbation
- Computer algebra for equation generation
- Numerical results and visualization



$$\frac{a}{R} = 0.1$$

Model features and assumptions

- 3D unsteady fully developed flow
- Planar curved axis (toroidal shape)
- Elastic wall

Wall model (membrane)

$$\operatorname{div}\mathbf{S} + \mathbf{b} = 0$$

$$\mathbf{b} = -\mathbf{T}\,\mathbf{n} - \rho_w\,\ddot{\mathbf{u}}$$

ρ_w wall mass density

$\mathbf{S}$ membrane stress tensor

$\mathbf{T}$ fluid stress tensor

Linear elastic isotropic wall

$$\hat{\mathbf{S}}(\mathbf{E}) = \begin{pmatrix} \dfrac{hE}{1-\sigma^2}(\epsilon_{\theta\theta} + \sigma\epsilon_{\psi\psi}) & 2hG\,\epsilon_{\theta\psi} \\ \\ 2hG\,\epsilon_{\theta\psi} & \dfrac{hE}{1-\sigma^2}(\epsilon_{\psi\psi} + \sigma\epsilon_{\theta\theta}) \end{pmatrix}$$

E Young modulus
G shear modulus
σ Poisson ratio
h wall thickness

Blood model (newtonian fluid)

$$\operatorname{div} \mathbf{T} - \rho\, \mathbf{a} = 0$$

$$\mathbf{T} = -p\, \mathbf{I} + \mu \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T \right)$$

$$\operatorname{div} \mathbf{v} = 0$$

ρ mass density

μ viscosity

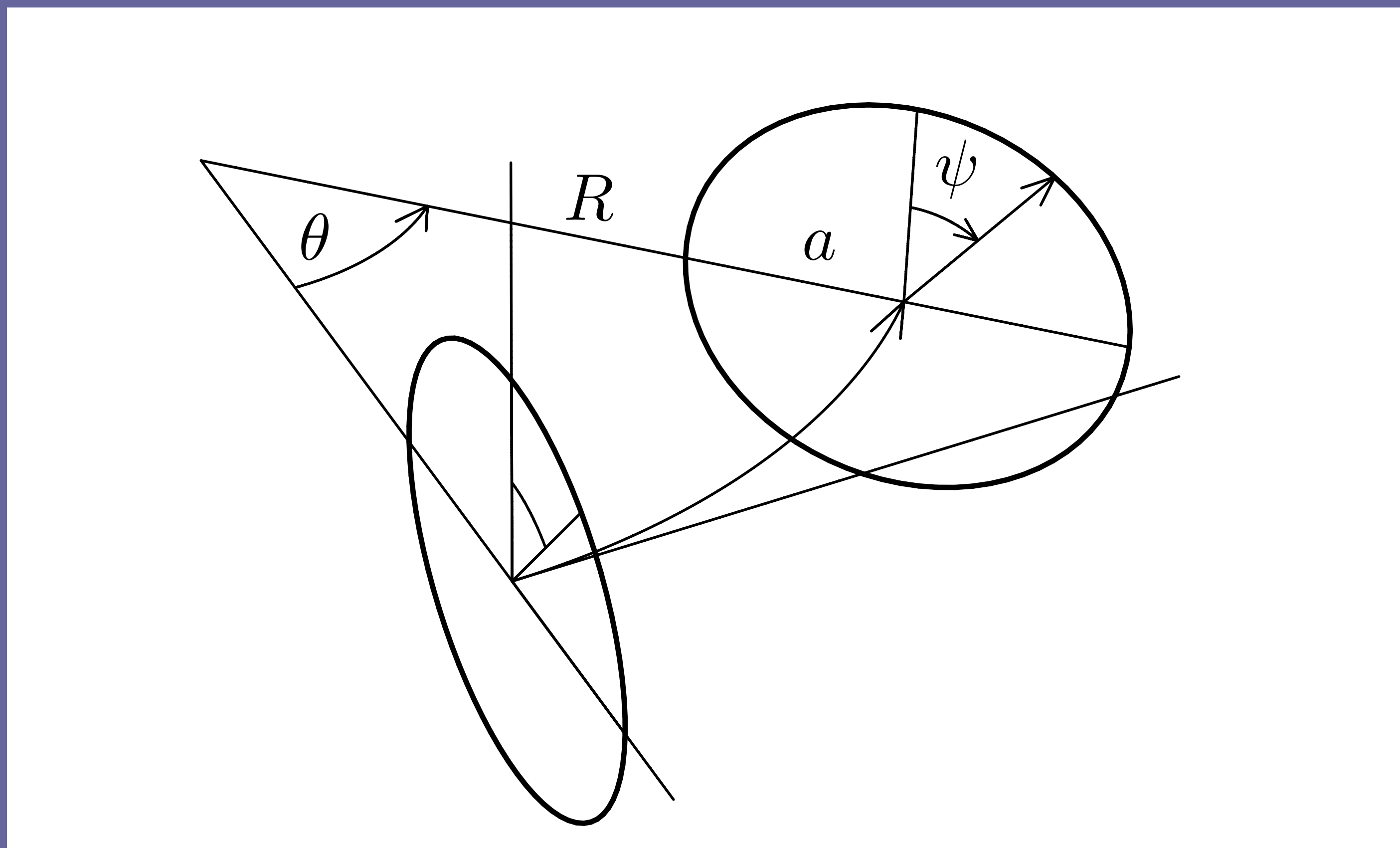
Blood model (Navier-Stokes)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}\right) = -\nabla p + \mu\,\Delta \mathbf{v}$$

$$\operatorname{div} \mathbf{v} = 0$$

$$\mathbf{v} = \dot{\mathbf{u}} \quad \text{(no slip condition at the wall)}$$

Toroidal coordinate system



Wave propagation

Wave propagation

steady flow in a curved tube

$+$

small oscillatory motion

Wave propagation

steady flow in a curved tube

$+$

small oscillatory motion

$$\bar{\chi}(r,\psi) + \tilde{\chi}(r,\psi)e^{i(\omega t - kz)}$$

Wave propagation

steady flow in a curved tube

$+$

small oscillatory motion

$$\bar{\chi}(r,\psi) + \tilde{\chi}(r,\psi)e^{i(\omega t - kz)}$$

Linearization of the flow equations over $\bar{\chi}$.

Perturbation method

$$\chi = \chi_0 + \varepsilon\chi_1 + \varepsilon^2\chi_2 + \varepsilon^3\chi_3 + \ldots$$

$$\varepsilon := \frac{a}{R}$$ (Curvature parameter)

$$\left(\bar{\chi}_0 + \varepsilon\bar{\chi}_1\right) + \left(\tilde{\chi}_0 + \varepsilon\tilde{\chi}_1\right) e^{i(\omega t - kz)}$$

Scaling

$$\lambda = \frac{\omega}{k}, \qquad \frac{a}{\lambda} \ll 1$$

$$\frac{u}{w} = O\left(\frac{a}{\lambda}\right), \quad \frac{v}{w} = O\left(\frac{a}{\lambda}\right), \quad \frac{\frac{\partial w}{\partial z}}{w} = O\left(\frac{a}{\lambda}\right)$$

$$\mathbf{v} \mapsto (u, v, w)$$

Different stages

- Linearization (wave amplitude)
- Perturbation (curvature)
- Scaling (wave length)

Linearized flow equations

$$\mathbf{v} \mapsto (u, v, w)$$

$$\rho\left\{\frac{\partial u}{\partial t} - \frac{2\,\bar{\boldsymbol{w}}\,w\sin\psi}{R + r\sin\psi}\right\} =$$
$$-\frac{\partial p}{\partial r} + \mu\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \psi^2} + \frac{\sin\psi}{R + r\sin\psi}\frac{\partial u}{\partial r}\right.$$
$$+\frac{\cos\psi}{r\,(R + r\sin\psi)}\frac{\partial u}{\partial \psi} - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \psi} - \frac{v R\cos\psi}{r\,(R + r\sin\psi)^2}$$
$$\left. - \frac{2\sin\psi}{(R + r\sin\psi)^2}\frac{\partial w}{\partial \theta} - \frac{u\,\sin^2\psi}{(R + r\sin\psi)^2} - \frac{2v\,\sin\psi\cos\psi}{(R + r\sin\psi)^2}\right\} \quad \text{(radial)}$$

Linearized flow equations

$$\rho\left\{\frac{\partial v}{\partial t}-\frac{2\,\bar{\boldsymbol{w}}\,w\cos\psi}{R+r\sin\psi}\right\}=-\frac{1}{r}\frac{\partial p}{\partial\psi}+\mu\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\psi^2}+\frac{2}{r^2}\frac{\partial u}{\partial\psi}\right.$$
$$+\frac{uR\cos\psi}{r\left(R+r\sin\psi\right)^2}+\frac{\sin\psi}{R+r\sin\psi}\frac{\partial v}{\partial r}+\frac{\cos\psi}{r\left(R+r\sin\psi\right)}\frac{\partial v}{\partial\psi}-\frac{v}{r^2}$$
$$\left.-\frac{2\cos\psi}{\left(R+r\sin\psi\right)^2}\frac{\partial w}{\partial\theta}-\frac{v\ \cos^2\psi}{(R+r\sin\psi)^2}\right\}$$

(circumferential)

Linearized flow equations

$$\rho\frac{\partial w}{\partial t} = -\frac{R}{R + r\sin\psi}\frac{\partial p}{\partial z} + \mu\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \psi^2} + \frac{\sin\psi}{R + r\sin\psi}\frac{\partial w}{\partial r}\right.$$

$$\left. + \frac{\cos\psi}{r\,(R + r\sin\psi)}\frac{\partial w}{\partial \psi} - \frac{w}{(R + r\sin\psi)^2}\right\} \quad \text{(axial)}$$

Wall equations

$$\mathbf{u} \mapsto (\eta, \xi, \zeta)$$

$$\rho_w h \frac{\partial^2 \eta}{\partial t^2} = \left[p - 2\mu \frac{\partial u}{\partial r} \right]_{r=a} - \frac{hE}{1-\sigma^2} \left[\frac{\eta + \dfrac{\partial \xi}{\partial \psi}}{a^2} + \frac{\sin\psi \left(\eta \sin\psi + \xi \cos\psi + \dfrac{\partial \zeta}{\partial \theta} \right)}{(R + a\sin\psi)^2} \right]$$

$$- \frac{\sigma h E}{1-\sigma^2} \left[\frac{\sin\psi \left(2\eta + \dfrac{\partial \xi}{\partial \psi} \right) + \xi \cos\psi + \dfrac{\partial \zeta}{\partial \theta}}{a(R + a\sin\psi)} \right]$$

(radial)

Wall equations

$$\rho_w h \frac{\partial^2 \xi}{\partial t^2} = -\mu \left[\frac{1}{r}\frac{\partial u}{\partial \psi} - \frac{v}{r} + \frac{\partial v}{\partial r} \right]_{r=a} + \frac{hE}{1-\sigma^2} \left[\frac{\dfrac{\partial \eta}{\partial \psi} + \dfrac{\partial^2 \xi}{\partial \psi^2}}{a^2} + \cos\psi \left(\frac{\eta + \dfrac{\partial \xi}{\partial \psi}}{a(R + a\sin\psi)} \right.\right.$$

$$\left.\left. - \frac{\eta \sin\psi + \xi\cos\psi + \dfrac{\partial \zeta}{\partial \theta}}{(R + a\sin\psi)^2} \right) \right] + \frac{\sigma h E}{1-\sigma^2} \left[\frac{-\xi\sin\psi + \sin\psi \dfrac{\partial \eta}{\partial \psi} + \dfrac{\partial^2 \zeta}{\partial \psi \partial \theta}}{a(R + a\sin\psi)} \right]$$

$$+hG \left[\frac{\dfrac{\partial^2 \zeta}{\partial \theta \partial \psi}}{a(R + a\sin\psi)} + \frac{\dfrac{\partial^2 \xi}{\partial \theta^2} - \cos\psi \dfrac{\partial \zeta}{\partial \theta}}{(R + a\sin\psi)^2} \right]$$

(circumferential)

Wall equations

$$\rho_w h \frac{\partial^2 \zeta}{\partial t^2} = -\mu \left[\frac{1}{R + a\sin\psi} \frac{\partial u}{\partial \theta} - \frac{w \sin\psi}{R + a\sin\psi} + \frac{\partial w}{\partial r} \right]_{r=a}$$

$$+ \frac{hE}{1-\sigma^2} \left[\frac{\sin\psi \frac{\partial \eta}{\partial \theta} + \cos\psi \frac{\partial \xi}{\partial \theta} + \frac{\partial^2 \zeta}{\partial \theta^2}}{(R + a\sin\psi)^2} \right] + \frac{\sigma h E}{1-\sigma^2} \left[\frac{\frac{\partial \eta}{\partial \theta} + \frac{\partial^2 \xi}{\partial \psi \partial \theta}}{a(R + a\sin\psi)} \right]$$

$$+ hG \left[\frac{1}{a^2} \frac{\partial^2 \zeta}{\partial \psi^2} + \frac{\frac{\partial^2 \xi}{\partial \psi \partial \theta} + \zeta \sin\psi + \cos\psi \frac{\partial \zeta}{\partial \psi}}{a(R + a\sin\psi)} + \frac{\cos\psi \frac{\partial \xi}{\partial \theta} - \zeta\cos^2\psi}{(R + a\sin\psi)^2} \right]$$

(axial)

Perturbation method

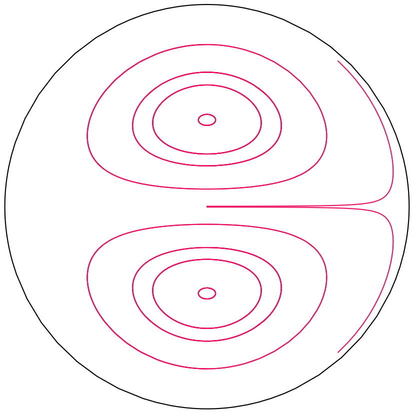
$$\chi = \chi_0 + \varepsilon\chi_1 + \varepsilon^2\chi_2 + \varepsilon^3\chi_3 + \ldots$$

$$\varepsilon := \frac{a}{R} \qquad \text{(Curvature parameter)}$$

$$\left(\bar{\chi}_0 + \varepsilon\bar{\chi}_1\right) + \left(\tilde{\chi}_0 + \varepsilon\tilde{\chi}_1\right) e^{i(\omega t - kz)}$$

0-th order and 1-st order solutions (steady flow)

Dean's solution



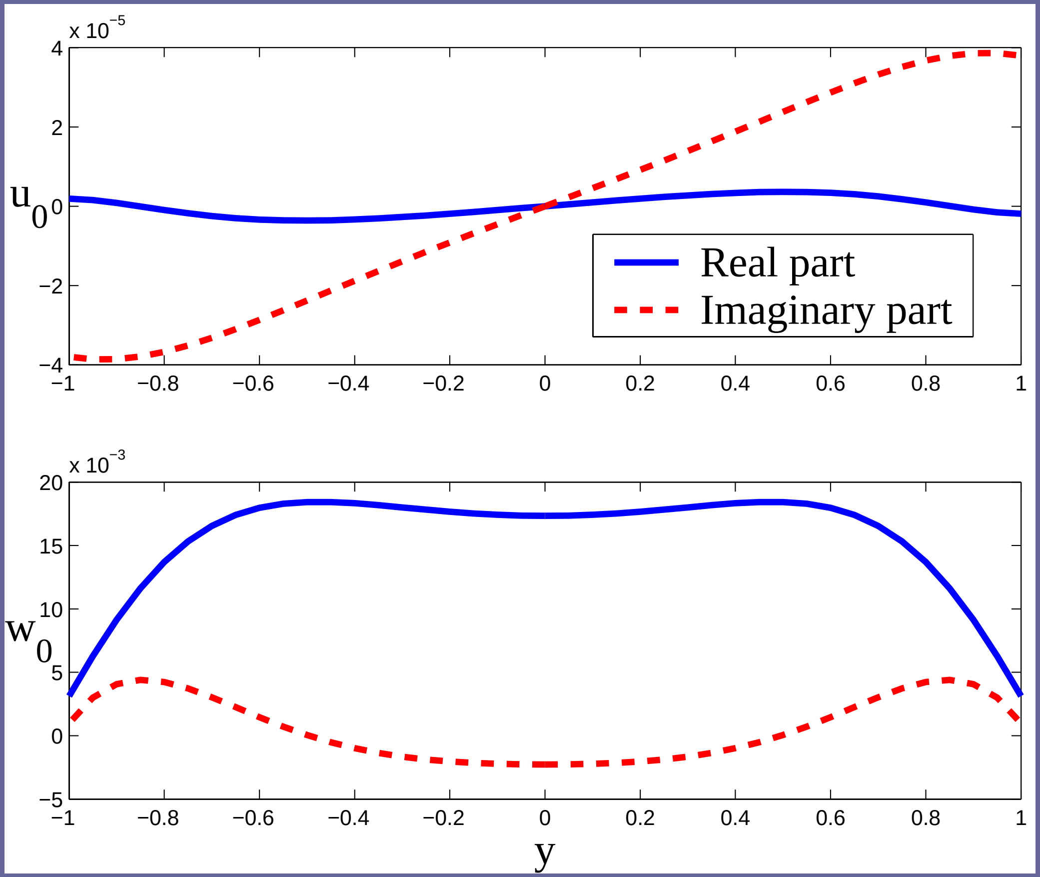
0-th order solution (unsteady flow)

Womersley's solution

- Straight axis
- Homogeneous equations
- Axisymmetric solution
- Frequency equation

Womersley's solution

Axisymmetric flow in a straight elastic tube



1-th order solution (unsteady flow)

$$\frac{du_1}{dy} + \frac{u_1}{y} - \frac{v_1}{y} - ikaw_1 = -(ikayw_0 + u_0)$$

$$\frac{d^2u_1}{dy^2} + \frac{1}{y}\frac{du_1}{dy} - \left(\frac{2}{y^2} + i\alpha^2\right)u_1 + \frac{2v_1}{y^2} - \frac{a}{\mu}\frac{dp_1}{dy} = -\left(\frac{du_0}{dy} + 2ika\,w_0 + \frac{2a\bar{w}_0}{\nu}w_0\right)$$

$$\frac{d^2v_1}{dy^2} + \frac{1}{y}\frac{dv_1}{dy} - \left(\frac{2}{y^2} + i\alpha^2\right)v_1 + \frac{2u_1}{y^2} - \frac{ap_1}{\mu y} = -\left(\frac{u_0}{y} + 2ikaw_0 + \frac{2a\bar{w}_0}{\nu}w_0\right)$$

$$\frac{d^2w_1}{dy^2} + \frac{1}{y}\frac{dw_1}{dy} - \left(\frac{1}{y^2} + i\alpha^2\right)w_1 + \frac{ika^2}{\mu}p_1 = -\left(\frac{dw_0}{dy} - \frac{ika^2y}{\mu}p_0\right)$$

$\left(\alpha := a\sqrt{\frac{\rho\,\omega}{\mu}}\right.$ Womersley number)

Wave solution

Assembling the 0-th and 1-th order solutions

$$(\chi_0 + \varepsilon\chi_1)\, e^{i(\omega t - kz)}$$

Full solution

Superposing the steady solution

$$(\bar{\chi}_0 + \varepsilon \bar{\chi}_1) + (\chi_0 + \varepsilon \chi_1)\, e^{i(\omega t - kz)}$$

Steady flow Unsteady flow

Harmonic form

$$\chi = \bar{\chi} + \left[\mathtt{Re}(\tilde{\chi})\, \cos\left(\omega t - \mathtt{Re}(k)\, z\right) - \mathtt{Im}(\tilde{\chi})\, \sin\left(\omega t - \mathtt{Re}(k)\, z\right)\right] e^{\mathtt{Im}(k)\, z}$$

Numerical results

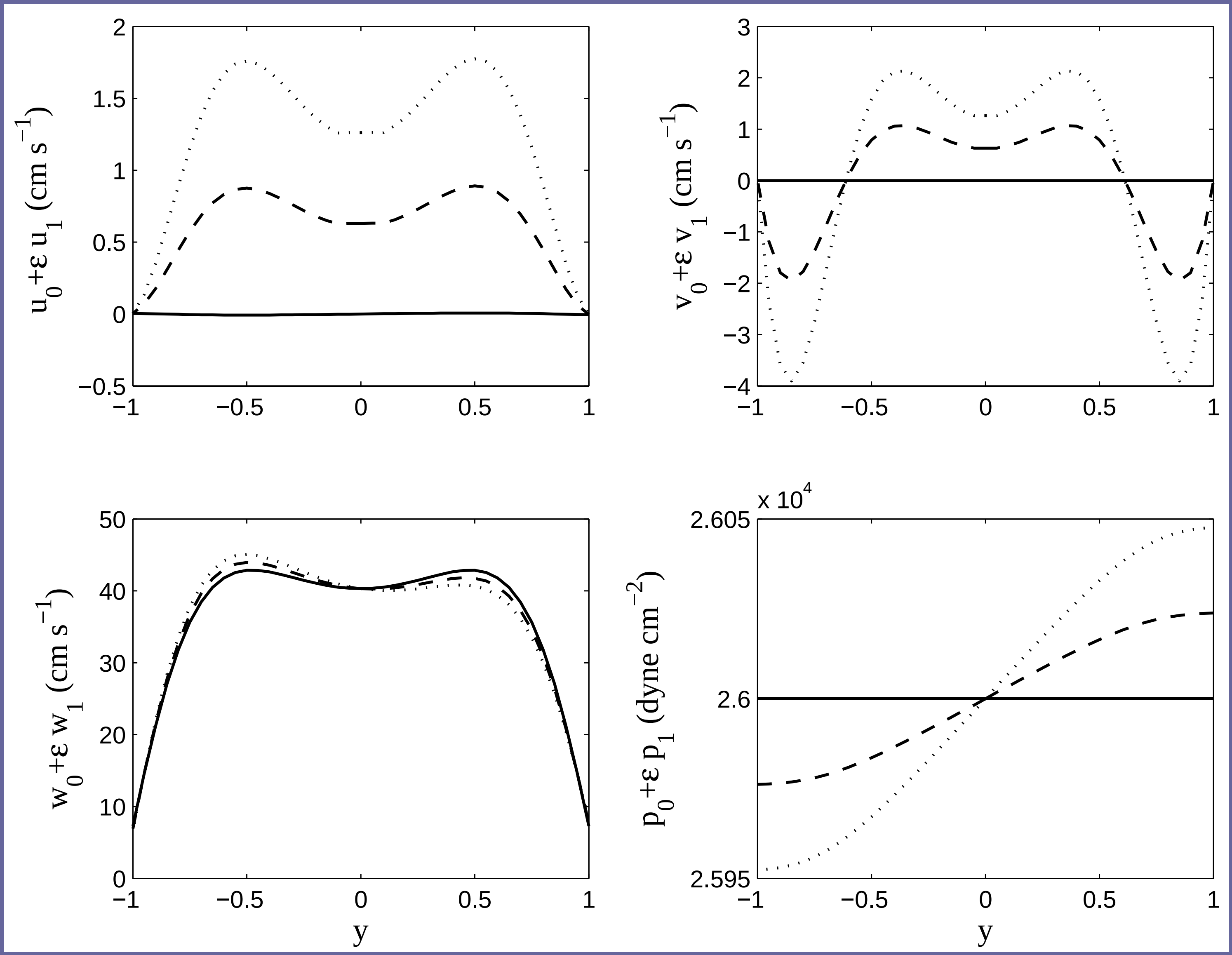
$$E = 10^7 \text{ dynes/cm}^2 \qquad h = 0.05 \text{ cm} \qquad \sigma = 0.5$$

$$\omega = 2\pi \text{ s}^{-1} \qquad a = 0.5 \,\text{cm}$$

$$\mu = 0.04 \text{ g/ cm } s \qquad \rho = \rho_w = 1 \text{ g/cm}^3$$

$$A = 26000 \,\text{dyne/cm}^2 \qquad \frac{d\bar{p}_0}{dz} = 7 \,\text{dyne/cm}^3$$

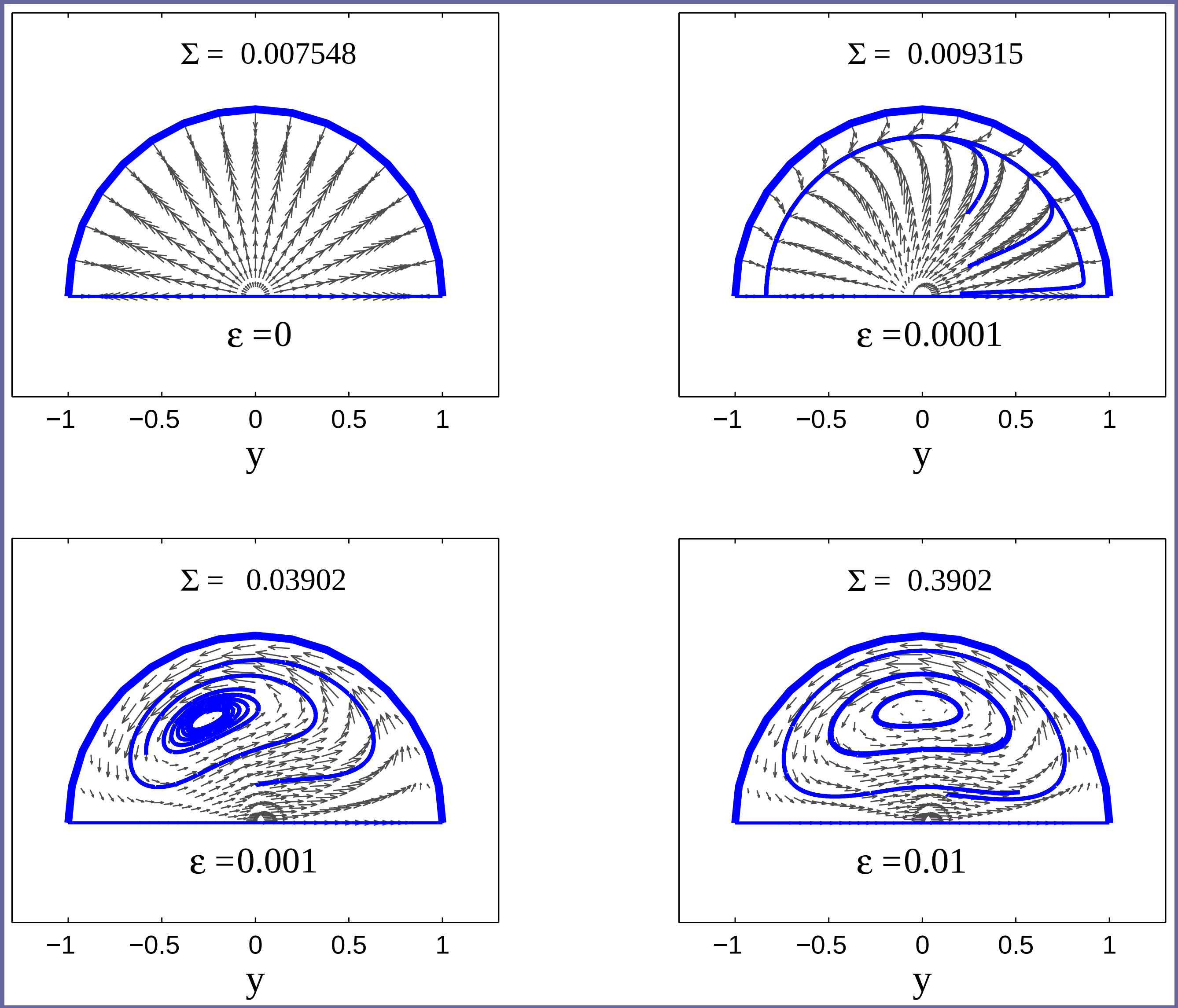
$$\Delta y = 0.02$$



Unsteady solution: $\tilde{\chi}_0 + \varepsilon\tilde{\chi}_1$

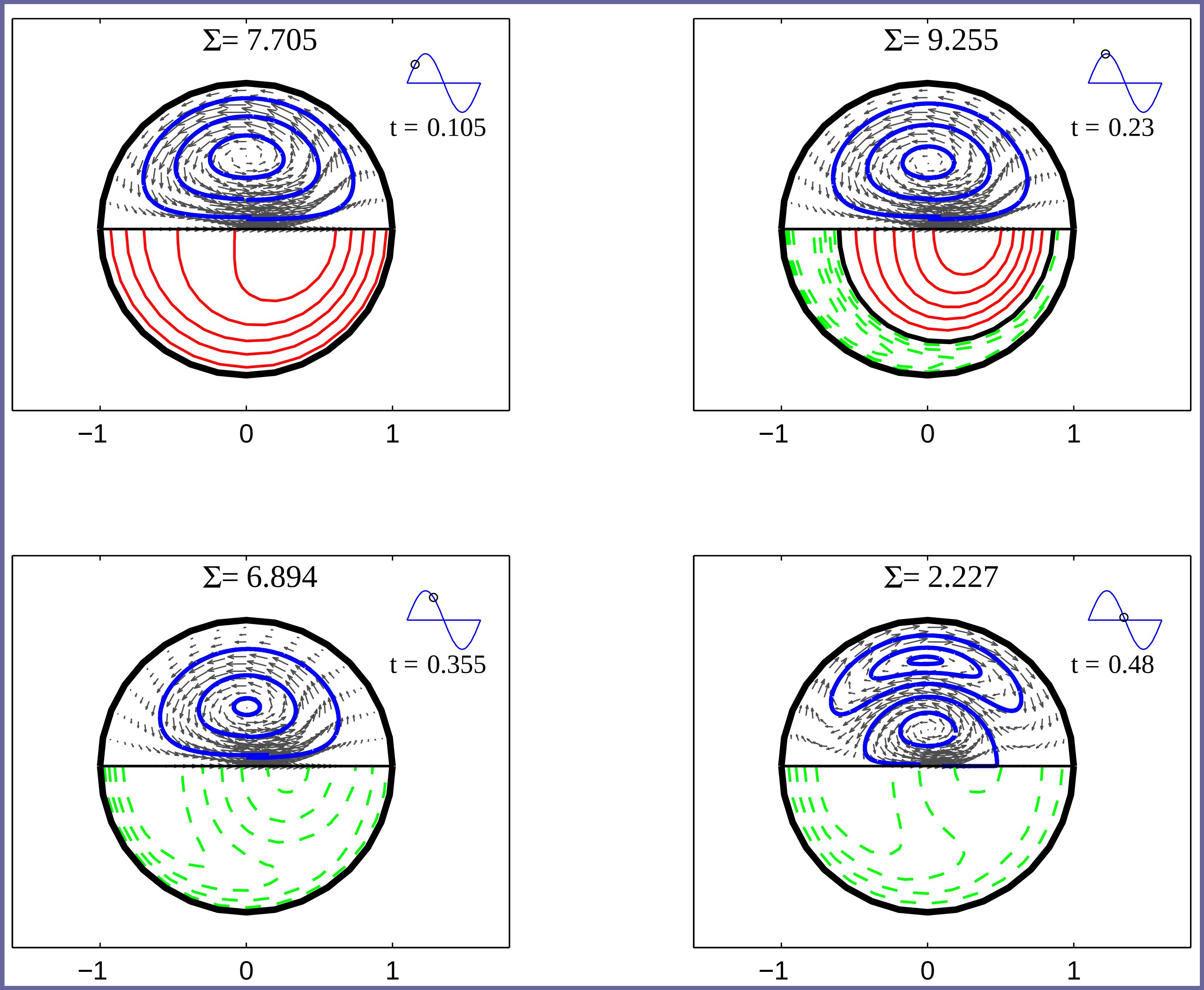
$\varepsilon = 0$ (continuous line), $\varepsilon = 0.05$ (dashed line), $\varepsilon = 0.1$ (dotted line).

Secondary flow at $t = 0, z = 0$: $\tilde{\chi}_0 + \varepsilon\tilde{\chi}_1$

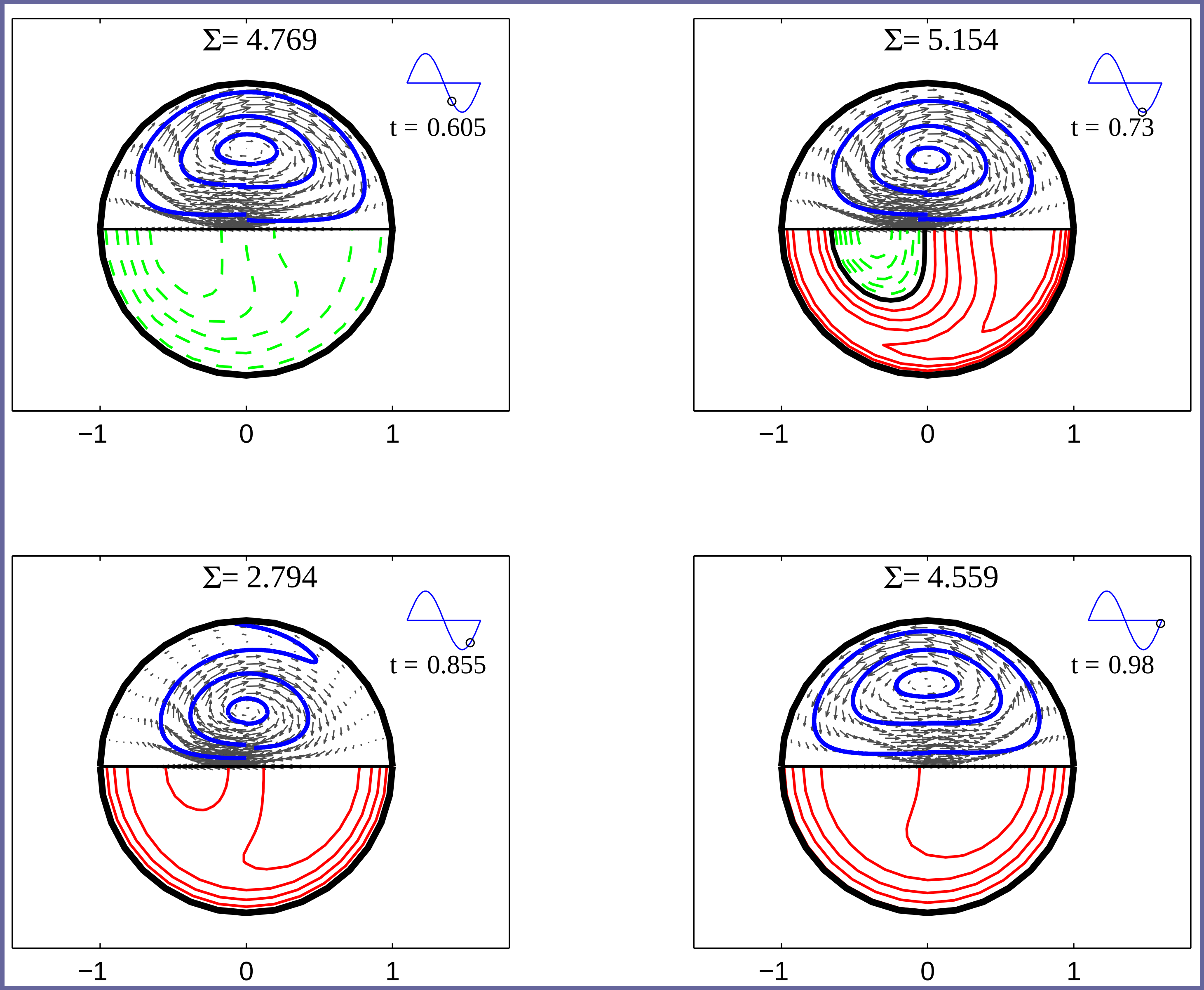


$$\Sigma = \max_{r,\psi} \sqrt{(\mathtt{Re}\tilde{u})^2 + (\mathtt{Re}\tilde{v})^2}.$$

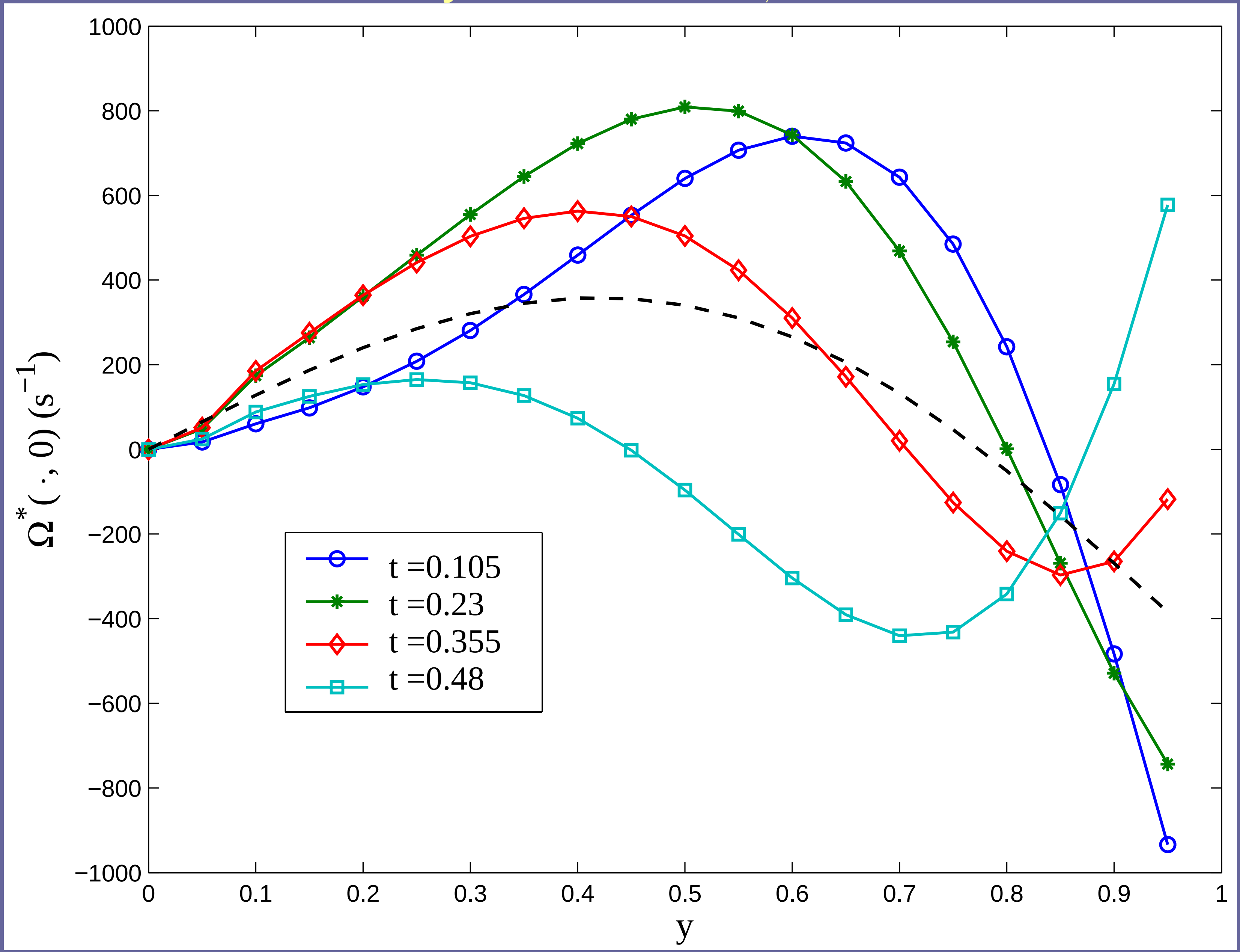
Secondary flow at $z = 0$: $\bar{\chi} + \tilde{\chi} e^{i(\omega t - kz)}, \varepsilon = 0.1$



Secondary flow at $z = 0$: $\bar{\chi} + \tilde{\chi}e^{i(\omega t - kz)}, \varepsilon = 0.1$



Vorticity curves at $z = 0, \varepsilon = 0.1$



Influence of the wall elasticity

–	$E = 5 \cdot 10^5$	$E = 10^7$	$E = 10^9$	$E = 10^{11}$	$E = 10^{13}$
k	0.0422	0.0095	$9.45 \cdot 10^{-4}$	$9.45 \cdot 10^{-5}$	$9.45 \cdot 10^{-6}$
$\max\limits_{\psi} \lvert\tilde{\eta}\rvert$	0.2349	$1.172 \cdot 10^{-2}$	$1.172 \cdot 10^{-4}$	$1.172 \cdot 10^{-6}$	$1.172 \cdot 10^{-8}$
$\max\limits_{\psi} \lvert\tilde{\xi}\rvert$	0.00512	$2.72 \cdot 10^{-4}$	$2.76 \cdot 10^{-6}$	$2.76 \cdot 10^{-8}$	$2.76 \cdot 10^{-10}$
$\max\limits_{\psi} \lvert\tilde{\zeta}\rvert$	1.7836	0.3988	0.0398	$3.98 \cdot 10^{-3}$	$3.98 \cdot 10^{-4}$
$\max\limits_{y,\psi} \lvert\tilde{w}\rvert$	200.64	45.061	4.511	0.451	0.0451
$\max\limits_{y} \lvert\tilde{\Omega}^*(\cdot,0)\rvert$	4234.67	946.80	94.675	9.4671	0.9467
$\max\limits_{\psi} \lvert\tilde{\tau}_\psi\rvert$	16.93	3.787	0.378	0.037	0.0037
$\max\limits_{\psi} \lvert\tilde{\tau}_z\rvert$	58.35	13.125	1.3146	0.1315	0.0131