Contact mechanics in an affine nutshell

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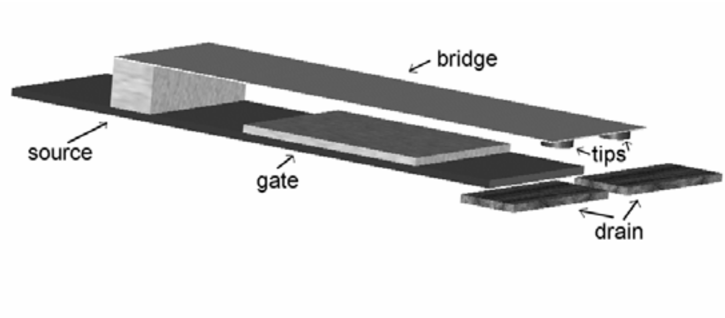
Abstract

A nice tutorial in contact mechanics is readily obtained by considering an incompressible Mooney-Rivlin affine body undergoing short-range interactions with a flat, rigid surface – interactions meant to model impact, damping, friction and adhesion. Numerical integration of a handful of ODE's provides vivid simulations of the motion of an affine body – stiff or soft, heavy or light – that bounces, rocks, rolls, slides or gets stuck on a flat floor or a vertical wall.


bridge
source
gate
tips
drain

Outline

- Rigid body
- Affine body
- Equations of motion
- Piola stress
  - Mooney-Rivlin material
  - Incompressibility
  - Dissipation
- Contact forces constitutive laws
  - Repulsion
  - Damping
  - Friction
  - Adhesion
- Bouncing, sliding, rolling, etc. (numerical simulations)

Rigid body

A motion of the body $\mathcal{B}$ is described at each time t by a placement defined on the *paragon shape* $\mathcal{D}$:

$$p : \mathcal{D} \times \mathcal{I} \to \mathcal{E}$$

characterized by the following representation:

$$p(x,t) = p_o(t) + R(t)(x - x_o)$$

where $R(t) : \mathcal{V} \to \mathcal{V}$ is a rotation in the translation space of $\mathcal{E}$.

Test velocity fields

$$w(x) = w_o + W\,R(t)(x - x_o), \quad \text{with} \quad \operatorname{sym} W = 0$$

Rigid body

Balance principle

$$\int_{\mathcal{D}} b \cdot w \, dV + \int_{\partial\mathcal{D}} q \cdot w \, dA = 0 \qquad \forall w, \, \forall t$$

Equations of motion

$$-m \, \ddot{p}_o(t) - m \, g + f(t) = 0$$

$$\mathrm{skw}\big(- \ddot{R}(t) \, J \, R(t)^T + M(t) \, R(t)^T \big) = 0$$

$$m := \int_{\mathcal{D}} \rho \, dV; \quad J := \int_{\mathcal{D}} \rho (x - x_o) \otimes (x - x_o) \, dV$$

$$f(t) := \sum \int_{\partial\mathcal{D}} q_j(x,t) \, dA; \quad M(t) := \sum \int_{\partial\mathcal{D}} (x - x_o) \otimes q_j(x,t) dA$$

Affine body

A motion of the body $\mathcal{B}$ is described at each time t by a placement defined on the *paragon shape* $\mathcal{D}$:

$$p : \mathcal{D} \times \mathcal{I} \rightarrow \mathcal{E}$$

characterized by the following representation:

$$p(x,t) = p_o(t) + F(t)(x - x_o)$$

where $F(t) : \mathcal{V} \rightarrow \mathcal{V}$ is linear and such that $\det F(t) > 0$

Test velocity fields

$$w(x) = w_o + G\,F(t)(x - x_o)$$

Affine body

Balance principle

$$-S \cdot GF \operatorname{vol} \mathcal{D} + \int_{\mathcal{D}} b \cdot w \, dV + \int_{\partial \mathcal{D}} q \cdot w \, dA = 0 \qquad \forall w, \forall t$$

Equations of motion

$$-m \, \ddot{p}_o(t) - m \, g + f(t) = 0$$

$$-\ddot{F}(t) \, J \, F(t)^T + \left(M(t) - S(t) \operatorname{vol} \mathcal{D} \right) F(t)^T = 0$$

Piola stress and material properties

Frame indifference

$$S \cdot WF = 0 \quad \forall W \mid \operatorname{sym} W = 0 \quad \Rightarrow \quad \operatorname{skw} SF^T = 0$$

Mooney-Rivlin strain energy (incompressible material)

$$\varphi(F) := c_{10}(\imath_1(C) - 3) + c_{01}(\imath_2(C) - 3)$$

Stress response (energetic + reactive + dissipative)

$$SF^T = \widehat{S}(F)F^T - \pi\, I + \mu\, \dot{F}F^{-1}$$

$$\widehat{S}(F) \cdot \dot{F} = d\varphi(F)/dt$$

$$\Rightarrow \quad \widehat{S}(F)\, F^T = 2(c_{10}F\, F^T - c_{01}F^{-T}F^{-1})$$

Dissipation principle

$$S \cdot \dot{F} - d\varphi(F)/dt \geq 0 \quad \Rightarrow \quad \mu \geq 0$$

Contact force constitutive laws

Repulsive force

$$q_r(x,t) = \alpha_r \, d(x,t)^{-\nu_r} \, e_2$$

Damping force

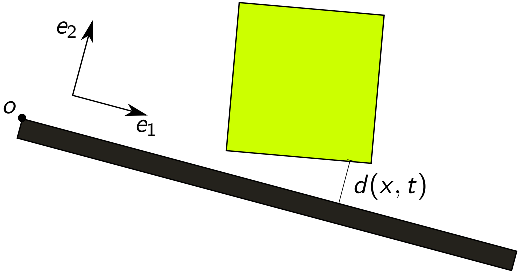
$$q_d(x,t) = -\alpha_d \, d(x,t)^{-\nu_d} \, (\dot{p}(x,t) \cdot e_2) \, e_2$$

Frictional force

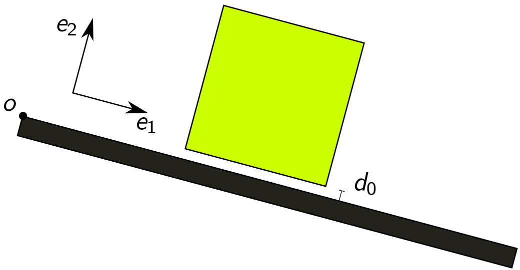
$$q_f(x,t) = -\alpha_f \, d(x,t)^{-\nu_f} \, (\dot{p}(x,t) \cdot e_1) \, e_1$$

Adhesive force

$$q_a(x,t) = -\alpha_a \, d(x,t)^{-\nu_a} \, e_2$$



$$d(x,t) := ((p(x,t) - o) \cdot e_2)$$



$$d(x,t) := ((p(x,t) - o) \cdot e_2)\, e_2$$

Contact force working

The contact forces are surface forces per unit deformed area:

$$\int_{\partial \mathcal{D}} q(x,t) \cdot w(x,t)\; k(x,t)\, dA$$

Area change factor:

$$k(x,t) := \|F(t)^{-T} n(x)\| \det(F(t))$$

$n(x)$ outward unit normal vector on $\partial \mathcal{D}$

Numerical simulations

rigid block; contact: repulsion, damping, no friction

001 | 002 | 003

rigid block; contact: repulsion, damping, friction

011 | 012 | 013

rigid block on a sloping plane

021 | 022 | 023

rigid disk

031 | 032 | 033 | 034 | 035

affine disk

041 | 042

Appendix

Cauchy stress

$$T = SF^T \frac{1}{\det F}$$

Pressure π

It is the *reactive* part of T. In an incompressible solid/fluid the velocity fields are said to be *isochoric*. The trace of the velocity gradient turns out to be zero.
A reactive stress, whose power is zero for any isochoric velocity field, has to be a spherical tensor $-\pi\, I$:

$$\pi\, I \cdot G = \pi \operatorname{tr} G = 0$$

Appendix

Mooney-Rivlin

It is a hyperelastic material model used for rubber-like materials as well as for biological tissues.
The *principal invariants* of $C := FF^T$ are defined as

$$\imath_1(C) := F \cdot F, \quad \imath_2(C) := F^\star \cdot F^\star$$

where $F^\star := F^{-T} \det F$ is the *cofactor* of F.

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