Evolution of spherical aneurysms

Stress-driven remodeling and control laws

A. Tatone

July 13, 2007

People involved in this project

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Background references

- A. Di Carlo, S. Quiligotti, Growth and balance, *Mechanics Research Communications*, 29, 449–456, 2002.
- A. Di Carlo, Surface and bulk growth unified, *Mechanics of Material Forces* (P. Steinmann & G.A. Maugin, eds.), 53–64, Springer, New York, NY, 2005.
- M. Tringelová, P. Nardinocchi, L. Teresi and A. DiCarlo, The cardiovascular system as a smart system, in *Topics on Mathematics for Smart Systems*, eds. V. Valente and B. Miara (World Scientific, Singapore, 2007).

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1. A possible natural history for saccular aneurysms
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   - Working & balance
   - Energetics & constitutive issues

3. Saccular aneurysms
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   - Working & balance
   - Energetics & constitutive issues

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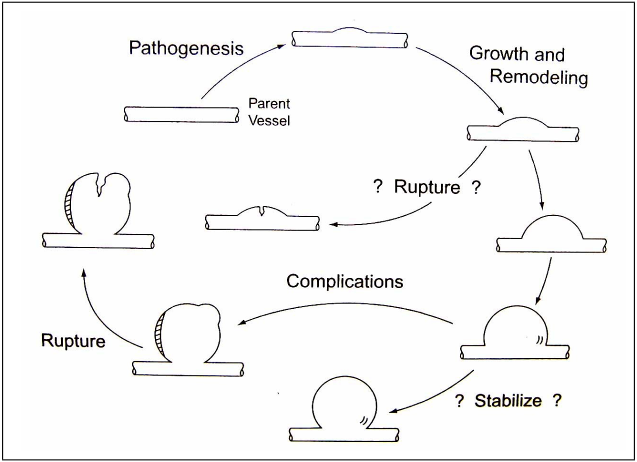
Part I

Aneurysms

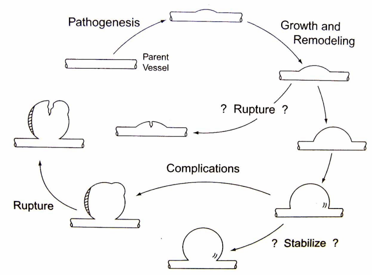
A possible natural history for saccular aneurysms

Intracranial saccular aneurysms are dilatations of the arterial wall.



[J.D.Humphrey, Cardiovascular Solid Mechanics, 2001]

A possible natural history for saccular aneurysms



- An initial insult may cause a local weakening of the wall and thus a mild dilatation.
- This raises the local stress field above normal values, thus setting into motion a growth and remodeling process that attempts to reduce the stress toward values that are homeostatic for the parent vessel.
- If degradation and deposition of collagen are well balanced, this could produce a larger, but stable lesion.
- If degradation exceedes deposition at any time, this could yield a

A recent survey in Japan

EBM of Neurosurgical Disease in Japan

Table 2 Summary of patients with ruptured aneurysms

Case No.	Sex	Age		Location of aneurysm	Motive for examination	Past history	Size of aneurysm (mm) 6 months	12 months	24 months	At rupture	Treatment
1	F	53	single	rt ICA	ex. for intracranial disease	hypertension	2.9	2.9	2.9	5.7	embolization (27 months)
2	F	71	multiple	AcomA	ex. for intracranial disease	hypertension, pituitary adenoma	4.9	4.9	4.9	5.9	clipping (7 months)
3	F	77	multiple	lt MCA	ex. for anxiety	hypertension, heart disease	4.5	4.5		4.5	dead (7 months)
4	F	42	multiple	lt MCA	ex. for anxiety	hypertension	4	4		7.0	clipping (18 months)

AcomA: anterior communicating artery, ex.: examination, ICA: internal carotid artery, MCA: middle cerebral artery.

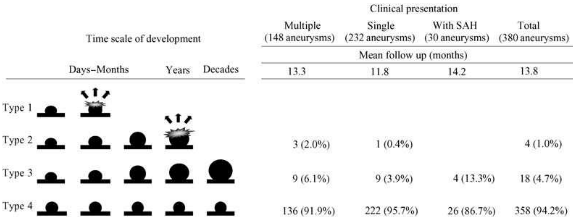
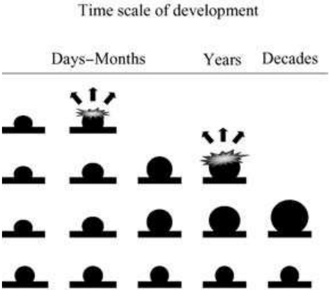


Fig. 1 Process of growth and rupture of aneurysms. Type 1: aneurysm ruptures within a time span as short as several days to several months after formation, Type 2: aneurysm builds up slowly for a few years after formation and ruptures in this process, Type 3: aneurysm keeps growing slowly for many years without rupturing, Type 4: aneurysm grows up to a certain size, probably under 5

A recent survey in Japan



Multiple (148 aneurysms)	Single (232 aneurysms)	With SAH (30 aneurysms)	Total (380 aneurysms)
Mean follow up (months)			
13.3	11.8	14.2	13.8
3 (2.0%)	1 (0.4%)		4 (1.0%)
9 (6.1%)	9 (3.9%)	4 (13.3%)	18 (4.7%)
136 (91.9%)	222 (95.7%)	26 (86.7%)	358 (94.2%)

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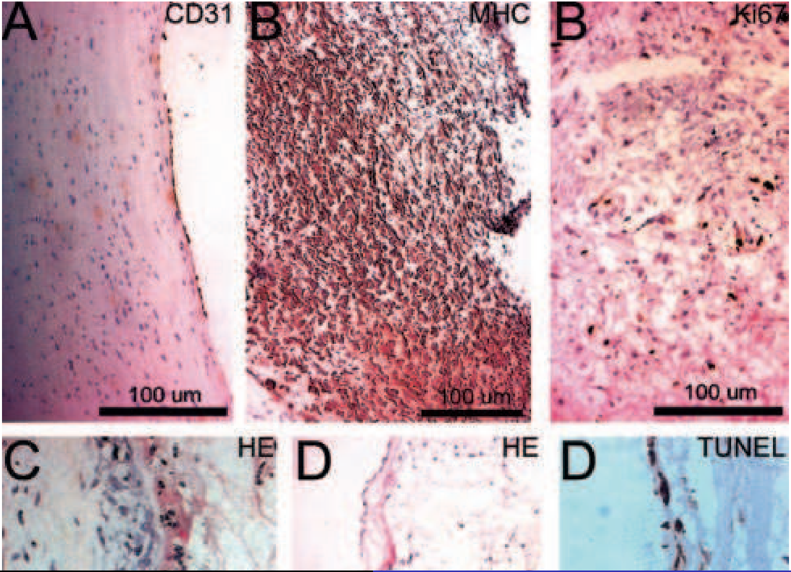
Stroke

JOURNAL OF THE AMERICAN HEART ASSOCIATION

American Stroke Association[SM]
A Division of American Heart Association

Remodeling of Saccular Cerebral Artery Aneurysm Wall Is Associated With Rupture: Histological Analysis of 24 Unruptured and 42 Ruptured Cases
Juhana Frösen, Anna Piippo, Anders Paetau, Marko Kangasniemi, Mika Niemelä, Juha Hernesniemi and Juha Jääskeläinen
Stroke 2004;35;2287-2293; originally published online Aug 19, 2004;


A
CD31
100 um
B
MHC
100 um
B
Ki67
100 um
C
HE
D
HE
D
TUNEL

Before rupture, the wall of saccular cerebral artery aneurysms undergoes morphological changes associated with remodeling of the aneurysm wall. Some of these changes, like SMC [smooth muscle cell] proliferation and macrophage infiltration, likely reflect ongoing repair attempts that could be enhanced with pharmacological therapy.

The morphological changes that result from the MH [myointimal hyperplasia] and matrix destruction are collectively referred to as remodeling of the vascular wall. Although MH is an adaptation mechanism of arteries to hemodynamic stress, in SAH [subarachnoid hemorrhage] patients, for undefined reasons, vascular wall remodeling [is] insufficient to prevent SCAA [saccular cerebral artery aneurysm] rupture.

Part II

Mechanical Model

Kinematics: gross and refined motions

gross placement

$$p : \mathscr{B} \to \mathscr{E}$$

body gradient

$$\nabla p|_b : \mathrm{T}_b\mathscr{B} \to \mathrm{V}\mathscr{E}$$

element-wise configuration (*prototype*)

$$\mathbb{P}|_b : \mathrm{T}\mathscr{B}|_b \to \mathrm{V}\mathscr{E},$$

Kinematics: gross and refined motions

refined motion

$$(p(\tau), \mathbb{P}(\tau)) : \mathscr{B} \to \mathscr{E} \times (\mathrm{V}\mathscr{E} \otimes \mathrm{V}\mathscr{E})$$

realized velocity

$$(\dot{p}(\tau), \dot{\mathbb{P}}(\tau)\mathbb{P}(\tau)^{-1}) : \mathscr{B} \to \mathrm{V}\mathscr{E} \times (\mathrm{V}\mathscr{E} \otimes \mathrm{V}\mathscr{E}) \,.$$

test velocities

$$(\mathrm{v}, \mathbb{V}) : \mathscr{B} \to \mathrm{V}\mathscr{E} \times (\mathrm{V}\mathscr{E} \otimes \mathrm{V}\mathscr{E})$$

(*gross velocity*, *growth velocity*)

Working

total working

$$\int_{\mathscr{B}} \left(\mathrm{A}^{\mathfrak{i}} \cdot \mathbb{V} - \mathrm{S} \cdot \mathrm{Dv} \right) + \int_{\mathscr{B}} \left(\mathrm{b} \cdot \mathrm{v} + \mathrm{A}^{\mathfrak{o}} \cdot \mathbb{V} \right) + \int_{\partial\mathscr{B}} \mathrm{t}_{\partial\mathscr{B}} \cdot \mathrm{v}$$

(integrals taken with respect to *prototypal volume* and *prototypal area*)

prototypal gradient

$$\mathrm{Dv} := (\nabla \mathrm{v})\mathbb{P}^{-1}$$

Balance principle

$$\int_{\mathscr{B}} \left(\mathrm{A}^{\mathfrak{i}} \cdot \mathbb{V} - \mathrm{S} \cdot \mathrm{Dv} \right) + \int_{\mathscr{B}} (\mathrm{b} \cdot \mathrm{v} + \mathrm{A}^{\mathfrak{o}} \cdot \mathbb{V}) + \int_{\partial\mathscr{B}} \mathrm{t}_{\partial\mathscr{B}} \cdot \mathrm{v} = 0$$

balance of brute forces

$$\mathrm{Div}\,\mathrm{S} + \mathrm{b} = 0 \quad \text{on } \mathscr{B}$$

$$\mathrm{S}\,\mathrm{n}_{\partial\mathscr{B}} = \mathrm{t}_{\partial\mathscr{B}} \quad \text{on } \partial\mathscr{B}$$

balance of accretive couples

$$\mathrm{A}^{\mathfrak{i}} + \mathrm{A}^{\mathfrak{o}} = 0 \quad \text{on } \mathscr{B}$$

Energetics

free energy

$$\Psi(\mathscr{P}) = \int_{\mathscr{P}} \psi$$

(ψ free energy per unit prototypal volume)

Constitutive issues: theory and recipes

The constitutive theory of inner forces rests on two main pillars, altogether independent of balance:

- *the principle of material indifference to change in observer*
- *the dissipation principle*

Both of them deliver strict selection rules on admissible constitutive recipes for the inner force. None of them applies to the outer force, which has to be regarded as an adjustable control on the motion.

Constitutive issues: theory and recipes

$$\int_{\mathscr{B}} \left(\mathrm{A}^{\mathfrak{i}} \cdot \mathbb{V} - \mathrm{S} \cdot \mathrm{Dv} \right) + \int_{\mathscr{B}} (\, \mathrm{b} \cdot \mathrm{v} + \mathrm{A}^{\mathfrak{o}} \cdot \mathbb{V} \,) + \int_{\partial \mathscr{B}} \mathrm{t}_{\partial \mathscr{B}} \cdot \mathrm{v} = 0$$

In our theory of the biomechanics of growth, the outer accretive couple $\mathrm{A}^{\mathfrak{o}}$ plays a primary role, representing the mechanical effects of the biochemical control system, smartly distributed within the body itself: ignoring the chemical degrees of freedom does not make negligible their feedback on mechanics.

Material indifference to change in observer

Change in observer

$$\widetilde{p}(b,\tau) = \widetilde{\mathrm{x}}_{\mathrm{o}}(\tau) + \widetilde{\mathrm{Q}}(\tau)\,(\,p\,(b,\tau) - x_{\mathrm{o}}(\tau))$$
$$\widetilde{\mathbb{P}}(b,\tau) = \mathbb{P}(b,\tau)$$

$$\widetilde{\mathrm{v}}(b) = \widetilde{\mathrm{Q}}(\tau)\,\mathrm{v}(b) + \widetilde{\mathrm{w}}(\tau) + \widetilde{\mathrm{W}}(\tau)\,(\,\widetilde{p}\,(b,\tau) - \widetilde{\mathrm{x}}_{\mathrm{o}}(\tau))$$
$$\widetilde{\mathbb{V}}(b) = \mathbb{V}(b)$$

Material indifference to change in observer

The working expended over each body-part on each test velocity $(\mathrm{v}, \mathbb{V})$ by the inner force constitutively related to each refined motion $(p, \mathbb{P})$ should be invariant under all change in observer.

The free energy Ψ should be constitutively prescribed in such a way as to be invariant under all change in observer.

Material indifference to change in observer

brute Cauchy stress

$$\mathrm{T} := (\det \mathrm{F})^{-1}\, \mathrm{S}\, \mathrm{F}^\top$$

$$\mathrm{T}^\top = \mathrm{T}$$

warp

$$\mathrm{F} := \mathrm{D}p = (\nabla p)\, \mathbb{P}^{-1}$$

(measures how the body gradient of the gross placement differs from the prototypal stance)

Material indifference to change in observer

If we further assume that the response of the body element at b filters off from $(p, \mathbb{P})$ all information other than

$$p|_b, \ \nabla p|_b, \ \mathbb{P}|_b$$

we obtain the following *reduction theorem*: there are constitutive mappings $\widehat{\mathrm{S}}_b$, $\widehat{\mathrm{A}}^{\mathrm{i}}_b$ and $\widehat{\psi}_b$ such that

$$\begin{aligned} \mathrm{S}(b,\tau) &= \mathrm{R}(b,\tau)\,\widehat{\mathrm{S}}_b(\ell_b,\tau) \\ \mathrm{A}^{\mathrm{i}}(b,\tau) &= \widehat{\mathrm{A}}^{\mathrm{i}}_b(\ell_b,\tau) \\ \psi(b,\tau) &= \widehat{\psi}_b(\ell_b,\tau) \end{aligned}$$

$$\begin{aligned} \ell_b &:= (\,\mathrm{U}|_b, \mathbb{P}|_b) \\ \mathrm{F} &= \mathrm{R}\,\mathrm{U} \end{aligned}$$

Dissipation principle

$$(\mathrm{S}\cdot(\mathrm{D}\,\dot{p}) - \mathrm{A}^{\mathrm{i}}\cdot(\dot{\mathbb{P}}\mathbb{P}^{-1})) - (\dot{\psi} + \psi\,\mathrm{I}\cdot(\dot{\mathbb{P}}\mathbb{P}^{-1})) \geq 0$$

power expended : $-${working expended by the inner force constitutively related to the motion on the velocity realized along the motion}

power dissipated : {*power expended* along a refined motion} $-${time derivative of the *free energy* along that motion}

Dissipation principle : *the power dissipated should be non-negative, for all body-parts, at all times*

Time derivative of the free energy

$$\dot{\Psi}(\mathscr{P}) = \Big(\int_{\mathscr{P}} \psi\,\omega\Big)^{\cdot} = \int_{\mathscr{P}} (\psi\,\omega)^{\cdot}$$
$$= \int_{\mathscr{P}} (\dot{\psi}\,\omega + \psi\,\dot{\omega})$$
$$= \int_{\mathscr{P}} \Big(\dot{\psi} + \psi\,\mathrm{I}\cdot(\dot{\mathbb{P}}\mathbb{P}^{-1})\Big)\,\omega.$$

(the prototypal-volume form ω evolves in time with $\dot{\mathbb{P}}\mathbb{P}^{-1}$)

Constitutive assumptions: free energy and inner force

$$\psi(b,\tau) = \widehat{\psi}_b(\mathrm{U}|_b, \mathbb{P}|_b, \tau) = \phi_b(\mathrm{F}(b,\tau))$$

The dissipation principle is fulfilled if and only if for each b the mappings $\widehat{\mathrm{S}}$ and $\widehat{\mathrm{A}}^{\mathrm{i}}$ satisfy

$$\widehat{\mathrm{S}} = \partial\phi + \overset{+}{\mathrm{S}}\,, \qquad \widehat{\mathrm{A}}^{\mathrm{i}} = \mathbb{E} + \overset{+}{\mathrm{A}}$$

together with the *reduced dissipation inequality*

$$\overset{+}{\mathrm{A}}(\ell,\tau)\cdot(\dot{\mathbb{P}}(\tau)\mathbb{P}(\tau)^{-1}) - \overset{+}{\mathrm{S}}(\ell,\tau)\cdot\dot{\mathrm{F}}(\tau) \leq 0$$

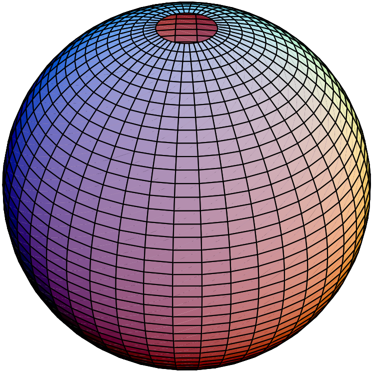
Eshelby couple

$$\mathbb{E} := \mathrm{F}^{\top}\widehat{\mathrm{S}} - \phi\,\mathrm{I}$$

extra-energetic responses

$$\overset{+}{\mathrm{S}},\ \overset{+}{\mathrm{A}}$$

Saccular aneurysms



Geometry & kinematics

paragon shape $\mathcal{D}$ of $\mathscr{B}$ centered at

$$\mathsf{x}_\mathsf{o} \in \mathscr{E}$$

spherical coordinates

$$(\widehat{\xi}(x), \widehat{\vartheta}(x), \widehat{\varphi}(x))$$

radius of x

$$\widehat{\xi}(x) = \|x - \mathsf{x}_\mathsf{o}\|$$

gross placement

$$\begin{aligned} \mathsf{p} : \mathcal{D} &\rightarrow \mathscr{E} \\ x &\mapsto \mathsf{x}_\mathsf{o} + \rho\,(\widehat{\xi}(x))\,\mathrm{e}_\mathsf{r}(\widehat{\vartheta}(x), \widehat{\varphi}(x)) \end{aligned}$$

actual radius

$$\rho : [\xi_-, \xi_+] \rightarrow \mathbb{R}$$

Geometry & kinematics

spherically symmetric vector fields

$$\mathrm{v} : \mathcal{D} \to \mathrm{V}\mathscr{E}$$

radial component of v

$$v : [\xi_-, \xi_+] \to \mathbb{R}$$

$$\mathrm{v}(x) = v(\xi)\,\mathrm{e}_\mathsf{r}(\vartheta, \varphi)\,.$$

Geometry & kinematics

spherically symmetric tensor fields

$$\mathscr{L}_\varepsilon : \mathcal{D} \rightarrow \mathrm{V}\mathscr{E} \otimes \mathrm{V}\mathscr{E}$$

$$\mathscr{L}_\varepsilon(x) = \mathrm{L}_\mathsf{r}(\xi)\,\mathsf{P}_\mathsf{r}(\vartheta, \varphi) + \mathrm{L}_\mathsf{h}(\xi)\,\mathsf{P}_\mathsf{h}(\vartheta, \varphi)$$

orthogonal projectors

$$\mathsf{P}_\mathsf{r}(x) := \mathrm{e}_\mathsf{r}(\vartheta, \varphi) \otimes \mathrm{e}_\mathsf{r}(\vartheta, \varphi)$$
$$\mathsf{P}_\mathsf{h}(x) := \mathsf{I} - \mathsf{P}_\mathsf{r}(x)$$

Refined motion

gradient of the gross placement

$$\nabla \mathsf{p}|_x = \rho'(\xi)\,\mathsf{P}_\mathsf{r}(\vartheta,\varphi) + \frac{\rho(\xi)}{\xi}\,\mathsf{P}_\mathsf{h}(\vartheta,\varphi)\,,$$

prototype

$$\mathbb{P}(x,\tau) = \alpha_\mathsf{r}(\xi,\tau)\,\mathsf{P}_\mathsf{r}(\vartheta,\varphi) + \alpha_\mathsf{h}(\xi,\tau)\,\mathsf{P}_\mathsf{h}(\vartheta,\varphi)\,.$$

warp (*Kröner-Lee decomposition*)

$$\mathrm{F} := (\nabla \mathsf{p})\,\mathbb{P}^{-1} = \lambda_\mathsf{r}\,\mathsf{P}_\mathsf{r} + \lambda_\mathsf{h}\mathsf{P}_\mathsf{h}$$

$$\lambda_\mathsf{r}(\xi,\tau) = \frac{\rho'(\xi,\tau)}{\alpha_\mathsf{r}(\xi,\tau)}\,, \qquad \lambda_\mathsf{h}(\xi,\tau) = \frac{\rho(\xi,\tau)}{\xi\alpha_\mathsf{h}(\xi,\tau)}\,.$$

Refined motion

gross velocity and *growth velocity*

$$\dot{\mathsf{p}}(x,\tau) \;=\; \dot{\rho}(\xi,\tau)\,\mathrm{e}_{\mathsf{r}}(\vartheta,\varphi)$$

$$\dot{\mathbb{P}}\,\mathbb{P}^{-1}(x,\tau) \;=\; \frac{\dot{\alpha}_{\mathsf{r}}(\xi,\tau)}{\alpha_{\mathsf{r}}(\xi,\tau)}\,\mathsf{P}_{\mathsf{r}}(\vartheta,\varphi) + \frac{\dot{\alpha}_{\mathsf{h}}(\xi,\tau)}{\alpha_{\mathsf{h}}(\xi,\tau)}\,\mathsf{P}_{\mathsf{h}}(\vartheta,\varphi)$$

Dynamics: brute and accretive forces; balance principle

working

$$\int_{\mathcal{D}} \Big(\mathbb{A}^{\mathfrak{i}} \cdot \mathbb{V} - \mathrm{S} \cdot \nabla \mathrm{v} \Big) + \int_{\mathcal{D}} \mathbb{A}^{\mathfrak{o}} \cdot \mathbb{V} + \int_{\partial\mathcal{D}} \mathrm{t}_{\partial\mathcal{D}} \cdot \mathrm{v} \,,$$

(integrals taken with respect to the *paragon volume* and *paragon area*)

$$\int_{\xi_-}^{\xi_+} \Big(\mathrm{A}_{\mathsf{r}} \mathrm{V}_{\mathsf{r}} + 2\, \mathrm{A}_{\mathsf{h}} \mathrm{V}_{\mathsf{h}} - \mathrm{S}_{\mathsf{r}}\, v' - 2\, \mathrm{S}_{\mathsf{h}}\, v/\xi \Big)\, 4\,\pi\,\xi^2 d\xi + \Big(4\,\pi\,\xi^2\, t\, v \Big)\Big|_{\xi_\mp}$$

$$\mathbb{A} := \mathbb{A}^{\mathfrak{i}} + \mathbb{A}^{\mathfrak{o}} = \mathrm{A}_{\mathsf{r}}\, \mathsf{P}_{\mathsf{r}} + \mathrm{A}_{\mathsf{h}}\, \mathsf{P}_{\mathsf{h}} \,.$$

Balance laws

$$\left.\begin{array}{rcl} 2\big(\mathrm{S}_\mathsf{r}(\xi)-\mathrm{S}_\mathsf{h}(\xi)\big)+\xi\,\mathrm{S}_\mathsf{r}'(\xi) & = & 0 \\ \mathrm{A}_\mathsf{r}(\xi)=\mathrm{A}_\mathsf{h}(\xi) & = & 0 \end{array}\right\}\quad (\xi_- < \xi < \xi_+)$$

$$\mp\mathrm{S}_\mathsf{r}(\xi_\mp) = t_\mp$$

Energetics

$$\Psi(\mathscr{P}) = \int_{\mathscr{P}} \mathrm{J}\,\psi$$

(ψ free energy per unit *prototypal* volume)

$$\mathrm{J} := \det(\mathbb{P}) = \alpha_{\mathsf{r}}\,\alpha_{\mathsf{h}}^2 > 0$$

($\mathrm{J}\,\psi$ free energy per unit *paragon* volume)

Dissipation principle

$$\mathrm{S} \cdot (\nabla \dot{\mathsf{p}}) - \mathbb{A}^{\mathsf{i}} \cdot (\dot{\mathbb{P}}\, \mathbb{P}^{-1}) - (\mathrm{J}\,\psi)^{\cdot} \geq 0$$

$$\psi(x,\tau) = \phi\,(\,\lambda_{\mathsf{r}}(\xi,\tau), \lambda_{\mathsf{h}}(\xi,\tau); \xi\,)$$

$$\mathrm{S}_{\mathsf{r}} = \mathrm{J}\,\phi_{,\mathsf{r}}/\alpha_{\mathsf{r}} + \overset{+}{\mathrm{S}}_{\mathsf{r}} \qquad \mathrm{S}_{\mathsf{h}} = \mathrm{J}\,\phi_{,\mathsf{h}}/\alpha_{\mathsf{h}} + \overset{+}{\mathrm{S}}_{\mathsf{h}}$$

$$\mathrm{A}^{\mathsf{i}}_{\mathsf{r}} = \mathrm{J}\,[\,\mathrm{S}_{\mathsf{r}}\,\alpha_{\mathsf{r}}\,\lambda_{\mathsf{r}}/\mathrm{J} - \phi\,] + \overset{+}{\mathrm{A}}_{\mathsf{r}} \qquad \mathrm{A}^{\mathsf{i}}_{\mathsf{h}} = \mathrm{J}\,[\,\mathrm{S}_{\mathsf{h}}\,\alpha_{\mathsf{h}}\,\lambda_{\mathsf{h}}/\mathrm{J} - \phi\,] + \overset{+}{\mathrm{A}}_{\mathsf{h}}$$

reduced dissipation inequality

$$\overset{+}{\mathrm{S}}_{\mathsf{r}}\,\alpha_{\mathsf{r}}\,\dot{\lambda}_{\mathsf{r}} + 2\overset{+}{\mathrm{S}}_{\mathsf{h}}\,\alpha_{\mathsf{h}}\,\dot{\lambda}_{\mathsf{h}} - \overset{+}{\mathrm{A}}_{\mathsf{r}}\,\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} - 2\overset{+}{\mathrm{A}}_{\mathsf{h}}\,\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} \geq 0$$

Constitutive prescriptions

$$\overset{+}{\mathrm{S}}_{\mathsf{r}}\,\alpha_{\mathsf{r}}\,\dot{\lambda}_{\mathsf{r}} + 2\overset{+}{\mathrm{S}}_{\mathsf{h}}\,\alpha_{\mathsf{h}}\,\dot{\lambda}_{\mathsf{h}} - \overset{+}{\mathrm{A}}_{\mathsf{r}}\,\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} - 2\overset{+}{\mathrm{A}}_{\mathsf{h}}\,\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} \geq 0$$

$$\overset{+}{\mathrm{S}}_{\mathsf{r}} = \overset{+}{\mathrm{S}}_{\mathsf{h}} = 0$$

$$\overset{+}{\mathrm{A}}_{\mathsf{r}} = -\mathrm{J}\,D_{\mathsf{r}}\,\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}}\,, \qquad \overset{+}{\mathrm{A}}_{\mathsf{h}} = -\mathrm{J}\,D_{\mathsf{h}}\,\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}}$$

$$D_{\mathsf{r}} > 0\,,\ D_{\mathsf{h}} > 0$$

Incompressible elasticity

incompressibility

$$\det \mathrm{F} = \lambda_{\mathsf{r}}\,\lambda_{\mathsf{h}}^2 = 1 \quad \Longleftrightarrow \quad \lambda_{\mathsf{r}} = 1/\lambda_{\mathsf{h}}^2\,.$$

reactive inner force

$$\overset{\bowtie}{\mathrm{S}} = \mathrm{J}\,\overset{\bowtie}{\pi}\left(\frac{1}{\alpha_{\mathsf{r}}\,\lambda_{\mathsf{r}}}\,\mathsf{P}_{\mathsf{r}} + \frac{1}{\alpha_{\mathsf{h}}\,\lambda_{\mathsf{h}}}\,\mathsf{P}_{\mathsf{h}}\right), \qquad \overset{\bowtie}{\mathbb{A}} = \mathrm{J}\,\overset{\bowtie}{\pi}\,\mathsf{I}$$

free-energy restriction

$$\widetilde{\phi} : \lambda \mapsto \phi\,(1/\lambda^2,\,\lambda\,)$$

Incompressible elasticity

active and reactive components

$$\mathrm{S}_\mathsf{r} = \frac{\mathrm{J}}{\alpha_\mathsf{r}\,\lambda_\mathsf{r}}\left(\overset{\bowtie}{\pi} - (\lambda_\mathsf{h}/3)\,\widetilde{\phi}'\right) \qquad \mathrm{S}_\mathsf{h} = \frac{\mathrm{J}}{\alpha_\mathsf{h}\,\lambda_\mathsf{h}}\left(\overset{\bowtie}{\pi} + (\lambda_\mathsf{h}/6)\,\widetilde{\phi}'\right)$$

$$\mathrm{A}^i_\mathsf{r} = \mathrm{J}\left(\mathrm{T}_\mathsf{r} - \widetilde{\phi} - D_\mathsf{r}\,\dot{\alpha}_\mathsf{r}/\alpha_\mathsf{r}\right) \qquad \mathrm{A}^i_\mathsf{h} = \mathrm{J}\left(\mathrm{T}_\mathsf{h} - \widetilde{\phi} - D_\mathsf{h}\,\dot{\alpha}_\mathsf{h}/\alpha_\mathsf{h}\right)$$

Cauchy stress

$$\mathrm{T} = (\mathrm{J}\,\det(\mathrm{F}))^{-1}\,\mathrm{S}\,\mathbb{P}^\top\mathrm{F}^\top$$

radial and *hoop* components

$$\mathrm{T}_\mathsf{r} = \mathrm{J}^{-1}\mathrm{S}_\mathsf{r}\,\alpha_\mathsf{r}\,\lambda_\mathsf{r}$$

$$\mathrm{T}_\mathsf{h} = \mathrm{J}^{-1}\mathrm{S}_\mathsf{h}\,\alpha_\mathsf{h}\,\lambda_\mathsf{h}$$

Fung strain energy

$$\widetilde{\phi}(\lambda) = (c/\delta)\, \exp\Big((\Gamma/2)\,(\lambda^2 - 1)^2\Big)\,,$$

$$c := 0.88\; N/m$$

$$\delta := 27.8 \times 10^{-6}\; m$$

$$\Gamma := 12.99$$

Evolution laws

$$D_{\mathsf{r}}\,\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} = \left(\mathrm{T}_{\mathsf{r}} - \widetilde{\phi}\right) + \mathrm{A}^{\mathfrak{o}}_{\mathsf{r}}/\mathrm{J}$$
$$D_{\mathsf{h}}\,\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} = \left(\mathrm{T}_{\mathsf{h}} - \widetilde{\phi}\right) + \mathrm{A}^{\mathfrak{o}}_{\mathsf{h}}/\mathrm{J}$$

$$2\left(\mathrm{S}_{\mathsf{r}} - \mathrm{S}_{\mathsf{h}}\right) + \xi\,\mathrm{S}'_{\mathsf{r}} = 0$$

Part III

Numerical simulations

Growth driven by mean hoop stress value

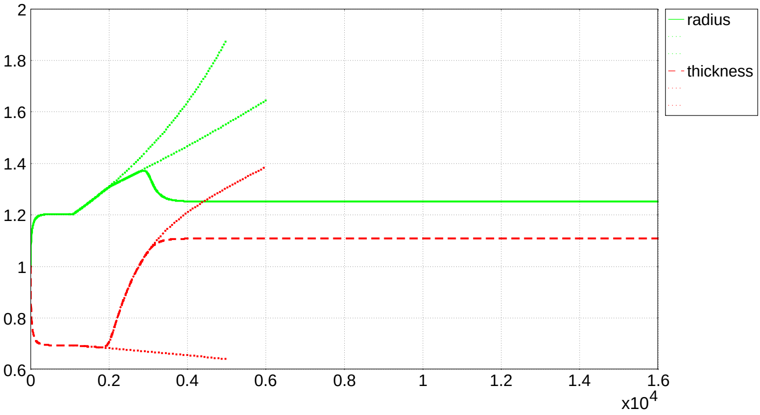
Outer accretive couple

$$\mathrm{A}_{\mathsf{r}}^{o} = \mathrm{J}\left(G_{\mathsf{r}}\,(\mathrm{T}_{\mathsf{h}}^{m} - \mathrm{T}^{\odot}) - \mathrm{T}_{\mathsf{r}} + \widetilde{\phi}\right)$$
$$\mathrm{A}_{\mathsf{h}}^{o} = \mathrm{J}\left(G_{\mathsf{h}}\,(\mathrm{T}^{\odot} - \mathrm{T}_{\mathsf{h}}^{m}) - \mathrm{T}_{\mathsf{h}} + \widetilde{\phi}\right)$$

Evolution laws

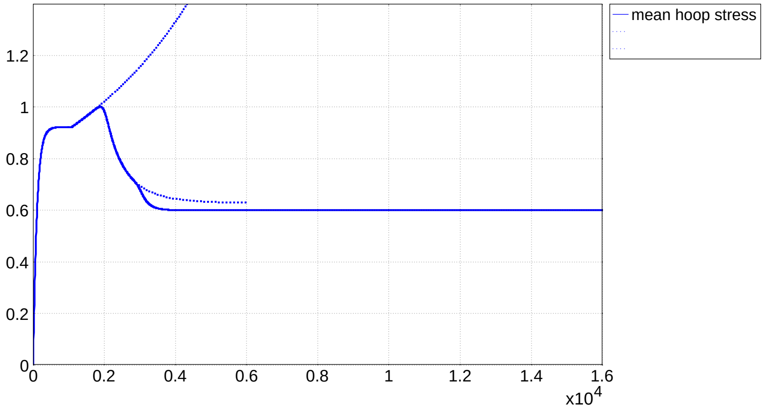
$$\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} = (G_{\mathsf{r}}/D_{\mathsf{r}})\,(\mathrm{T}_{\mathsf{h}}^{m} - \mathrm{T}^{\odot})$$
$$\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} = (G_{\mathsf{h}}/D_{\mathsf{h}})\,(\mathrm{T}^{\odot} - \mathrm{T}_{\mathsf{h}}^{m})$$

($\mathrm{T}_{\mathsf{h}}^{m}$ *mean hoop stress*, $\mathrm{T}^{\odot}$ *target hoop stress*)



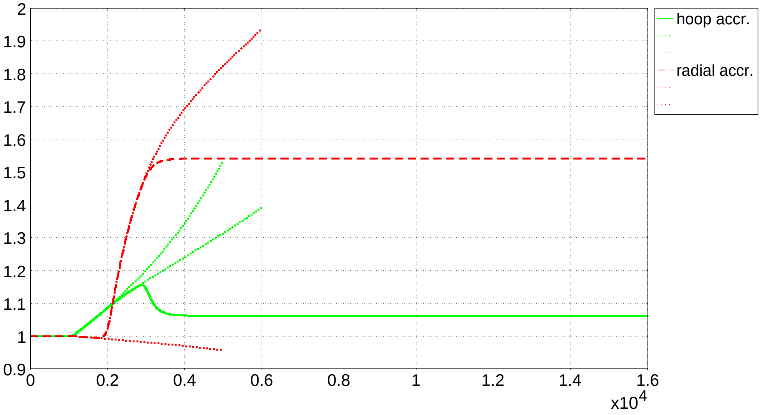
G_r / D_r	2000
G_h / D_h	5000
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-15-k]



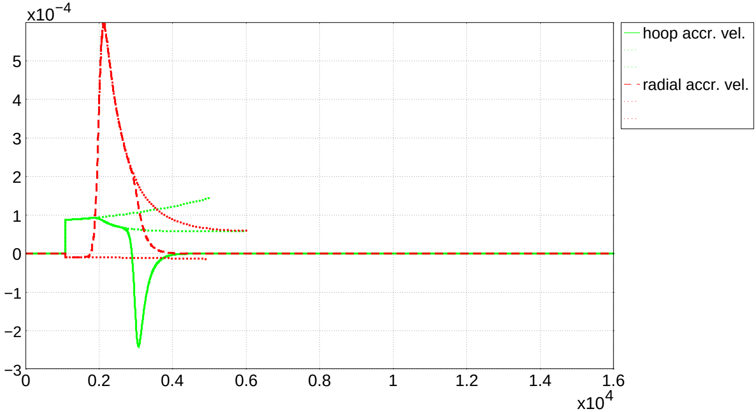
$G_r\,/\,D_r$	2000
$G_h\,/\,D_h$	5000
$D_r\,/\,D_h$	1
D_r	0.01
D_h	0.01

[case-15-k]



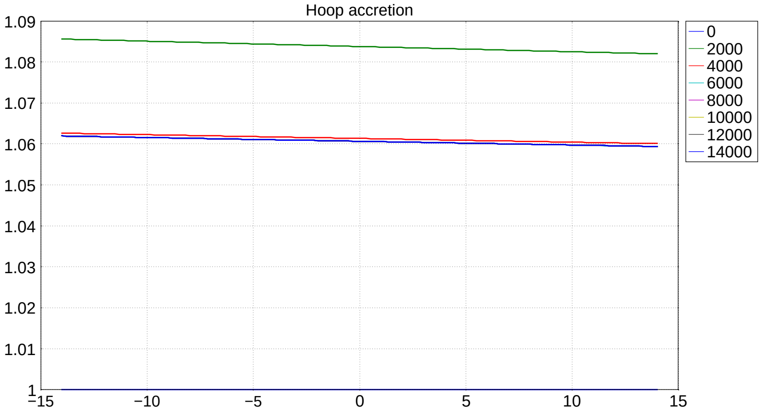
G_r / D_r	2000
G_h / D_h	5000
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-15-k]



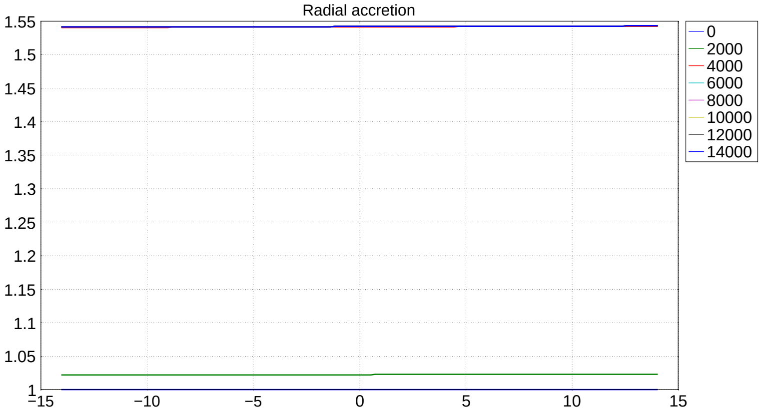
G_r / D_r	2000
G_h / D_h	5000
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-15-k]



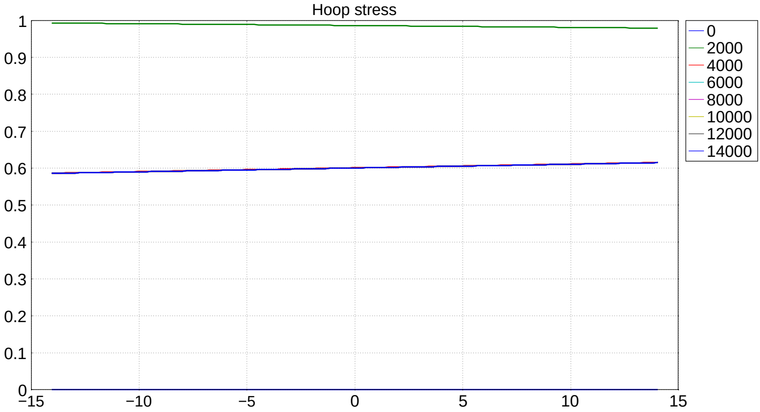
G_r / D_r	2000
G_h / D_h	5000
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-15-k]



G_r / D_r	2000
G_h / D_h	5000
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-15-k]

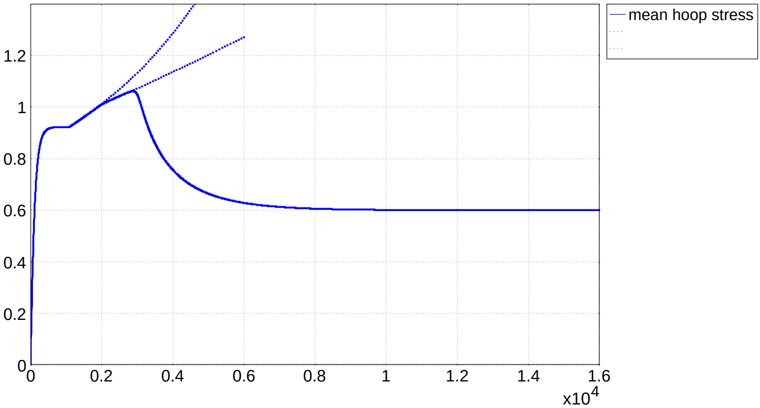


$G_r\,/\,D_r$	2000
$G_h\,/\,D_h$	5000
$D_r\,/\,D_h$	1
D_r	0.01
D_h	0.01

`[case-15-k]`

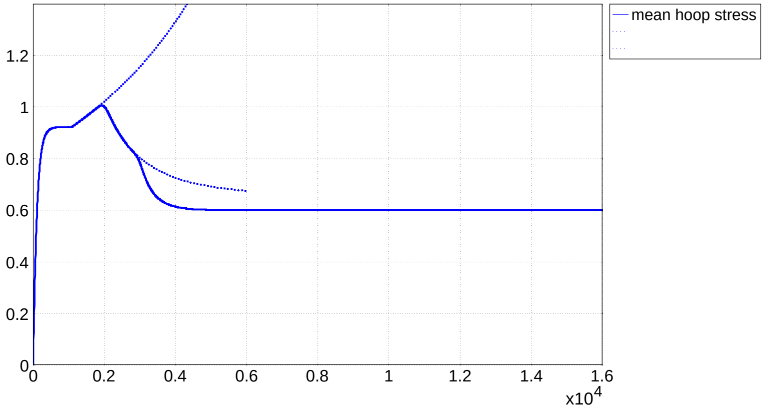
G_r / D_r	2000
G_h / D_h	5000
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-15-k]



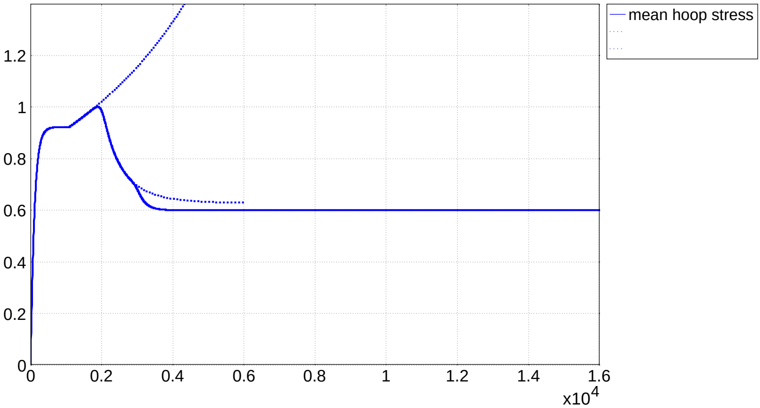
G_r / D_r	100
G_h / D_h	1000
D_r / D_h	10
D_r	0.1
D_h	0.01

[case-19-k]



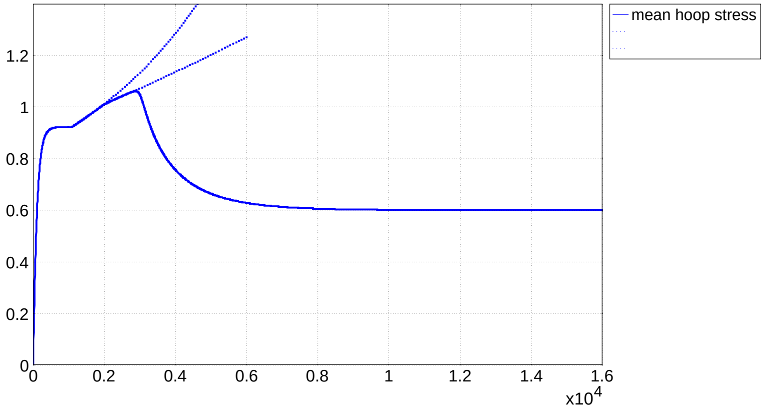
G_r / D_r	1000
G_h / D_h	2500
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-20-k]



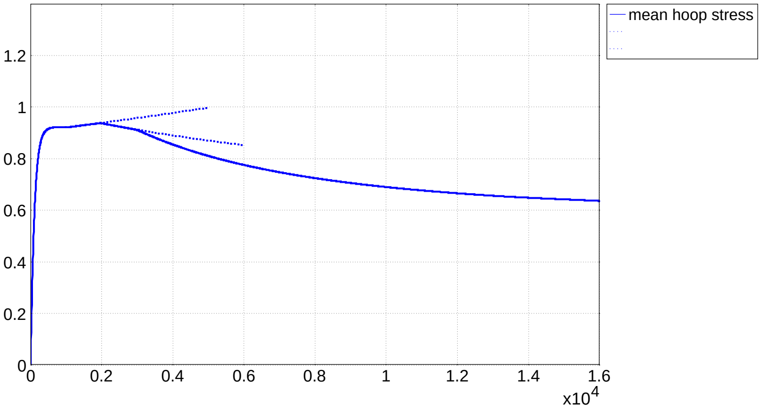
G_r / D_r	2000
G_h / D_h	5000
D_r / D_h	1
D_r	0.01
D_h	0.01

[case-15-k]



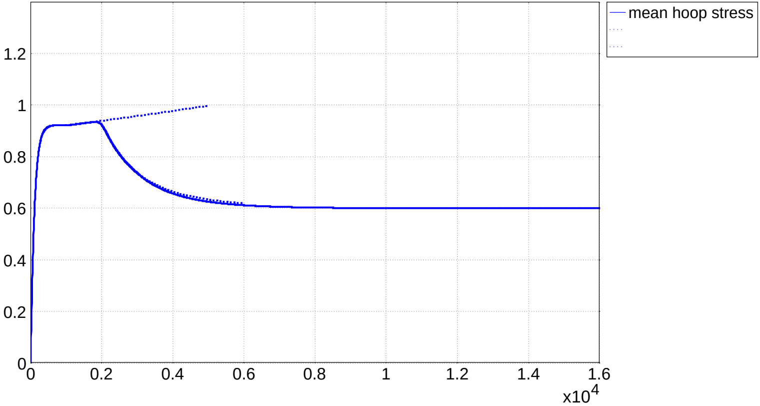
G_r / D_r	100
G_h / D_h	1000
D_r / D_h	10
D_r	0.1
D_h	0.01

[case-19-k]



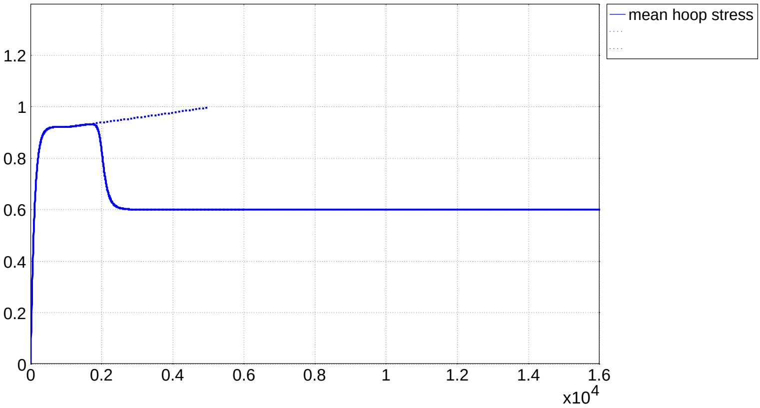
G_r / D_r	100
G_h / D_h	100
D_r / D_h	0.1
D_r	0.01
D_h	0.1

[case-17-k]



G_r / D_r	1000
G_h / D_h	100
D_r / D_h	0.1
D_r	0.01
D_h	0.1

[case-18-k]



G_r / D_r	10000
G_h / D_h	100
D_r / D_h	0.1
D_r	0.01
D_h	0.1

[case-16-k]

Growth driven by local hoop stress value

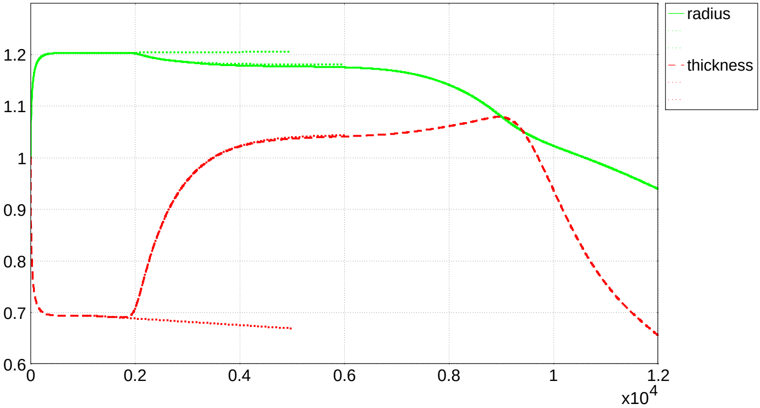
Outer accretive couple

$$\mathrm{A}_{\mathsf{r}}^{\mathfrak{o}} = \mathrm{J}\left(G_{\mathsf{r}}\left(\mathrm{T}_{\mathsf{h}} - \mathrm{T}^{\odot}\right) - \mathrm{T}_{\mathsf{r}} + \widetilde{\phi}\right)$$
$$\mathrm{A}_{\mathsf{h}}^{\mathfrak{o}} = \mathrm{J}\left(G_{\mathsf{h}}\left(\mathrm{T}^{\odot} - \mathrm{T}_{\mathsf{h}}\right) - \mathrm{T}_{\mathsf{h}} + \widetilde{\phi}\right)$$

Evolution laws

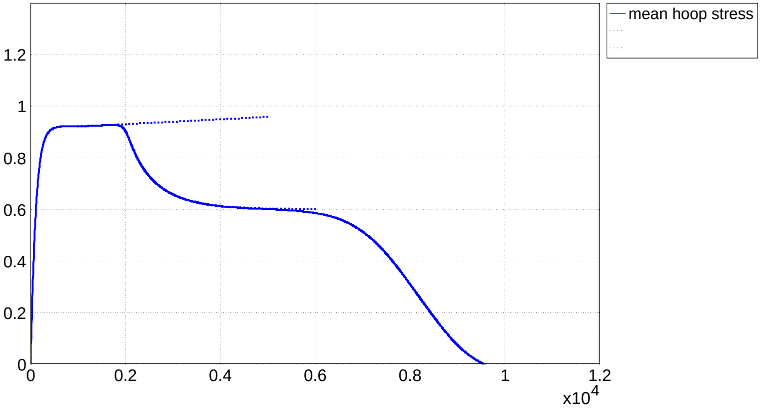
$$\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} = (G_{\mathsf{r}}/D_{\mathsf{r}})\left(\mathrm{T}_{\mathsf{h}} - \mathrm{T}^{\odot}\right)$$
$$\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} = (G_{\mathsf{h}}/D_{\mathsf{h}})\left(\mathrm{T}^{\odot} - \mathrm{T}_{\mathsf{h}}\right)$$

($\mathrm{T}_{\mathsf{h}}^{m}$ *mean hoop stress*, $\mathrm{T}^{\odot}$ *target hoop stress*)



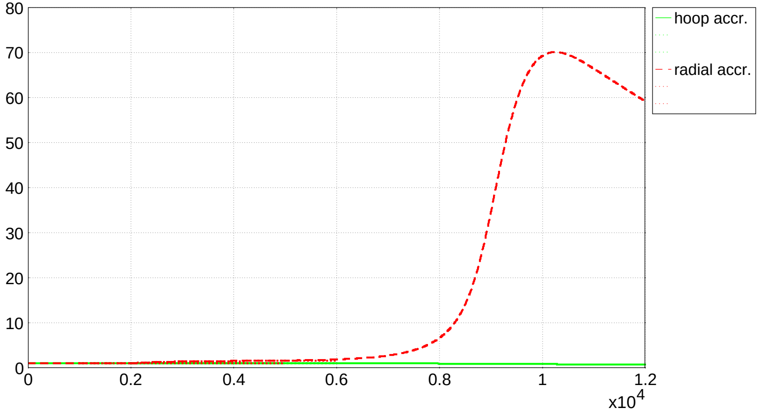
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



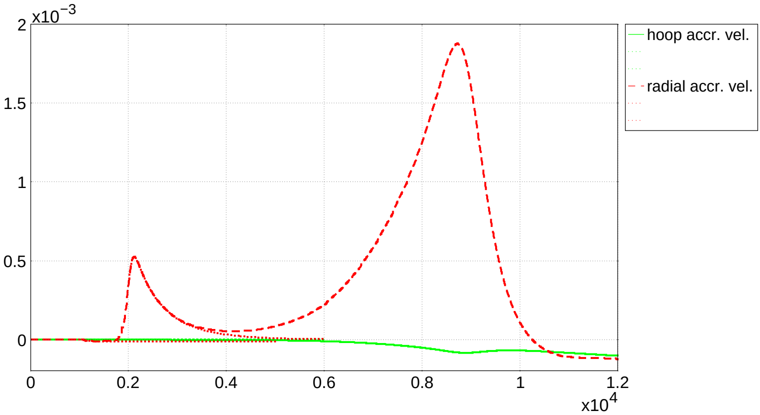
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



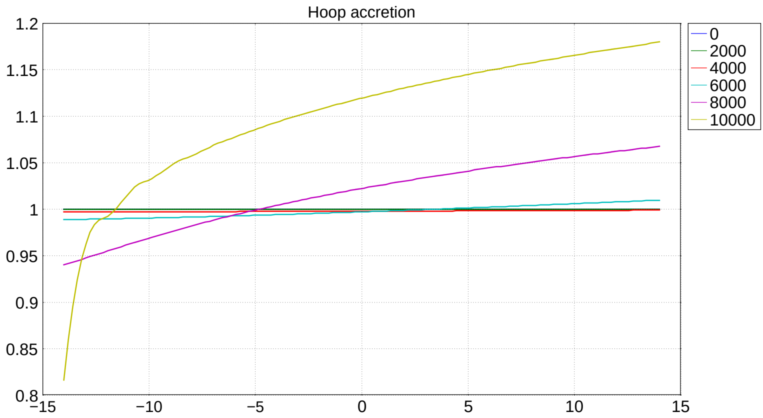
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



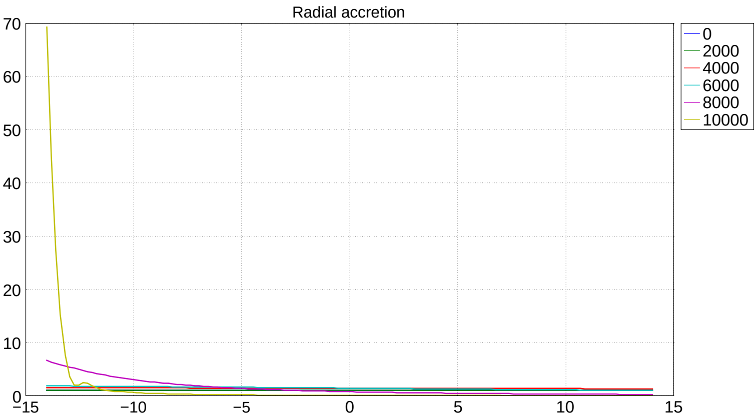
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



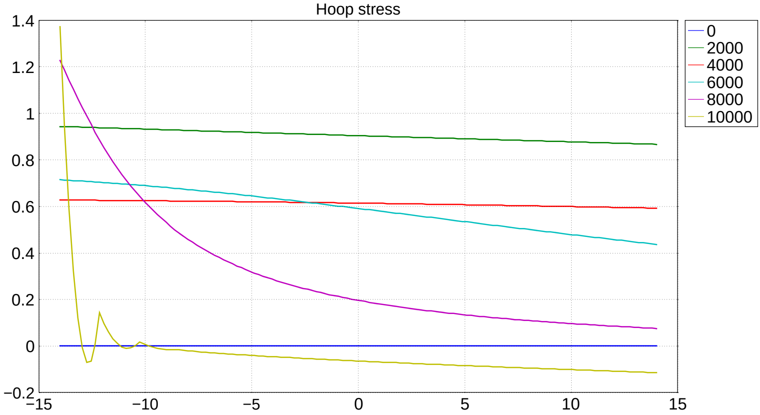
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



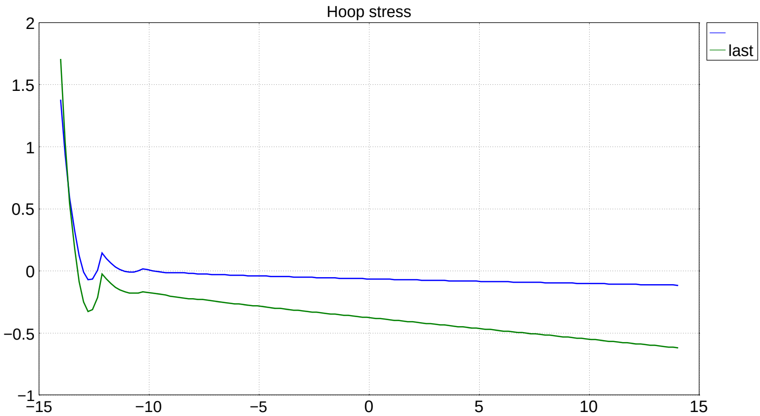
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



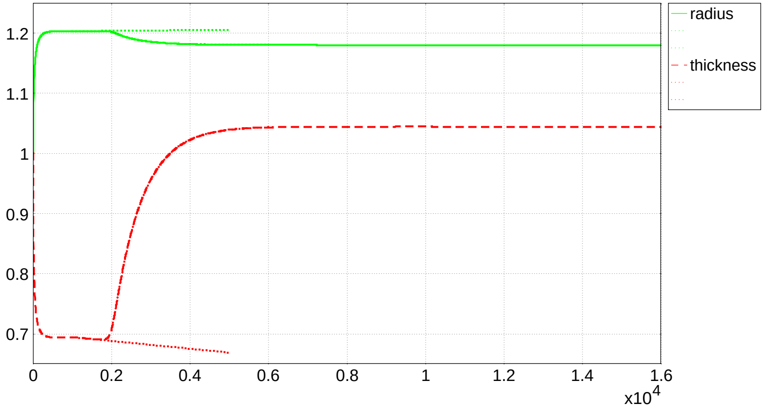
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



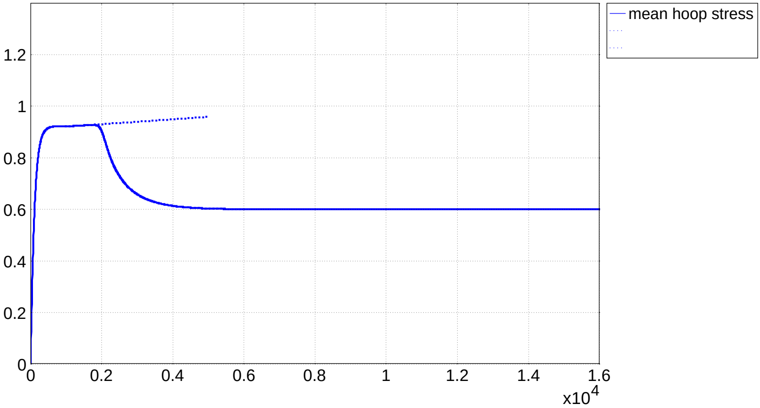
G_r / D_r	2000
G_h / D_h	80
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15e-kp]



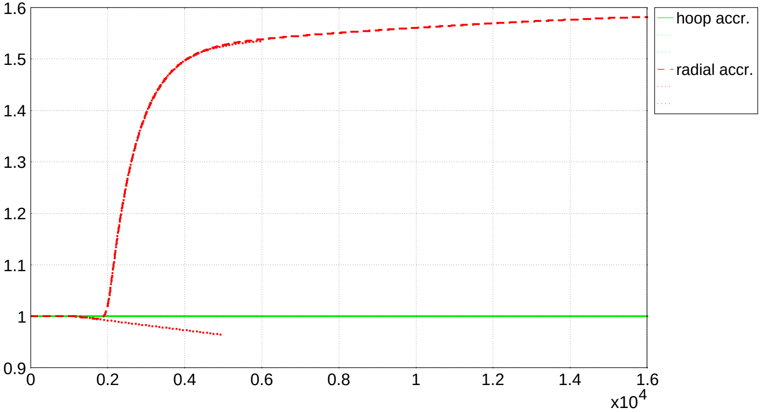
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



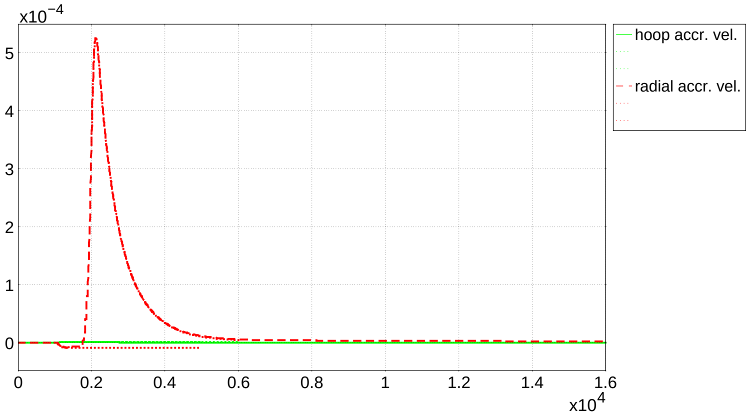
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



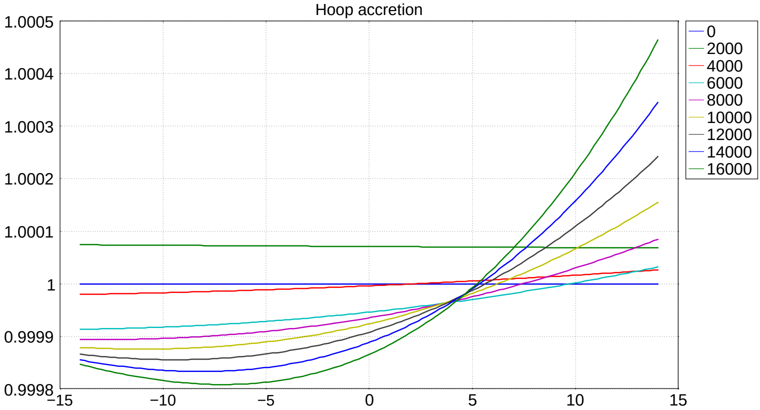
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



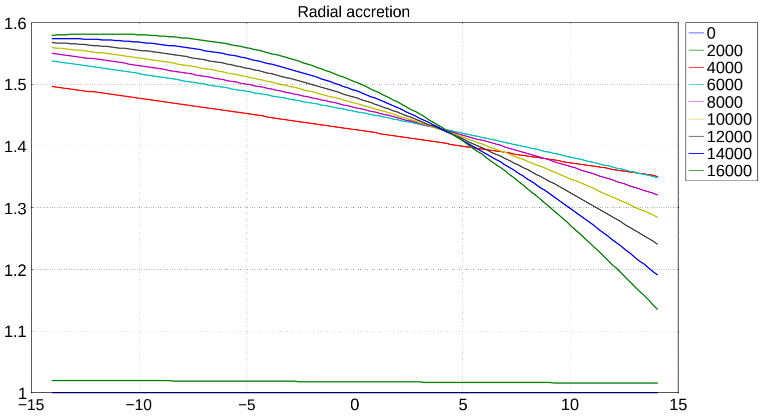
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



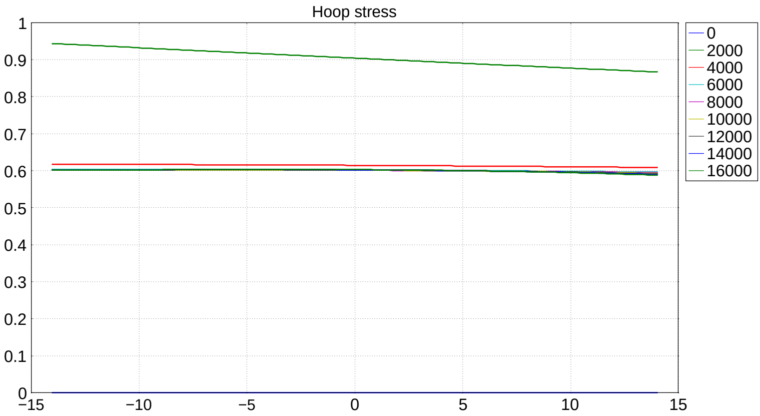
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



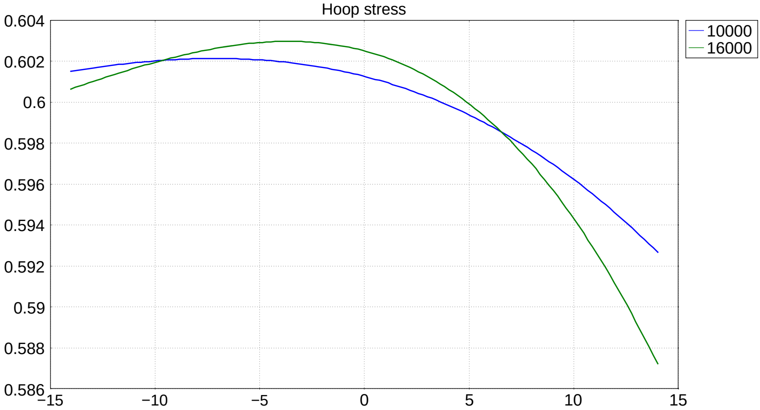
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



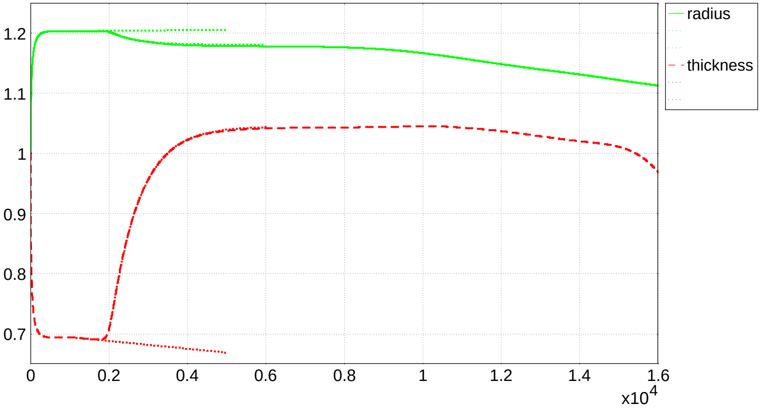
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



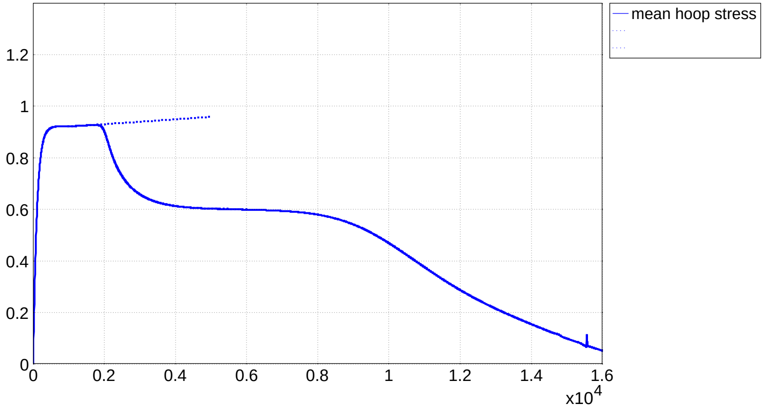
G_r / D_r	2000
G_h / D_h	5
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15b-kp]



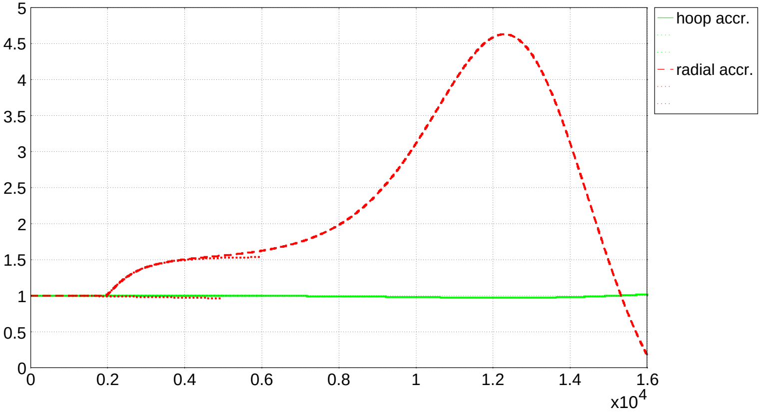
G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]



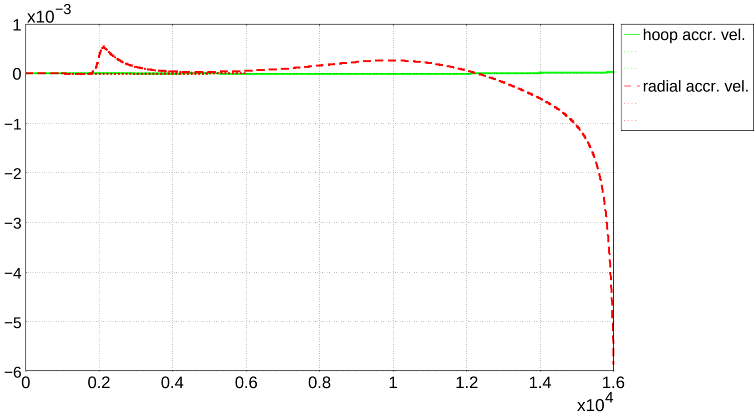
G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]



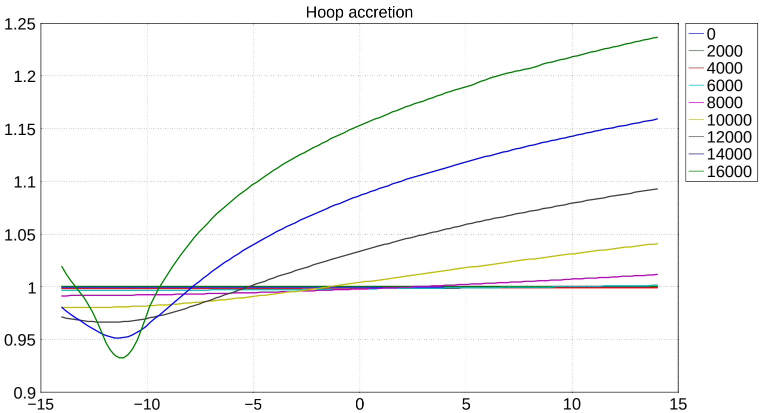
G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]



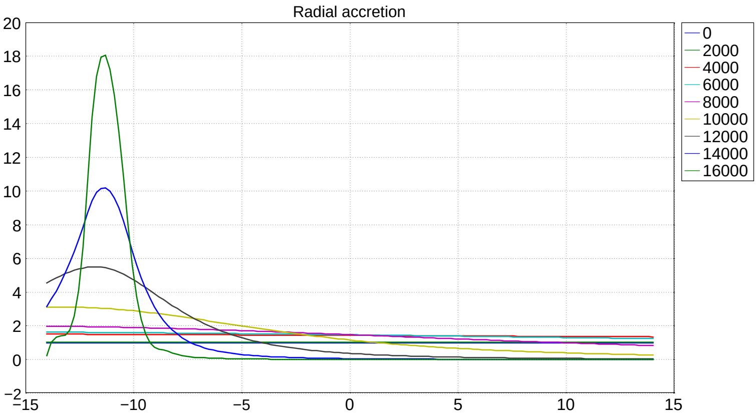
G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]



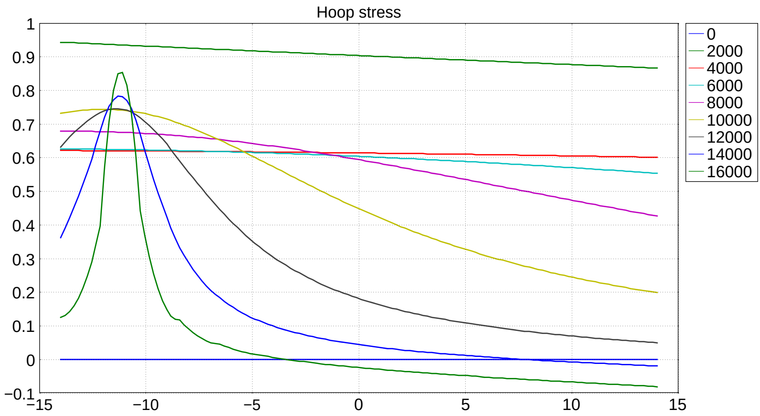
G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]



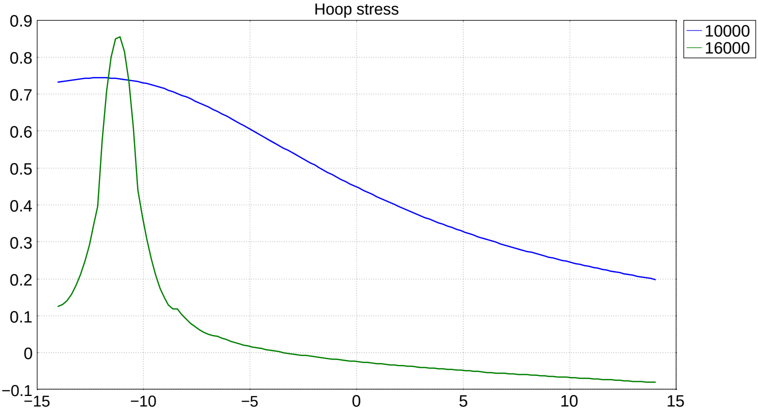
G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]



G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]



G_r / D_r	2000
G_h / D_h	50
D_r / D_h	0.001
D_r	0.01
D_h	10

[case-15c-kp]

Limited hoop growth

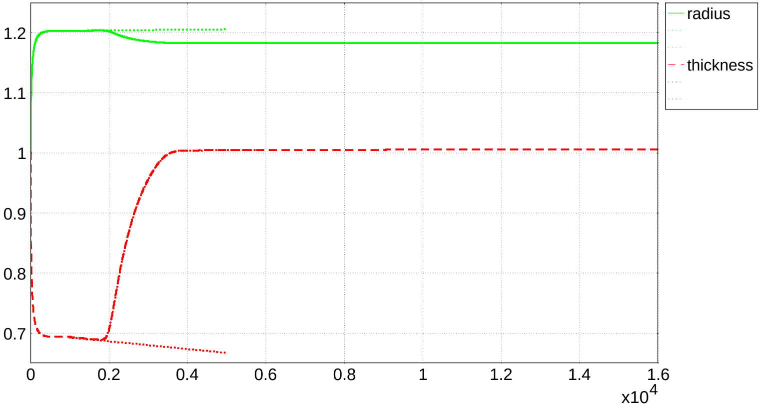
Evolution laws

$$
\begin{aligned}
D_{\mathsf{r}}\, \dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} &= \left(\mathrm{T}_{\mathsf{r}} - \widetilde{\phi}\right) + \mathrm{A}^{o}_{\mathsf{r}}/\mathrm{J} \\
D_{\mathsf{h}}\, \dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} &= \left(\mathrm{T}_{\mathsf{h}} - \widetilde{\phi}\right) + \mathrm{A}^{o}_{\mathsf{h}}/\mathrm{J}
\end{aligned}
$$

Outer accretive couple

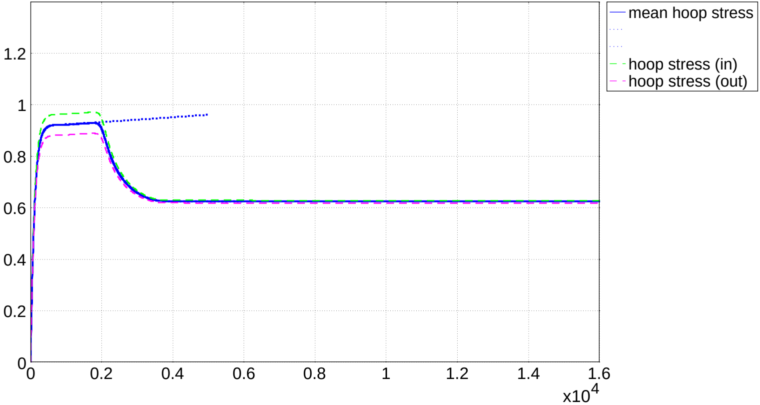
$$
\begin{aligned}
\mathrm{A}^{o}_{\mathsf{r}} &= \mathrm{J}\left(G_{\mathsf{r}}\left(\mathrm{T}_{\mathsf{h}} - \mathrm{T}^{\odot}\right) - \mathrm{T}_{\mathsf{r}} + \widetilde{\phi}\right) \\
\mathrm{A}^{o}_{\mathsf{h}} &= \mathrm{J}\left(-\mathrm{T}_{\mathsf{h}} + \widetilde{\phi}\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} &= \left(G_{\mathsf{r}}/D_{\mathsf{r}}\right)\left(\mathrm{T}_{\mathsf{h}} - \mathrm{T}^{\odot}\right) \\
\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} &= 0
\end{aligned}
$$



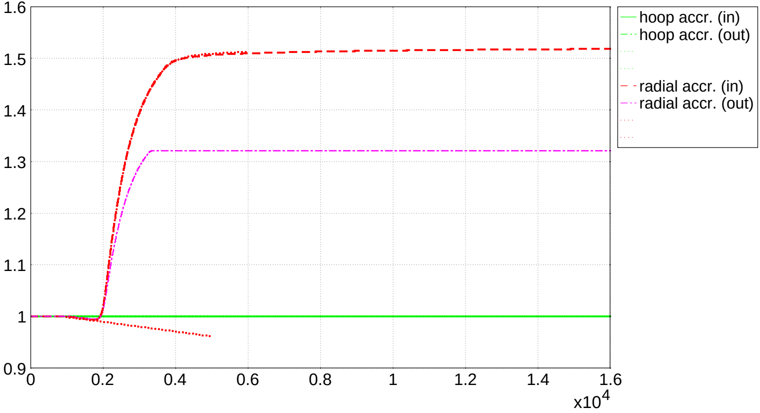
G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]



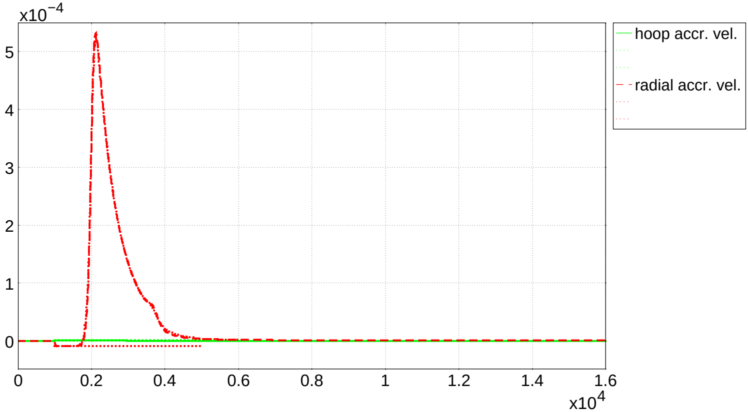
G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]



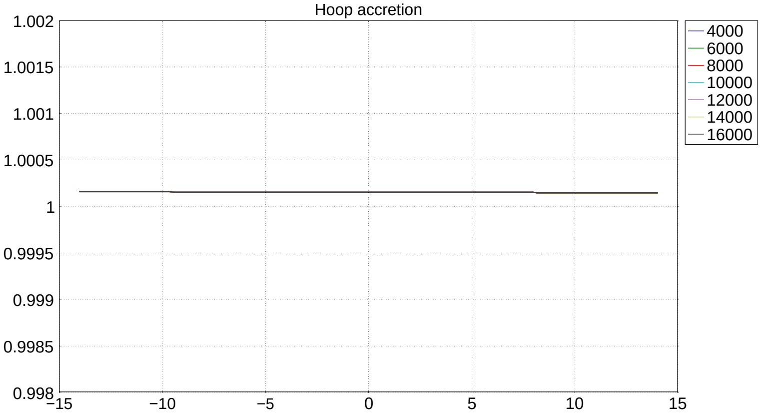
G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]



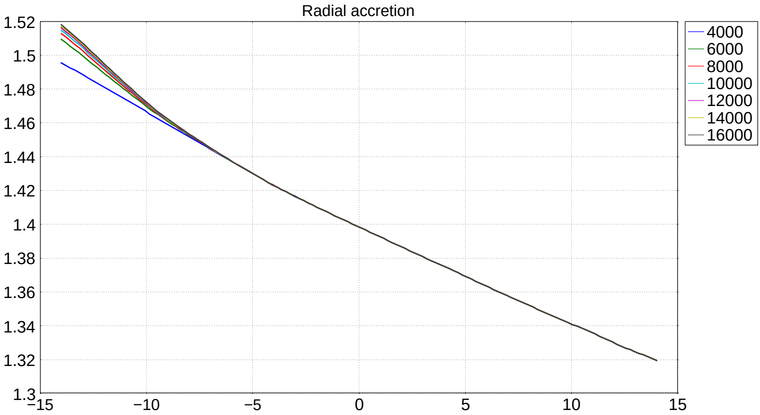
G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]



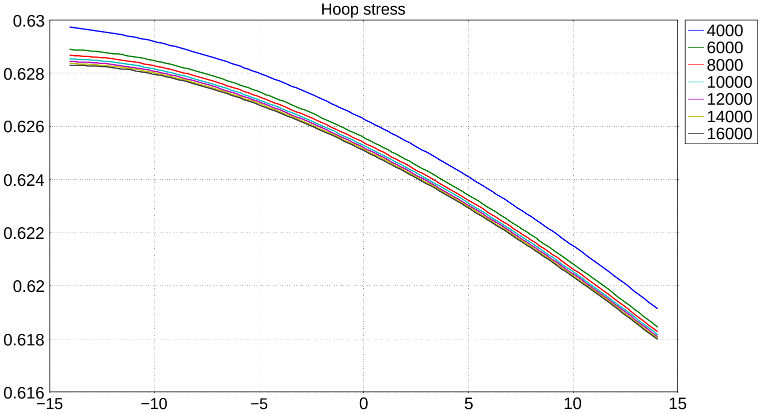
G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]



G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]



G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]

G_r / D_r	2000
D_r	0.01
D_h	10

[case-15b-brx]