Saccular aneurysms:
ill-fated or well-behaved

A. Tatone, V. Varano

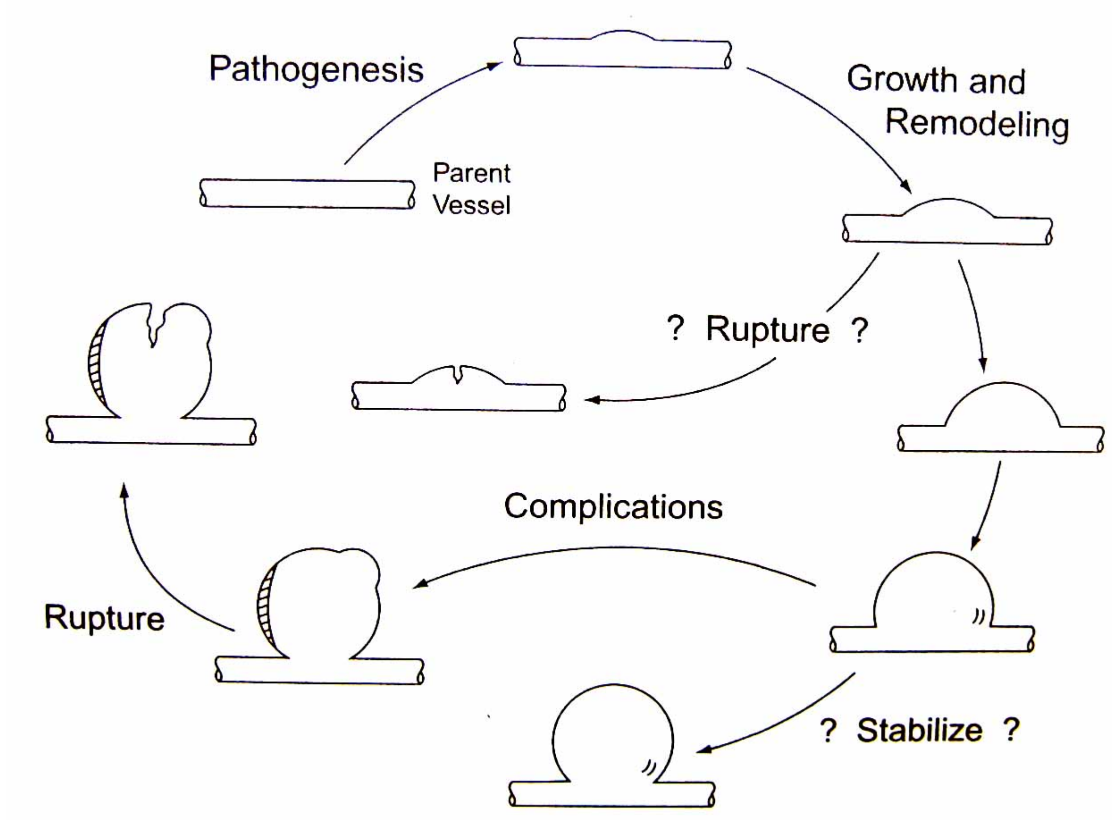
September 14, 2006

Based on a joint work by

A. Di Carlo [a], V. Sansalone [b], A. Tatone [c], V. Varano [a]

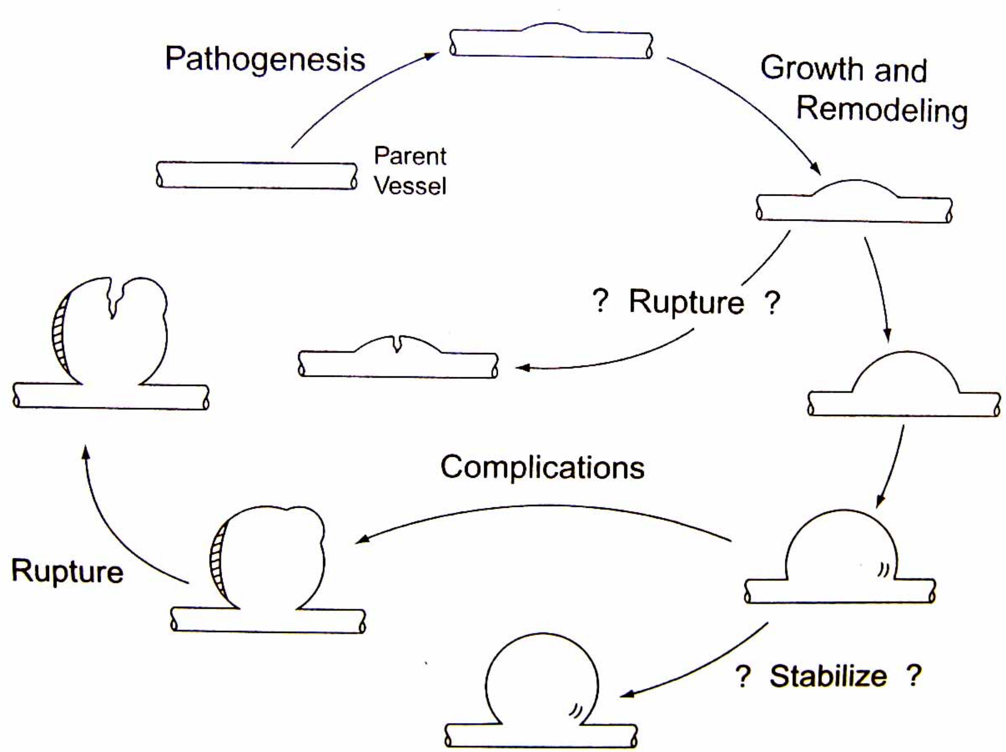
[a] Università degli Studi "Roma Tre", Roma, Italy
[b] Pôle Universitaire Léonard de Vinci, Paris La Défense, France
[c] Università degli Studi de L'Aquila, L'Aquila, Italy

Intracranial saccular aneurysms are dilatations of the arterial wall.

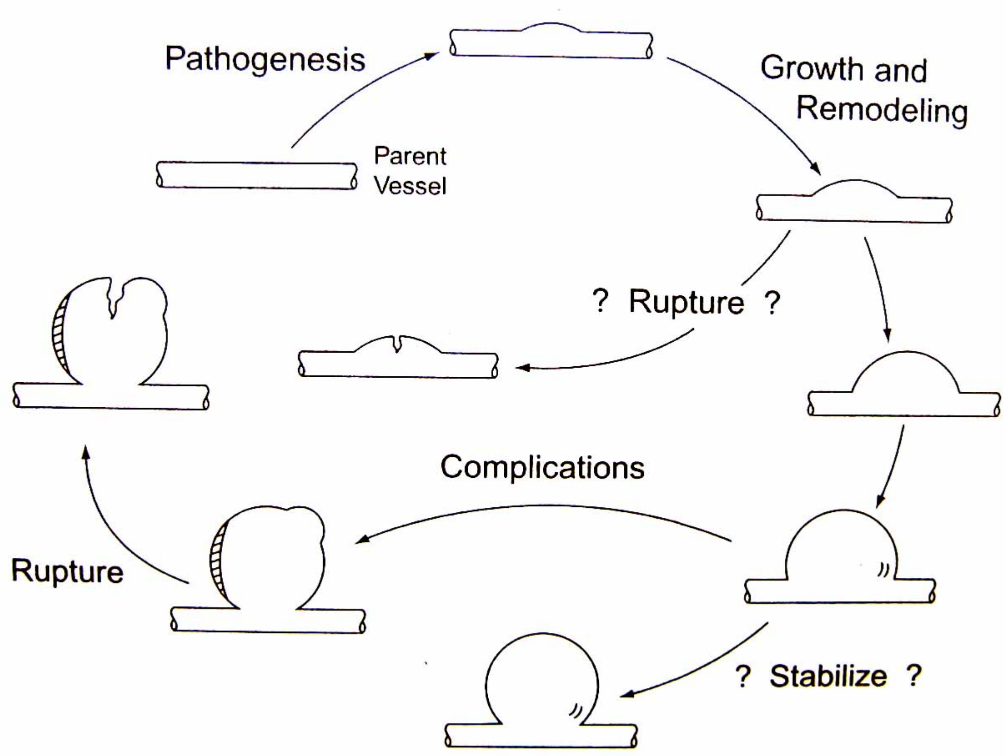


A possible natural history for saccular aneurysms

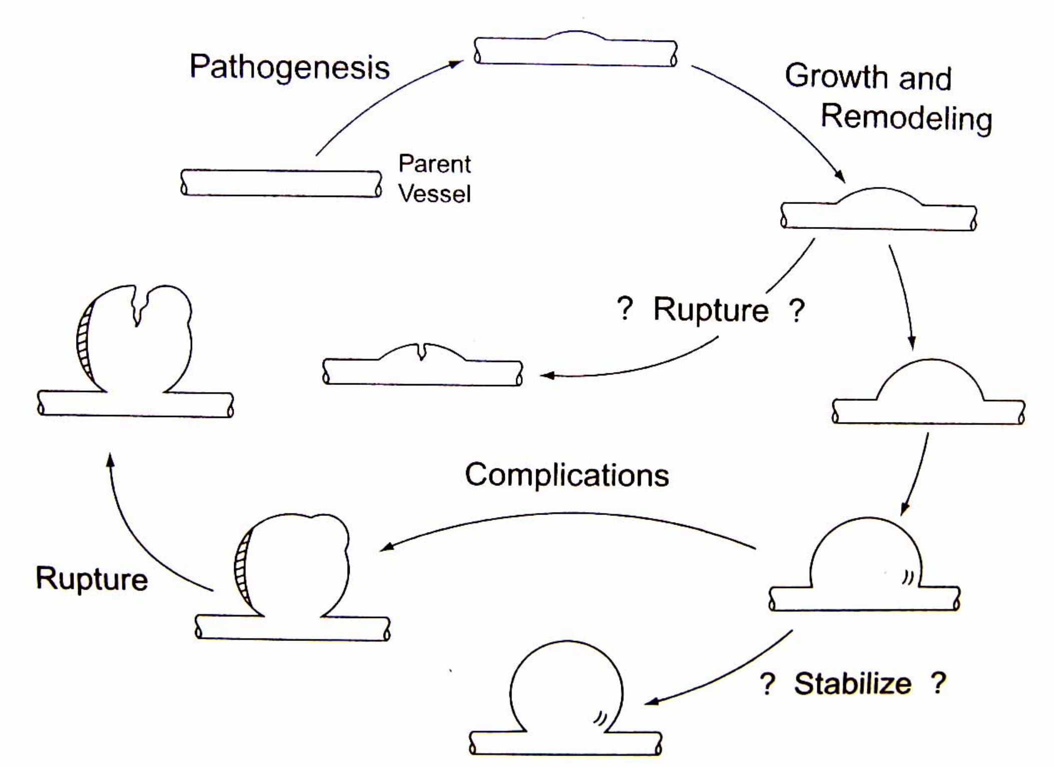
[J.D.Humphrey, Cardiovascular Solid Mechanics, 2001]



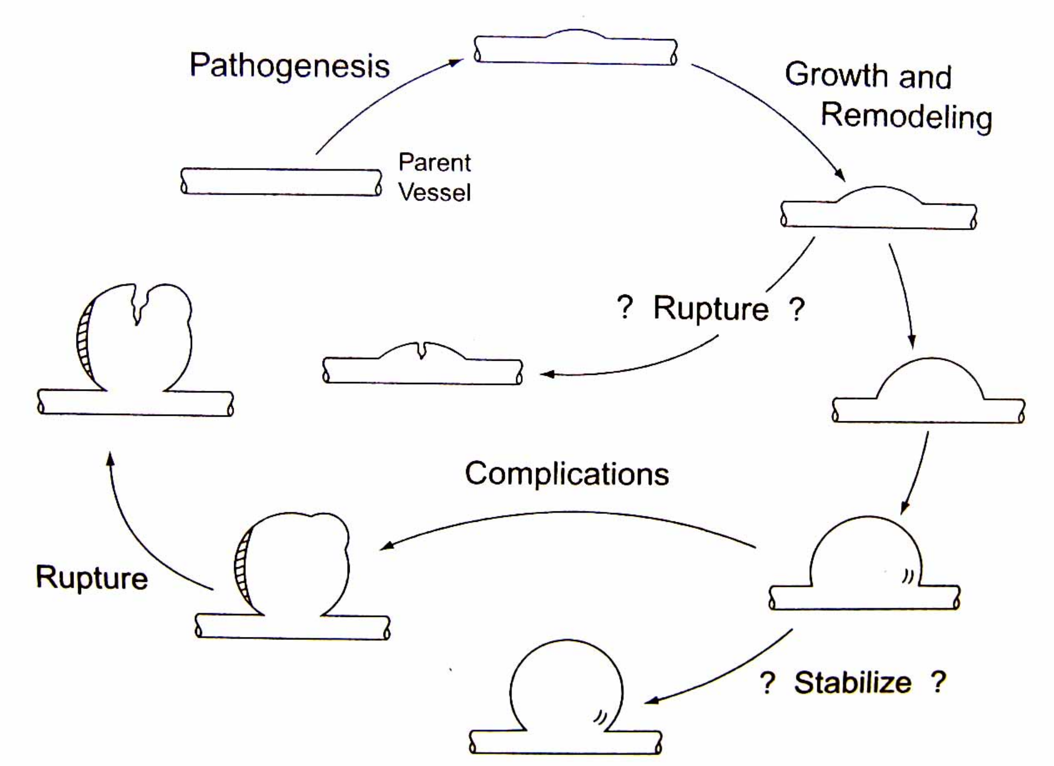
- An initial insult may cause a local weakening of the wall and thus a mild dilatation.



- This raises the local stress field above normal values, thus setting into motion a growth and remodeling process that attempts to reduce the stress toward values that are homeostatic for the parent vessel.

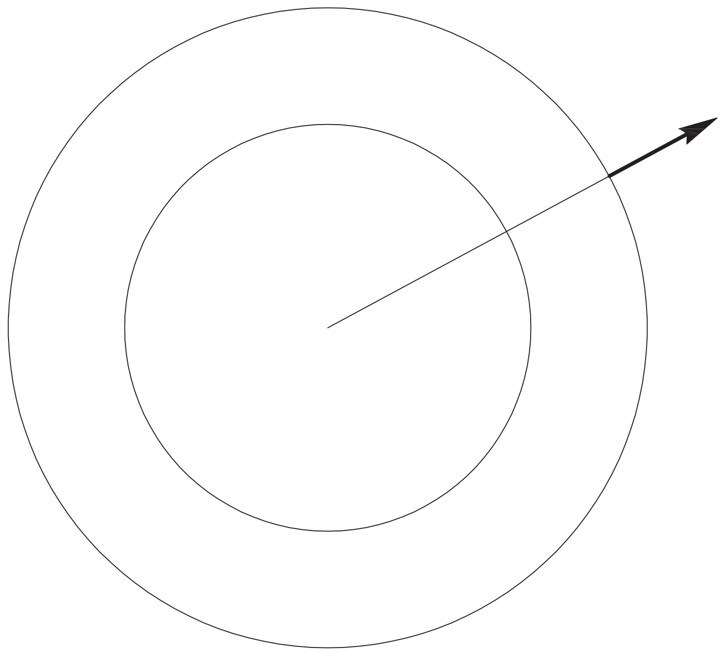


- If degradation and deposition of collagen are well balanced, this could produce a larger, but stable lesion.



- If degradation exceedes deposition at any time, this could yield a rupture.

Spherically symmetric shapes



Spherically symmetric placements

$$\mathsf{p}(x) = x_o + \rho(\xi)\,\mathsf{a}_r(x)$$

Gradient of p

$$\nabla \mathsf{p}|_x = \frac{\rho(\xi)}{\xi} P(x) + \rho'(\xi) N(x).$$

Prototype

$$\mathbb{P} = \alpha_h(\xi)\,P + \alpha_r(\xi)\,N$$

where

$$N(x) := \mathsf{a}_r(x) \otimes \mathsf{a}_r(x),$$
$$P(x) := I - N(x),$$

Warp

$$F = \nabla \mathsf{p}\, \mathbb{P}^{-1} = \lambda_h\, P + \lambda_r\, N$$

with

$$\lambda_h := \frac{\rho}{\xi \alpha_h}, \qquad \lambda_r := \frac{\rho'}{\alpha_r}$$

Hence

$$F^{\mathsf{T}} F = \lambda_h^2\, P + \lambda_r^2\, N$$

In a refined motion

$$\mathbb{P}(\tau) = \alpha_h(\tau)\, P + \alpha_r(\tau)\, N$$

$$\dot{\mathbb{P}}(\tau) = \dot{\alpha}_h(\tau) P + \dot{\alpha}_r(\tau) N$$

Prototype velocity

$$\mathbb{V} = \dot{\mathbb{P}}\mathbb{P}^{-1} = \beta_h(\tau)\, P + \beta_r(\tau)\, N$$

with

$$\beta_h(\tau) := \frac{\dot{\alpha}_h(\tau)}{\alpha_h(\tau)}, \quad \beta_r(\tau) := \frac{\dot{\alpha}_r(\tau)}{\alpha_r(\tau)}.$$

Gross velocity gradient

$$\nabla \mathsf{v} = \nabla \dot{\mathsf{p}} = \frac{v}{\xi} P + v' N$$

$$\mathsf{Dv} = \nabla \mathsf{v}\, \mathbb{P}^{-1} = \frac{v}{\xi \alpha_h} P + \frac{v'}{\alpha_r} N$$

with

$$v(\tau) := \dot{\rho}(\tau)$$

Working (bulk)

$$\int_{\mathcal{D}} \left(\mathsf{B} \cdot \nabla \tilde{\mathsf{v}} + \mathbb{A} \cdot \tilde{\mathbb{V}} \right) = \int_{\mathcal{D}} \left(2\mathsf{b}_h \frac{\tilde{v}}{\xi} + \mathsf{b}_r \, \tilde{v}' + 2\mathbb{a}_h \, \tilde{\beta}_h + \mathbb{a}_r \, \tilde{\beta}_r \right)$$

$$\int_{\mathcal{D}} \left((\mathbb{B} \cdot \mathsf{D}\tilde{\mathsf{v}})\mathbb{J} + \mathbb{A} \cdot \tilde{\mathbb{V}} \right) = \int_{\mathcal{D}} \left(2\lambda_h \mathbb{b}_h \mathbb{J} \, \frac{\tilde{v}}{\rho} + \lambda_r \mathbb{b}_r \mathbb{J} \, \frac{\tilde{v}'}{\rho'} + 2\mathbb{a}_h \, \tilde{\beta}_h + \mathbb{a}_r \, \tilde{\beta}_r \right)$$

$$\mathbb{J} = \det \mathbb{P} = \alpha_h^2 \alpha_r$$

$$\mathbb{B} = \mathbb{B}^i + \mathbb{B}^o = \mathbb{B}^i = \mathbb{S}, \qquad \mathsf{B} = \mathsf{B}^i + \mathsf{B}^o = \mathsf{B}^i = \mathsf{S}$$

$$\mathbb{S} = \frac{1}{\mathbb{J}} \mathsf{S} \, \mathbb{P}^{\mathsf{T}}$$

Balance equations

$$-2\mathsf{s}_h + 2\mathsf{s}_r + \xi\mathsf{s}_r' = 0$$

$$\mathbb{a}_h^i + \mathbb{a}_h^o = 0$$

$$\mathbb{a}_r^i + \mathbb{a}_r^o = 0$$

Hyperelastic stress $\breve{\mathbb{B}}^i$ and $\breve{\mathbb{A}}^i$

$$(\breve{\mathbb{B}}^i \cdot \mathsf{Dv})\mathbb{J} + \breve{\mathbb{A}}^i \cdot \mathbb{V} = \dot{\psi}(F, \mathbb{P})$$

$$\psi(\lambda_h, \lambda_r; \alpha_h, \alpha_r) = \varphi(\lambda_h, \lambda_r)\mathbb{J} = \varphi(\lambda_h, \lambda_r)\alpha_h^2 \alpha_r$$

In order for a potential ψ to exist, the following relations should hold

$$\breve{\mathbb{s}}_h = \frac{1}{2}\frac{\partial \varphi}{\partial \lambda_h}$$

$$\breve{\mathbb{s}}_r = \frac{\partial \varphi}{\partial \lambda_r}$$

$$\breve{\mathbb{a}}_h^i = \mathbb{J}(\varphi - \lambda_h \breve{\mathbb{s}}_h)$$

$$\breve{\mathbb{a}}_r^i = \mathbb{J}(\varphi - \lambda_r \breve{\mathbb{s}}_r)$$

Dissipation inequality

$$\varpi^{+} \geq 0$$

$$\varpi(F, \mathbb{P}) = \dot{\psi}(F, \mathbb{P}) + \varpi^{+}$$

$$\varpi(F, \mathbb{P}) := (\mathbb{S} \cdot \mathsf{Dv})\mathbb{J} + \mathbb{A} \cdot \mathbb{V}$$

$$\mathbb{S} = \check{\mathbb{S}} + \mathbb{S}^{+}$$

$$\mathbb{A} = \check{\mathbb{A}} + \mathbb{A}^{+}$$

$$\mathbb{s}_h = \frac{1}{2}\frac{\partial\varphi}{\partial\lambda_h} + \mathbb{s}_h{}^{+}$$

$$\mathbb{s}_r = \frac{\partial\varphi}{\partial\lambda_r} + \mathbb{s}_r{}^{+}$$

$$\mathbb{a}_h^i = \mathbb{J}(\varphi - \lambda_h \mathbb{s}_h) + \mathbb{a}_h^{i}{}^{+}$$

$$\mathbb{a}_r^i = \mathbb{J}(\varphi - \lambda_r \mathbb{s}_r) + \mathbb{a}_r^{i}{}^{+}$$

Extra energetic constitutive prescriptions

$$\mathbb{s}_h^{i\,+} := 0$$
$$\mathbb{s}_r^{i\,+} := 0$$

$$\mathbb{a}_h^{i\,+} := \frac{1}{m_h}\,\frac{\dot{\alpha}_h}{\alpha_h}$$
$$\mathbb{a}_r^{i\,+} := \frac{1}{m_r}\,\frac{\dot{\alpha}_r}{\alpha_r}$$

Evolution equations

$$\mathbb{a}_h^i + \mathbb{a}_h^o = 0$$
$$\mathbb{a}_r^i + \mathbb{a}_r^o = 0$$

$$\frac{\dot{\alpha}_h}{\alpha_h} = -m_h(\mathbb{J}(\varphi - \lambda_h \mathbb{s}_h) + \mathbb{a}_h^o)$$
$$\frac{\dot{\alpha}_r}{\alpha_r} = -m_r(\mathbb{J}(\varphi - \lambda_r \mathbb{s}_r) + \mathbb{a}_r^o)$$

Evolution equations

$$\frac{\dot{\alpha}_h}{\alpha_h} = -m_h(\mathbb{J}(\varphi - \lambda_h \mathbb{s}_h) + \mathbb{a}_h^o)$$

$$\frac{\dot{\alpha}_r}{\alpha_r} = -m_r(\mathbb{J}(\varphi - \lambda_r \mathbb{s}_r) + \mathbb{a}_r^o)$$

Passive growth

$$\mathbb{a}_h^o = 0, \quad \mathbb{a}_r^o = 0$$

Stress driven growth

$$\mathbb{a}_h^o = \widehat{\mathbb{a}}_h^o(\mathbb{s}_h, \mathbb{s}_r), \quad \mathbb{a}_r^o = \widehat{\mathbb{a}}_r^o(\mathbb{s}_h, \mathbb{s}_r)$$

Fung strain energy function

$$\varphi_{\mathsf{F}} := c(e^q - 1)$$

$$q := \frac{\Gamma}{2}(\lambda_h^2 - 1)^2$$

$$\varphi := \frac{\varphi_{\mathsf{F}}}{H}$$

Incompressibility

$$\lambda_h^2 \lambda_r = 1$$

Elastic constants

$$\Gamma = 13$$
$$c = 0.88 \text{ N/m}$$
$$H = 27.88 \times 10^{-6} \text{ m}$$

Aneurysm dimensions

$$h = 28 \times 10^{-6} \text{ m}$$
$$R = 3000 \times 10^{-6} \text{ m}$$

Inner pressure

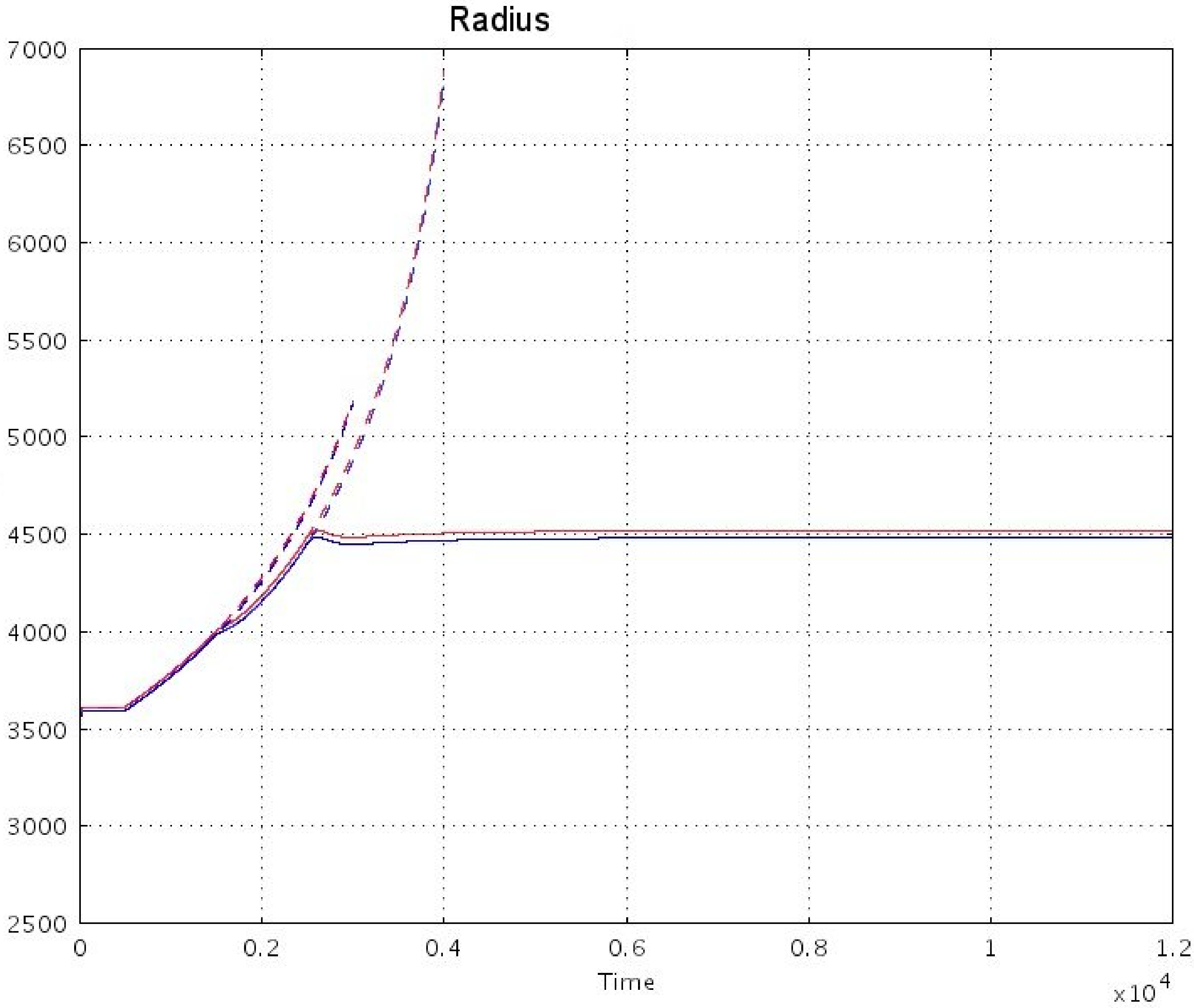
$$p = 10^4 \text{ Pa}$$

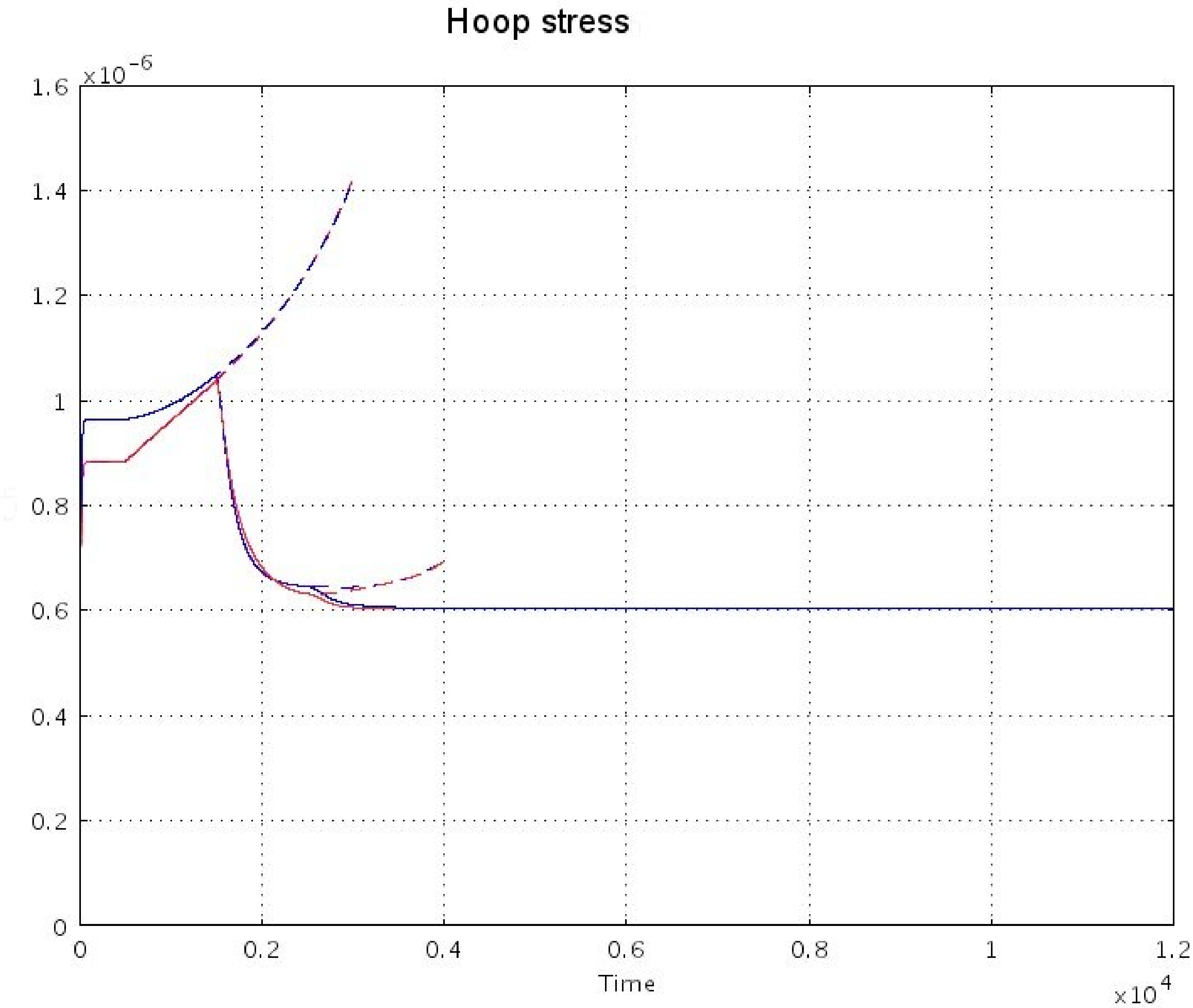
[J.D.Humphrey, Cardiovascular Solid Mechanics, 2001]

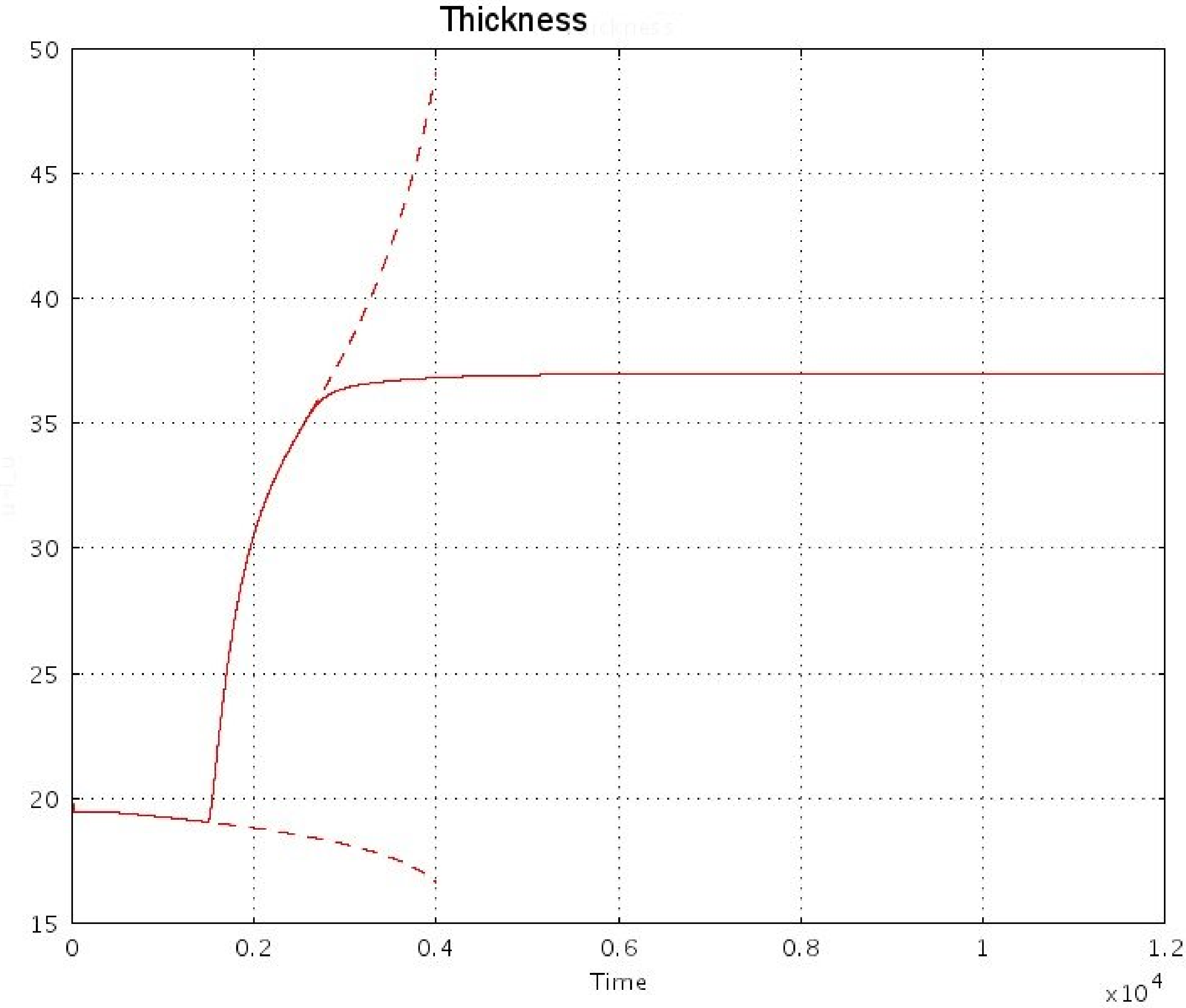
Outer accretive (control) forces

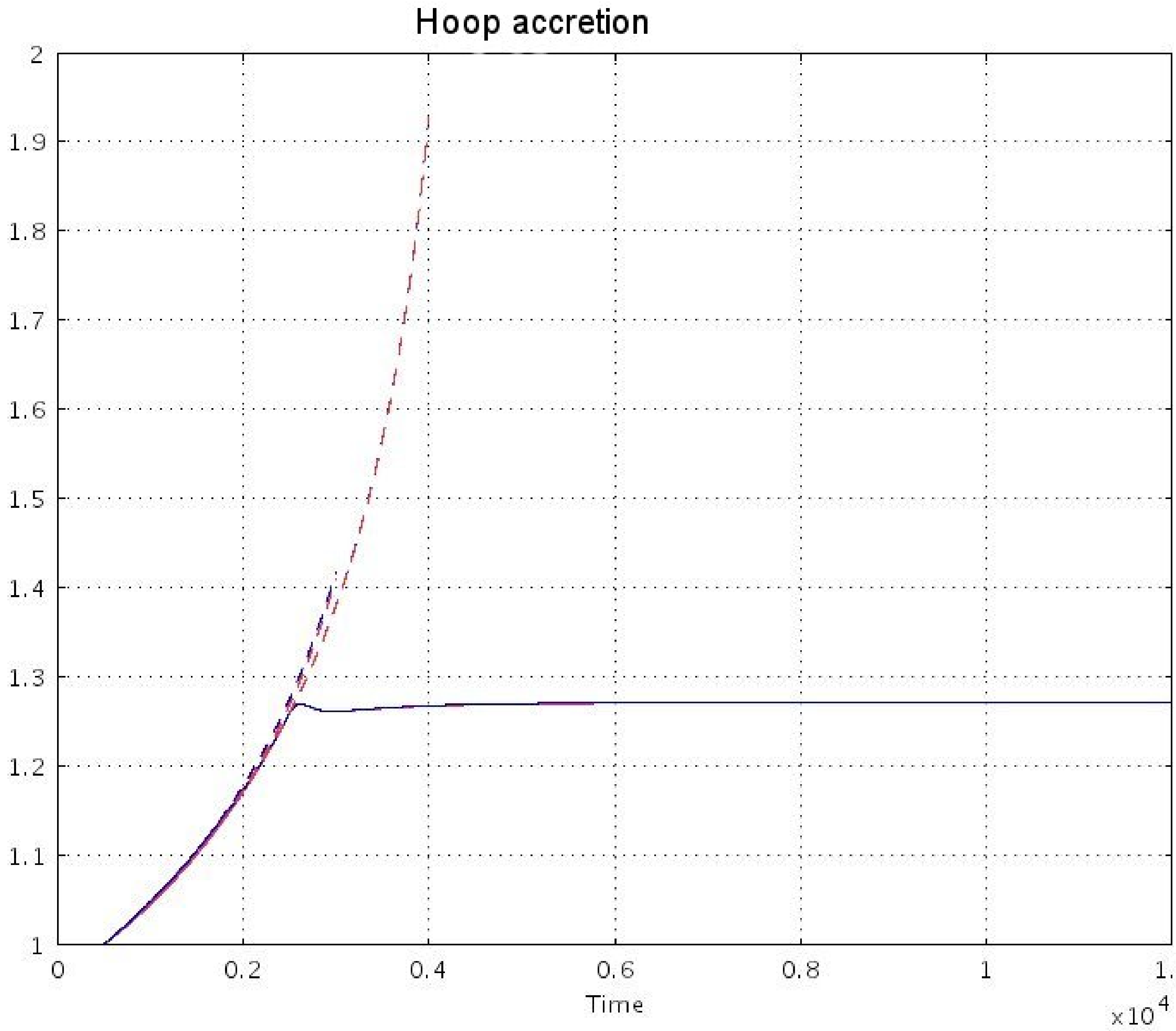
$$\mathbb{a}_h^o = 4.64 \times 10^2(\sigma_h^\star - \bar{\sigma}_h)$$
$$\mathbb{a}_r^o = -0.50 \times 10^2(\sigma_h^\star - \bar{\sigma}_h)$$

$$\sigma_h^\star = 0.60 \times 10^6 \text{ Pa}$$

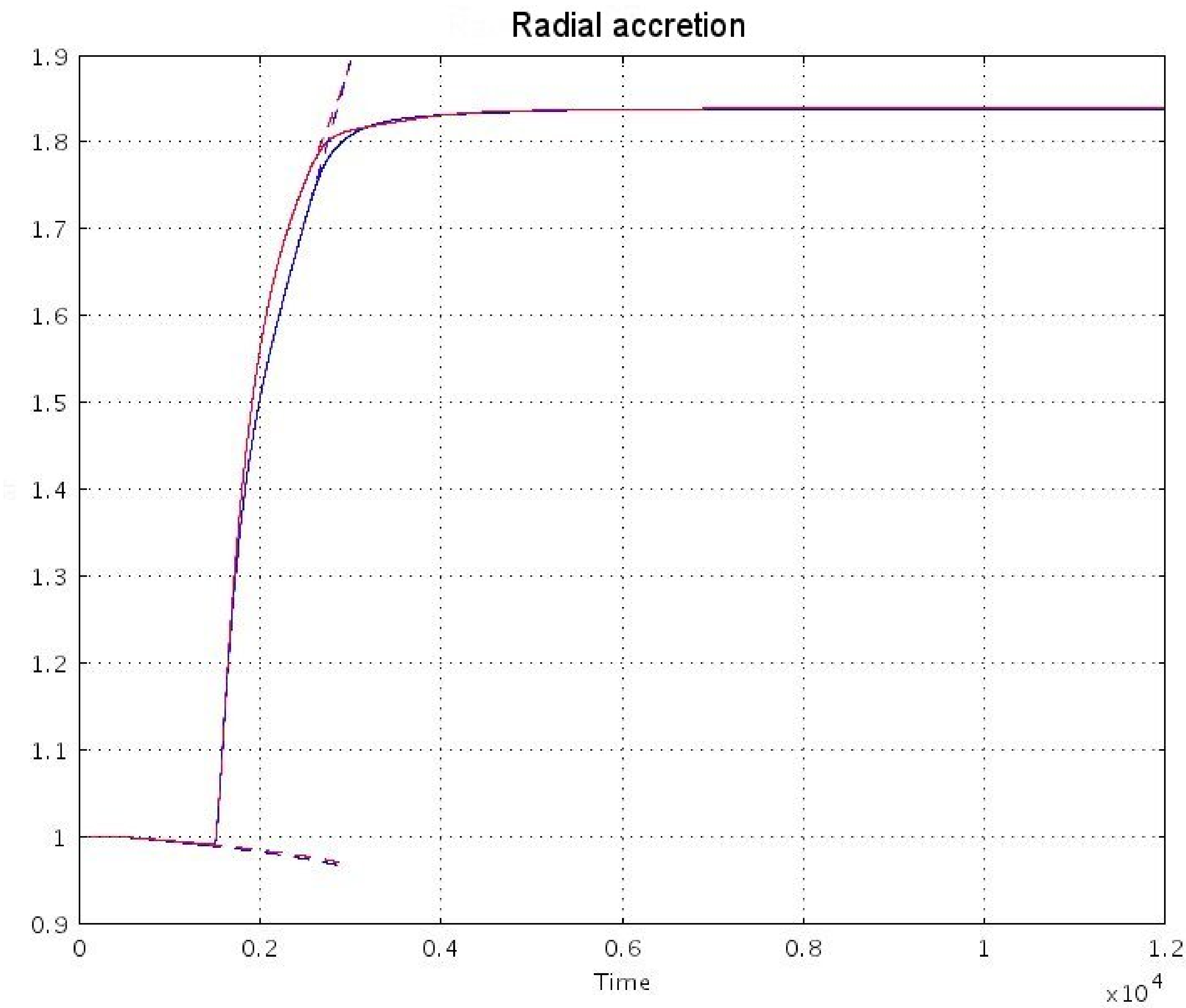

Radius
7000
6500
6000
5500
5000
4500
4000
3500
3000
2500
0
0.2
0.4
0.6
0.8
1
1.2
Time
x10^4


Hoop stress
x10^-6
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
Time
x10^4


Thickness
50
45
40
35
30
25
20
15
0
0.2
0.4
0.6
0.8
1
1.2
Time
x10^4



Hoop accretion
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0
0.2
0.4
0.6
0.8
1
1.2
Time
x10^4


Radial accretion
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0
0.2
0.4
0.6
0.8
1
1.2
Time
x10^4