Competing remodeling mechanisms in the development of saccular aneurysms

Stress-driven remodeling and control laws

V. Sansalone [a],

A. Di Carlo [b], A. Tatone [c], V. Varano [b]

[a] *Université Paris 12 – Val de Marne, France*
[b] *Università degli Studi "Roma Tre", Italy*
[c] *Università degli Studi dell'Aquila, Italy*

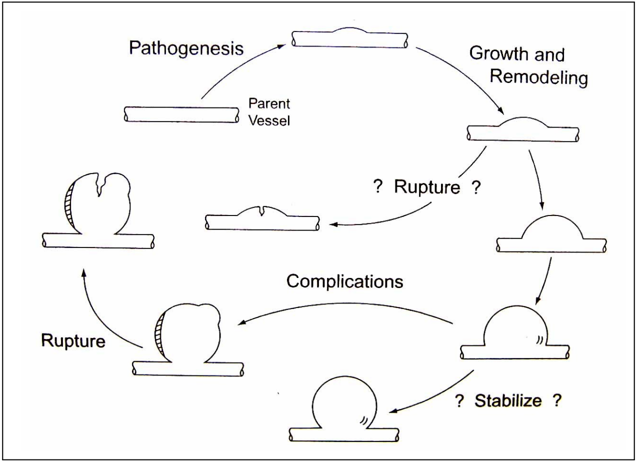
USNCCM9, July 25, 2007

Outline

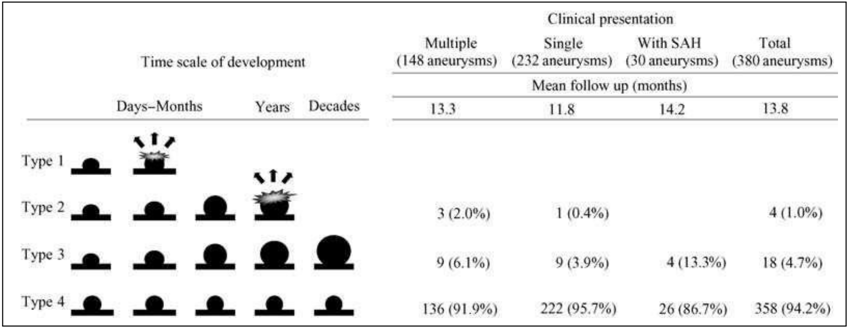
Part I

Aneurysms

Intracranial saccular aneurysms are dilatations of the arterial wall.



[J.D. Humphrey, *Cardiovascular Solid Mechanics*, 2001]



Time scale of development (Days–Months, Years, Decades)	Clinical presentation: Multiple (148 aneurysms)	Single (232 aneurysms)	With SAH (30 aneurysms)	Total (380 aneurysms)
Mean follow up (months)	13.3	11.8	14.2	13.8
Type 1				
Type 2	3 (2.0%)	1 (0.4%)		4 (1.0%)
Type 3	9 (6.1%)	9 (3.9%)	4 (13.3%)	18 (4.7%)
Type 4	136 (91.9%)	222 (95.7%)	26 (86.7%)	358 (94.2%)

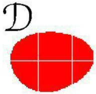
[M. Yonekura, *Neurologia medico-chirurgica*, 2004]

Part II

A mechanical model

Growth mechanics

Growth as *change in the zero-stress reference state.*



p : *gross placement*

$\nabla \mathrm{p}$: *gradient of the gross placement*

$\mathbb{P}$: *prototype*

F : *warp* (*Kröner-Lee decomposition*)

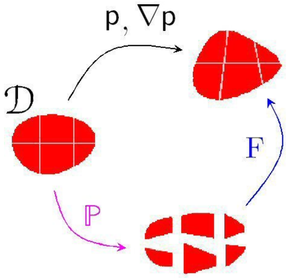
refined motion

$$
\begin{aligned}
(\mathrm{p}, \mathbb{P}) : \mathcal{D} \times \mathscr{T} &\rightarrow \mathscr{E} \times (V\mathscr{E} \otimes V\mathscr{E}) \\
(x, \tau) &\mapsto (\mathrm{p}(x, \tau), \mathbb{P}(x, \tau))
\end{aligned}
$$

($\mathcal{D}$: reference shape, $\mathscr{T}$: time line)

Growth mechanics

Growth as *change in the zero-stress reference state.*



p : *gross placement*

$\nabla \mathsf{p}$: *gradient of the gross placement*

$\mathbb{P}$: *prototype*

F : *warp* (*Kröner-Lee decomposition*)

refined motion

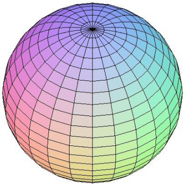
$$(\mathsf{p}, \mathbb{P}) : \mathfrak{D} \times \mathscr{T} \to \mathscr{E} \times (\mathrm{V}\mathscr{E} \otimes \mathrm{V}\mathscr{E})$$
$$(x, \tau) \mapsto (\mathsf{p}(x, \tau), \mathbb{P}(x, \tau))$$

($\mathfrak{D}$: reference shape, $\mathscr{T}$: time line)

Saccular aneurysms

paragon shape $\mathcal{D}$ of the vessel

$$\mathcal{B}(x_o, \xi_+) - \bar{\mathcal{B}}(x_o, \xi_-)$$



spherical coordinates

$$\widehat{\xi}(x), \widehat{\vartheta}(x), \widehat{\varphi}(x)$$

spherically symmetric vector fields

$$\mathrm{v}(x) = v(\xi)\, \mathbf{e}_r(\vartheta, \varphi)$$

spherically symmetric tensor fields

$$\mathrm{L}(x) = \mathrm{L}_r(\xi)\, \mathrm{P}_r(\vartheta, \varphi) + \mathrm{L}_h(\xi)\, \mathrm{P}_h(\vartheta, \varphi)$$

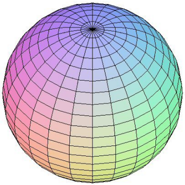
orthogonal projectors

$$\mathrm{P}_r := \mathbf{e}_r \otimes \mathbf{e}_r$$
$$\mathrm{P}_h := \mathrm{I} - \mathrm{P}_r$$

Saccular aneurysms

paragon shape $\mathcal{D}$ of the vessel

$$\mathcal{B}(x_o, \xi_+) - \bar{\mathcal{B}}(x_o, \xi_-)$$



spherical coordinates

$$\widehat{\xi}(x), \widehat{\vartheta}(x), \widehat{\varphi}(x)$$

spherically symmetric vector fields

$$\mathrm{v}(x) = v(\xi)\, \mathbf{e}_r(\vartheta, \varphi)$$

spherically symmetric tensor fields

$$\mathrm{L}(x) = \mathrm{L}_r(\xi)\, \mathsf{P}_r(\vartheta, \varphi) + \mathrm{L}_h(\xi)\, \mathsf{P}_h(\vartheta, \varphi)$$

orthogonal projectors

$$\mathsf{P}_r := \mathbf{e}_r \otimes \mathbf{e}_r$$
$$\mathsf{P}_h := \mathsf{I} - \mathsf{P}_r$$

Geometry & kinematics

gross placement

$$\mathsf{p} = x_o + \rho\ \mathbf{e}_r$$

gradient of the gross placement

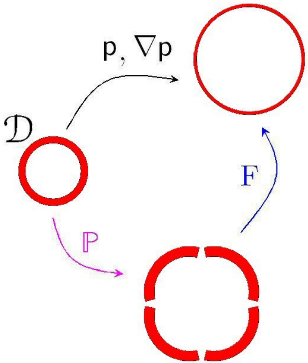
$$\nabla \mathsf{p} = \rho'\ \mathsf{P}_r + \frac{\rho}{\xi}\ \mathsf{P}_h$$

prototype

$$\mathbb{P} = \alpha_r\, \mathsf{P}_r + \alpha_h\, \mathsf{P}_h$$

warp

$$\mathrm{F} := (\nabla \mathsf{p})\, \mathbb{P}^{-1} = \lambda_r\, \mathsf{P}_r + \lambda_h \mathsf{P}_h$$



Refined motion

Refined motion: $(\mathsf{p}, \mathbb{P})$

Refined velocity: $(\dot{\mathsf{p}}, \dot{\mathbb{P}}\,\mathbb{P}^{-1})$

$$\dot{\mathsf{p}} = \dot{\rho}\,\mathbf{e}_{\mathsf{r}}$$

$$\dot{\mathbb{P}}\,\mathbb{P}^{-1} = \frac{\dot{\alpha}_{\mathsf{r}}}{\alpha_{\mathsf{r}}}\,\mathsf{P}_{\mathsf{r}} + \frac{\dot{\alpha}_{\mathsf{h}}}{\alpha_{\mathsf{h}}}\,\mathsf{P}_{\mathsf{h}}$$

Test velocity: $(\mathrm{v}, \mathbb{V})$

$$\mathrm{v} = v\,\mathbf{e}_{\mathsf{r}}$$

$$\mathbb{V} = \mathrm{V}_{\mathsf{r}}\,\mathsf{P}_{\mathsf{r}} + \mathrm{V}_{\mathsf{h}}\,\mathsf{P}_{\mathsf{h}}$$

(*gross velocity* and *growth velocity*)

Working

The basic balance structure of a mechanical theory is encoded in the way in which forces expend *working* on a general test velocity.

$$\int_{\mathcal{D}} \left(\mathbb{A}^{\mathfrak{i}} \cdot \mathbb{V} - \mathrm{S} \cdot \nabla \mathrm{v} \right) + \int_{\mathcal{D}} \mathbb{A}^{\mathfrak{o}} \cdot \mathbb{V} + \int_{\partial\mathcal{D}} \mathrm{t}_{\partial\mathcal{D}} \cdot \mathrm{v}$$

Balance laws

$$\left.\begin{array}{rcl} 2\big(\mathrm{S}_{\mathsf{r}}(\xi)-\mathrm{S}_{\mathsf{h}}(\xi)\big)+\xi\,\mathrm{S}'_{\mathsf{r}}(\xi) & = & 0 \\ \mathrm{A}^{\mathsf{i}}_{\mathsf{r}}(\xi)-\mathrm{A}^{\mathsf{o}}_{\mathsf{r}}(\xi) & = & 0 \\ \mathrm{A}^{\mathsf{i}}_{\mathsf{h}}(\xi)-\mathrm{A}^{\mathsf{o}}_{\mathsf{h}}(\xi) & = & 0 \end{array}\right\} \quad (\xi_{-}<\xi<\xi_{+})$$

$$\mp\mathrm{S}_{\mathsf{r}}(\xi_{\mp}) = t_{\mp}$$

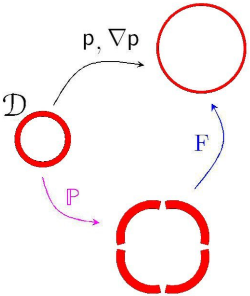
Energetics

$$\Psi(\mathscr{P}) = \int_{\mathscr{P}} \mathrm{J}\,\psi\,, \qquad \mathrm{J} := \det(\mathbb{P}) = \alpha_r\,\alpha_h^2 > 0$$

ψ free energy per unit *prototypal* volume
$\mathrm{J}\,\psi$ free energy per unit *paragon* volume

(H1): the value of the free energy $\psi(x)$ depends solely on the value of the warp $\mathrm{F}(x)$

$$\psi(x) = \phi\,(\,\lambda_r(\xi), \lambda_h(\xi); \xi\,)$$



Characterizing the passive mechanical response

(H2): incompressible elasticity

$$\det \mathrm{F} = \lambda_{\mathsf{r}}\, \lambda_{\mathsf{h}}^2 = 1 \quad \Longleftrightarrow \quad \lambda_{\mathsf{r}} = 1/\lambda_{\mathsf{h}}^2\,.$$

$$\widetilde{\phi} : \lambda \mapsto \phi\left(1/\lambda^2,\, \lambda\right)$$

Fung strain energy density

$$\widetilde{\phi}(\lambda) = (c/\delta)\, \exp\big((\Gamma/2)\,(\lambda^2 - 1)^2\big)$$

[J.D.Humphrey, *Cardiovascular Solid Mechanics*, 2001]

Dissipation principle

$$\mathrm{S} \cdot \nabla \dot{\mathsf{p}} - \mathbb{A}^{\mathsf{i}} \cdot \dot{\mathbb{P}}\, \mathbb{P}^{-1} - (\mathrm{J}\,\psi)^{\cdot} \geq 0$$

$$\mathrm{S}_{\mathsf{r}} = \mathrm{J}\phi_{,\mathsf{r}}/\alpha_{\mathsf{r}} + \overset{+}{\mathrm{S}}_{\mathsf{r}} \qquad \mathrm{S}_{\mathsf{h}} = \mathrm{J}\phi_{,\mathsf{h}}/(2\alpha_{\mathsf{h}}) + \overset{+}{\mathrm{S}}_{\mathsf{h}}$$

$$\mathrm{A}^{\mathsf{i}}_{\mathsf{r}} = \mathrm{J}\,[\,\mathrm{S}_{\mathsf{r}}\,\alpha_{\mathsf{r}}\,\lambda_{\mathsf{r}}/\mathrm{J} - \phi\,] + \overset{+}{\mathrm{A}}_{\mathsf{r}} \qquad \mathrm{A}^{\mathsf{i}}_{\mathsf{h}} = \mathrm{J}\,[\,\mathrm{S}_{\mathsf{h}}\,\alpha_{\mathsf{h}}\,\lambda_{\mathsf{h}}/\mathrm{J} - \phi\,] + \overset{+}{\mathrm{A}}_{\mathsf{h}}$$

reduced dissipation inequality

$$\overset{+}{\mathrm{S}}\,\mathbb{P}^{\top} \cdot \dot{\mathrm{F}} - \overset{+}{\mathbb{A}^{\mathsf{i}}} \cdot \dot{\mathbb{P}}\,\mathbb{P}^{-1} \geq 0$$

$$\overset{+}{\mathrm{S}}\,\mathbb{P}^{\top}\cdot\dot{\mathrm{F}}-\overset{+}{\mathbb{A}}^{i}\cdot\dot{\mathbb{P}}\,\mathbb{P}^{-1}\geq 0$$

In this framework, in a homeostatic state ($\mathbb{P}=\mathsf{I}$), no dissipation is associated with the remodeling.

$$\overset{+}{\mathrm{S}}\,\mathbb{P}^{\top}\cdot\dot{\mathrm{F}} - \overset{+}{\mathbb{A}}^{\mathrm{i}}\cdot\dot{\mathbb{P}}\,\mathbb{P}^{-1} \geq 0$$

In this framework, in a homeostatic state ($\mathbb{P} = \mathrm{I}$), no dissipation is associated with the remodeling.

But...

Even if the relaxed configuration does not evolve, some energy may be dissipated.

How to explain that?
How to deal with that?

Multiple (competing) remodeling mechanisms

(s) *slipping*, (c) *recovery*, and (p) *tissue apposition*

$$\mathbb{P} = \mathbb{P}_p \, \mathbb{P}_c \, \mathbb{P}_s$$

growth velocity

$$\dot{\mathbb{P}} \, \mathbb{P}^{-1} = \dot{\mathbb{P}}_p \, \mathbb{P}_p^{-1} + \dot{\mathbb{P}}_c \, \mathbb{P}_c^{-1} + \dot{\mathbb{P}}_s \, \mathbb{P}_s^{-1}$$

test velocity

$$\mathbb{V} = \mathbb{V}_p + \mathbb{V}_c + \mathbb{V}_s$$

working

$$\int_{\mathcal{D}} \left(\mathbb{A}^i_p \cdot \mathbb{V}_p + \mathbb{A}^i_c \cdot \mathbb{V}_c + \mathbb{A}^i_s \cdot \mathbb{V}_s - \mathrm{S} \cdot \nabla \mathrm{v} \right) + \int_{\mathcal{D}} \left(\mathbb{A}^o_p \cdot \mathbb{V}_p + \mathbb{A}^o_c \cdot \mathbb{V}_c + \mathbb{A}^o_s \cdot \mathbb{V}_s \right) + \int_{\partial\mathcal{D}} \mathrm{t}_{\partial\mathcal{D}} \cdot \mathrm{v}$$

Multiple (competing) remodeling mechanisms

(s) *slipping*, (c) *recovery*, and (p) *tissue apposition*

$$\mathbb{P} = \mathbb{P}_{\mathsf{p}}\,\mathbb{P}_{\mathsf{c}}\,\mathbb{P}_{\mathsf{s}}$$

growth velocity

$$\dot{\mathbb{P}}\,\mathbb{P}^{-1} = \dot{\mathbb{P}}_{\mathsf{p}}\,\mathbb{P}_{\mathsf{p}}^{-1} + \dot{\mathbb{P}}_{\mathsf{c}}\,\mathbb{P}_{\mathsf{c}}^{-1} + \dot{\mathbb{P}}_{\mathsf{s}}\,\mathbb{P}_{\mathsf{s}}^{-1}$$

test velocity

$$\mathbb{V} = \mathbb{V}_{\mathsf{p}} + \mathbb{V}_{\mathsf{c}} + \mathbb{V}_{\mathsf{s}}$$

working

$$\int_{\mathcal{D}} \Big(\mathbb{A}^{\mathfrak{i}}_{\mathsf{p}} \cdot \mathbb{V}_{\mathsf{p}} + \mathbb{A}^{\mathfrak{i}}_{\mathsf{c}} \cdot \mathbb{V}_{\mathsf{c}} + \mathbb{A}^{\mathfrak{i}}_{\mathsf{s}} \cdot \mathbb{V}_{\mathsf{s}} - \mathrm{S} \cdot \nabla \mathrm{v} \Big) + \int_{\mathcal{D}} \big(\mathbb{A}^{\mathfrak{o}}_{\mathsf{p}} \cdot \mathbb{V}_{\mathsf{p}} + \mathbb{A}^{\mathfrak{o}}_{\mathsf{c}} \cdot \mathbb{V}_{\mathsf{c}} + \mathbb{A}^{\mathfrak{o}}_{\mathsf{s}} \cdot \mathbb{V}_{\mathsf{s}} \big) + \int_{\partial\mathcal{D}} \mathrm{t}_{\partial\mathcal{D}} \cdot \mathrm{v}$$

Characterizing the remodeling mechanisms

(H3$_a$): We assume that only $\mathbb{P}_\mathsf{p}$ changes volume, while neither $\mathbb{P}_\mathsf{s}$ nor $\mathbb{P}_\mathsf{c}$ affects volume

$$J := \det(\mathbb{P}) = \det(\mathbb{P}_\mathsf{p}), \quad \det(\mathbb{P}_\mathsf{c}) = 1, \quad \det(\mathbb{P}_\mathsf{s}) = 1$$

(H3$_b$): We assume that tissue apposition is only radial

$$\mathbb{P}_\mathsf{p} = \alpha_\mathsf{r}^\mathsf{p}\, \mathsf{P}_\mathsf{r} + \mathsf{P}_\mathsf{h}$$

$$\mathbb{P}_\mathsf{c} = \alpha_\mathsf{r}^\mathsf{c}\, \mathsf{P}_\mathsf{r} + \alpha_\mathsf{h}^\mathsf{c}\, \mathsf{P}_\mathsf{h} \qquad \alpha_\mathsf{r}^\mathsf{c}(\alpha_\mathsf{h}^\mathsf{c})^2 = 1$$

$$\mathbb{P}_\mathsf{s} = \alpha_\mathsf{r}^\mathsf{s}\, \mathsf{P}_\mathsf{r} + \alpha_\mathsf{h}^\mathsf{s}\, \mathsf{P}_\mathsf{h} \qquad \alpha_\mathsf{r}^\mathsf{s}(\alpha_\mathsf{h}^\mathsf{s})^2 = 1$$

Dissipation principle

$$\mathrm{S}\cdot\nabla\dot{\mathsf{p}} - \left(\mathbb{A}^{\mathsf{i}}_{\mathsf{p}}\cdot\dot{\mathbb{P}}_{\mathsf{p}}\,\mathbb{P}_{\mathsf{p}}^{-1} + \mathbb{A}^{\mathsf{i}}_{\mathsf{c}}\cdot\dot{\mathbb{P}}_{\mathsf{c}}\,\mathbb{P}_{\mathsf{c}}^{-1} + \mathbb{A}^{\mathsf{i}}_{\mathsf{s}}\cdot\dot{\mathbb{P}}_{\mathsf{s}}\,\mathbb{P}_{\mathsf{s}}^{-1}\right) - (\mathrm{J}\,\psi)^{\cdot} \geq 0$$

$$\mathbb{A}^{\mathsf{i}}_{\mathsf{p}} = \left(\mathrm{F}^{\top}\mathrm{S}\,\mathbb{P}^{\top} - \mathrm{J}\,\phi\,\mathsf{I}\right) + \overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{p}}\,,$$

$$\mathbb{A}^{\mathsf{i}}_{\mathsf{c}} = \mathrm{F}^{\top}\mathrm{S}\,\mathbb{P}^{\top} + \overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{c}}\,,$$

$$\mathbb{A}^{\mathsf{i}}_{\mathsf{s}} = \mathrm{F}^{\top}\mathrm{S}\,\mathbb{P}^{\top} + \overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{s}}$$

reduced dissipation inequality

$$\overset{+}{\mathrm{S}}\,\mathbb{P}^{\top}\cdot\dot{\mathrm{F}} - \overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{p}}\cdot\dot{\mathbb{P}}_{\mathsf{p}}\,\mathbb{P}_{\mathsf{p}}^{-1} - \overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{c}}\cdot\dot{\mathbb{P}}_{\mathsf{c}}\,\mathbb{P}_{\mathsf{c}}^{-1} - \overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{s}}\cdot\dot{\mathbb{P}}_{\mathsf{s}}\,\mathbb{P}_{\mathsf{s}}^{-1} \geq 0$$

Characterizing the dissipation mechanisms

(H4): We assume that dissipation is only due to remodeling

$$\overset{+}{\mathrm{S}} = 0$$

$$\overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{p}} = -\mathrm{J}\, D^{\mathsf{p}}_{\mathsf{r}}\, \dot{\alpha}^{\mathsf{p}}_{\mathsf{r}} / \alpha^{\mathsf{p}}_{\mathsf{r}}\ \mathsf{P}_{\mathsf{r}}$$

$$\overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{c}} = -\mathrm{J}(D^{\mathsf{c}}_{\mathsf{r}}\, \dot{\alpha}^{\mathsf{c}}_{\mathsf{r}} / \alpha^{\mathsf{c}}_{\mathsf{r}}\ \mathsf{P}_{\mathsf{r}} + D^{\mathsf{c}}_{\mathsf{h}}\, \dot{\alpha}^{\mathsf{c}}_{\mathsf{h}} / \alpha^{\mathsf{c}}_{\mathsf{h}}\ \mathsf{P}_{\mathsf{h}})$$

$$\overset{+}{\mathbb{A}}{}^{\mathsf{i}}_{\mathsf{s}} = -\mathrm{J}(D^{\mathsf{s}}_{\mathsf{r}}\, \dot{\alpha}^{\mathsf{s}}_{\mathsf{r}} / \alpha^{\mathsf{s}}_{\mathsf{r}}\ \mathsf{P}_{\mathsf{r}} + D^{\mathsf{s}}_{\mathsf{h}}\, \dot{\alpha}^{\mathsf{s}}_{\mathsf{h}} / \alpha^{\mathsf{s}}_{\mathsf{h}}\ \mathsf{P}_{\mathsf{h}})$$

Evolution equations

remodeling laws

$$D^{\mathsf{p}}\,\dot\alpha^{\mathsf{p}}_{\mathsf{r}}/\alpha^{\mathsf{p}}_{\mathsf{r}} = (\mathrm{T}_{\mathsf{r}} - \widetilde{\phi}) + Q^{\mathsf{p}}$$
$$D^{\mathsf{c}}\,\dot\alpha^{\mathsf{c}}_{\mathsf{h}}/\alpha^{\mathsf{c}}_{\mathsf{h}} = (\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{r}}) + Q^{\mathsf{c}}$$
$$D^{\mathsf{s}}\,\dot\alpha^{\mathsf{s}}_{\mathsf{h}}/\alpha^{\mathsf{s}}_{\mathsf{h}} = (\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{r}}) + Q^{\mathsf{s}}$$

$$\mathbb{A}^{o}_{\mathsf{p}}/\mathrm{J} = Q^{\mathsf{p}}_{\mathsf{r}}\,\mathrm{P}_{\mathsf{r}} \qquad \mathbb{A}^{o}_{\mathsf{c}}/\mathrm{J} = Q^{\mathsf{c}}_{\mathsf{r}}\,\mathrm{P}_{\mathsf{r}} + Q^{\mathsf{c}}_{\mathsf{h}}\,\mathrm{P}_{\mathsf{h}} \qquad \mathbb{A}^{o}_{\mathsf{s}}/\mathrm{J} = Q^{\mathsf{s}}_{\mathsf{r}}\,\mathrm{P}_{\mathsf{r}} + Q^{\mathsf{s}}_{\mathsf{h}}\,\mathrm{P}_{\mathsf{h}}$$

$$Q^{\mathsf{p}} := Q^{\mathsf{p}}_{\mathsf{r}} \qquad Q^{\mathsf{c}} := (Q^{\mathsf{c}}_{\mathsf{h}} - Q^{\mathsf{c}}_{\mathsf{r}}) \qquad Q^{\mathsf{s}} := (Q^{\mathsf{s}}_{\mathsf{h}} - Q^{\mathsf{s}}_{\mathsf{r}})$$

$$D^{\mathsf{p}} := D^{\mathsf{p}}_{\mathsf{r}} \qquad D^{\mathsf{c}} := (2D^{\mathsf{c}}_{\mathsf{r}} + D^{\mathsf{c}}_{\mathsf{h}}) \qquad D^{\mathsf{s}} := (2D^{\mathsf{s}}_{\mathsf{r}} + D^{\mathsf{s}}_{\mathsf{h}})$$

$$\mathrm{T}_{\mathsf{r}} = \mathrm{J}^{-1}\mathrm{S}_{\mathsf{r}}\,\alpha_{\mathsf{r}}\,\lambda_{\mathsf{r}} \qquad \mathrm{T}_{\mathsf{h}} = \mathrm{J}^{-1}\mathrm{S}_{\mathsf{h}}\,\alpha_{\mathsf{h}}\,\lambda_{\mathsf{h}}$$

Characterizing controls

($H5_s$): null control on slipping mechanism

$$Q^s = 0$$

($H5_c$): recovery tuned with respect to slipping

$$Q^c = -\left(1 + \frac{D^c}{D^s}\right)\left(g\left(\mathrm{T}_h - \mathrm{T}_r\right) + (1 - g)\left(\mathrm{T}_h^\diamond - \mathrm{T}_r^\diamond\right)\right)$$

($H5_p$): radial apposition driven by hoop stress

$$Q^p = G^p\left(\mathrm{T}_h - \mathrm{T}_h^\diamond\right) - \left(\mathrm{T}_r - \widetilde{\phi}\right).$$

$\mathrm{T}_h^\diamond$, $\mathrm{T}_r^\diamond$: "target" values.

Evolution equations

$$
\begin{aligned}
\dot{\alpha}_{\mathsf{r}}/\alpha_{\mathsf{r}} &= -2\left((\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{r}})(1/D^{\mathsf{c}} + 1/D^{\mathsf{s}}) + Q^{\mathsf{c}}/D^{\mathsf{c}}\right) \\
&\quad + (\mathrm{T}_{\mathsf{r}} - \widetilde{\phi})/D^{\mathsf{p}} + Q^{\mathsf{p}}/D^{\mathsf{p}} \\
\dot{\alpha}_{\mathsf{h}}/\alpha_{\mathsf{h}} &= (\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{r}})(1/D^{\mathsf{c}} + 1/D^{\mathsf{s}}) + Q^{\mathsf{c}}/D^{\mathsf{c}}
\end{aligned}
$$

$$
\begin{aligned}
2\big(\mathrm{S}_{\mathsf{r}}(\xi) - \mathrm{S}_{\mathsf{h}}(\xi)\big) + \xi\,\mathrm{S}_{\mathsf{r}}'(\xi) &= 0 \\
\mp\mathrm{S}_{\mathsf{r}}(\xi_{\mp}) &= t_{\mp}
\end{aligned}
$$

Part III

Numerical simulations

Simulated natural histories

Let us assume that an aneurysm, subjected to a constant intramural pressure $p^\diamond$, has reached a spherical shape in a homeostatic state with hoop and radial stress:

$$\mathrm{T}_\mathsf{h}^\diamond \qquad \mathrm{T}_\mathsf{r}^\diamond$$

thanks to a full recovery control.

Let $Q^\diamond$ be the value of the control Q^c necessary to maintain this homeostatic state:

$$Q^\diamond := -\left(1 + \frac{D^\mathsf{c}}{D^\mathsf{s}}\right)\left(\mathrm{T}_\mathsf{h}^\diamond - \mathrm{T}_\mathsf{r}^\diamond\right)$$

Slipping

$$D^{\mathsf{s}}\,\dot{\alpha}^{\mathsf{s}}_{\mathsf{h}}/\alpha^{\mathsf{s}}_{\mathsf{h}} = (\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{r}}) + Q^{\mathsf{s}}$$

1. passive slipping:

$$Q^{\mathsf{s}}(t) = 0\,,$$

$$\dot{\alpha}^{\mathsf{s}}_{\mathsf{h}}/\alpha^{\mathsf{s}}_{\mathsf{h}} = 1/D^{\mathsf{s}}\,(\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{r}})\,.$$

Recovery

$$D^{c}\,\dot{\alpha}^{c}_{h}/\alpha^{c}_{h} = (T_{h} - T_{r}) + Q^{c}$$

1. sluggish recovery control–*stationary control* ($g = 0$):

$$Q^{c}(t) = Q^{\diamond}\,,$$

$$\dot{\alpha}^{c}_{h}/\alpha^{c}_{h} = 1/D^{c}\,(T_{h} - T_{r}) + Q^{c}/D^{c}\,;$$

2. prompt recovery control–*recovery immediately compensates slipping* ($g = 1$):

$$Q^{c}(t) = -\left(1 + \frac{D^{c}}{D^{s}}\right)(T_{h} - T_{r})\,,$$

$$\dot{\alpha}^{c}_{h}/\alpha^{c}_{h} = -1/D^{s}\,(T_{h} - T_{r})\,.$$

Tissue apposition

$$D^{\mathsf{p}}\,\dot{\alpha}^{\mathsf{p}}_{\mathsf{r}}/\alpha^{\mathsf{p}}_{\mathsf{r}} = (\mathrm{T}_{\mathsf{r}} - \widetilde{\phi}) + Q^{\mathsf{p}}$$

1. sluggish apposition control

$$Q^{\mathsf{p}}(t) = 0\,,$$

$$\dot{\alpha}^{\mathsf{p}}_{\mathsf{r}}/\alpha^{\mathsf{p}}_{\mathsf{r}} = 1/D^{\mathsf{p}}\,(\mathrm{T}_{\mathsf{r}} - \widetilde{\phi})\,;$$

2. apposition control parameterized by G^{p}:

$$Q^{\mathsf{p}}(t) = G^{\mathsf{p}}\big(\mathrm{T}_{\mathsf{h}} - \mathrm{T}^{\diamond}_{\mathsf{h}}\big) - \big(\mathrm{T}_{\mathsf{r}} - \widetilde{\phi}\big)\,,$$

$$\dot{\alpha}^{\mathsf{p}}_{\mathsf{r}}/\alpha^{\mathsf{p}}_{\mathsf{r}} = G^{\mathsf{p}}/D^{\mathsf{p}}\,\big(\mathrm{T}_{\mathsf{h}} - \mathrm{T}^{\diamond}_{\mathsf{h}}\big)\,.$$

History ♯1: slow recovery, null apposition

1. the intramural pressure experiences a short-time bump:

$$p(t) = p^{\diamond} + \delta p(t)\,;$$

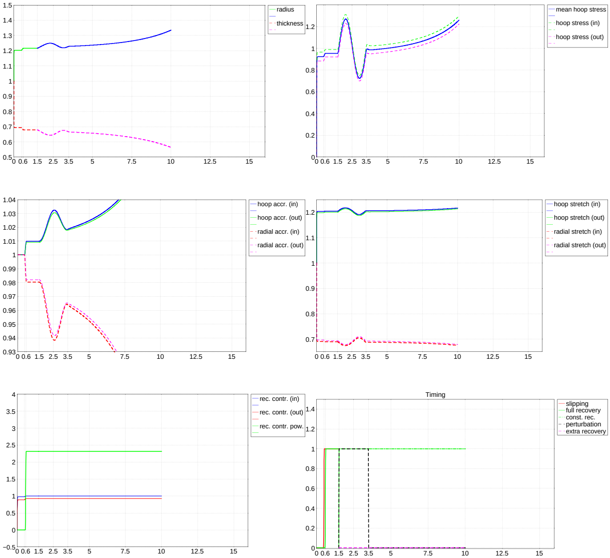
2. Q^{c} is held fixed to the previous value for the rest of the time:

$$Q^{\mathrm{c}}(t) = Q^{\diamond}\,,$$

   simulating the inability of the recovery control to keep pace with a sudden perturbation;

3. null tissue apposition:

$$Q^{\mathrm{p}} = 0\,.$$



SLOW RECOVERY

[case-42-001]

D^c / D^s	1000
D^c	0.01
D^s	1e-005
char time	10
δQ^c ampl	0
δQ^c period	0
δp ampl	0.25
δp period	2
Q^c factor g	0

A recovery control, held fixed to the previous homeostatic value, is unable to keep the aneurysm in a homeostatic state in response to a perturbation of the intramural pressure.

History ♯2: fast recovery, null apposition

1. the intramural pressure experiences a short-time bump:

$$p(t) = p^{\diamond} + \delta p(t)\,;$$

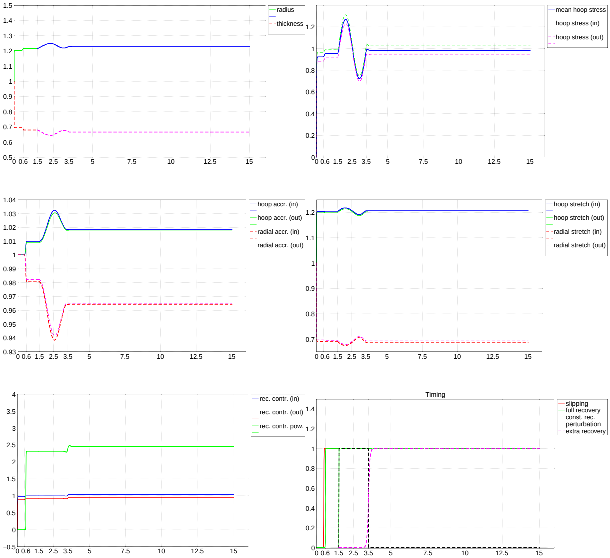
2. then Q^{c} is set to a full recovery control:

$$Q^{\mathrm{c}}(t) = -\left(1 + \frac{D^{\mathrm{c}}}{D^{\mathrm{s}}}\right)(\mathrm{T}_{\mathrm{h}} - \mathrm{T}_{\mathrm{r}})\,,$$

   simulating the capability of the recovery control to immediately keep pace with a sudden perturbation;

3. null tissue apposition:

$$Q^{\mathrm{p}} = 0\,.$$



FAST RECOVERY

[case-41-001]

D^c / D^s D^c D^s	1000 0.01 1e-005
char time	10
δQ^c ampl δQ^c period	0 0
δp ampl δp period	0.25 2
Q^c factor g	1

After the end of a short perturbation of the intramural pressure, a full recovery control drives the aneurysm to a new homeostatic state, with a higher hoop stress.

History ♯3: impaired recovery, null apposition

1. the intramural pressure experiences a short-time bump:

$$p(t) = p^{\diamond} + \delta p(t)\,;$$

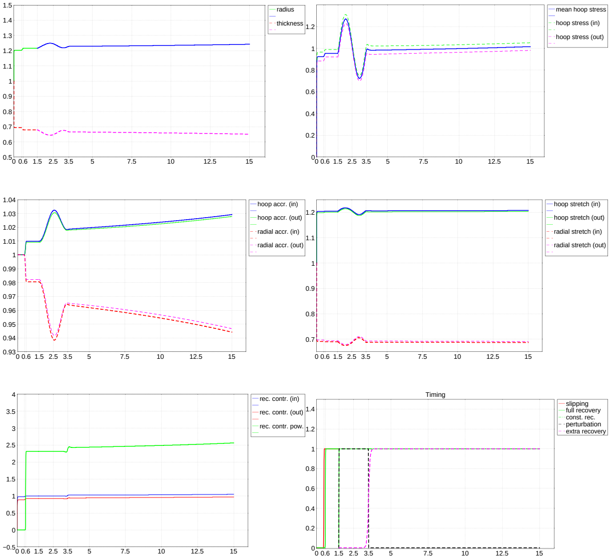
2. then Q^{c} is increased to a fraction of the value of a full recovery control:

$$Q^{\mathsf{c}} = Q^{\diamond} - \mathrm{g}\left(1 + \frac{D^{\mathsf{c}}}{D^{\mathsf{s}}}\right)\left((\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{h}}^{\diamond}) + (\mathrm{T}_{\mathsf{r}} - \mathrm{T}_{\mathsf{r}}^{\diamond})\right)\,,$$

which is meant to simulate an impaired recovery control.

3. null tissue apposition:

$$Q^{\mathsf{p}} = 0\,.$$



IMPAIRED RECOVERY

[case-41-003]

D^c / D^s	1000
D^c	0.01
D^s	1e-005
char time	10
δQ^c ampl	0
δQ^c period	0
δp ampl	0.25
δp period	2
Q^c factor g	0.8

After the end of a short perturbation of the intramural pressure, a recovery control, though higher than the previous homeostatic value but lower than the optimal value, cannot prevent the unlimited increase of the radius.

History ♯4: slow recovery, tissue apposition

1. the intramural pressure experiences a short-time bump:

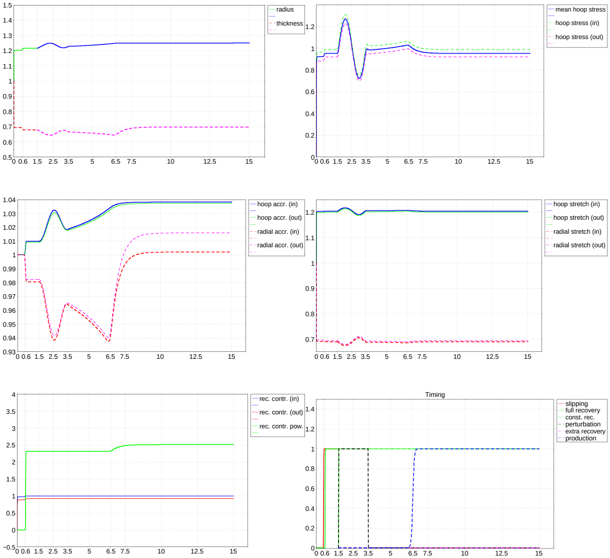
$$p(t) = p^{\diamond} + \delta p(t)\,;$$

2. Q^{c} is held fixed to the previous value for the rest of the time:

$$Q^{\mathsf{c}}(t) = Q^{\diamond}\,;$$

3. radial tissue apposition goes into action through a stress-driven control law:

$$Q^{\mathsf{p}} = G^{\mathsf{p}}\big(\mathrm{T}_{\mathsf{h}} - \mathrm{T}_{\mathsf{h}}^{\diamond}\big) - \big(\mathrm{T}_{\mathsf{r}} - \widetilde{\phi}\big)\,.$$

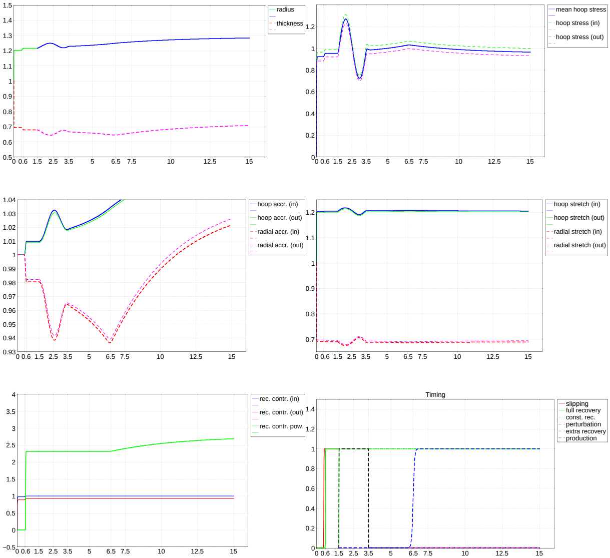


SLOW RECOVERY STRONG APPOSITION

[case-42-004]

D^c/D^s D^c D^s	1000 0.01 1e-005
char time	10
δQ^c ampl δQ^c period	0 0
δp ampl δp period	0.25 2
Q^c factor g	0
G^p D^p	4000 0.002

After the end of a short perturbation of the intramural pressure, radial tissue apposition goes into action making the aneurysm thicken and driving it to a new homeostatic state at the starting value of the hoop stress.



SLOW RECOVERY WEAK APPOSITION

[case-42-005]

D^c/D^s D^c D^s	1000 0.01 1e-005
char time	10
δQ^c ampl δQ^c period	0 0
δp ampl δp period	0.25 2
Q^c factor g	0
G^p D^p	1000 0.002

After the end of a short perturbation of the intramural pressure, a radial tissue apposition goes into action making the aneurysm thicken but failing to drive it quickly to a new homeostatic state.

Summary

- Growth of saccular aneurysms
  - elastic deformation;
  - change of relaxed configuration.
- Multiple remodeling mechanisms
  - *slipping*: only passive;
  - *recovery*: slow/fast control;
  - *tissue apposition*: hoop stress driven control.
- Numerical evidence
  - only recovery control is unable to keep the aneurysm in a homeostatic state;
  - control on tissue apposition plays a central role.

Future work

- Better characterization of material properties
  - evolution of elastic stiffness;
  - non uniform remodeling parameters.
- Weaker assumptions on symmetry
- Quantitative calibration and model validation

Background references

1. A. Di Carlo, S. Quiligotti, Growth and balance, *Mechanics Research Communications*, 29, 449–456, 2002.
    - Conceptual framework of growth and remodeling theory.
2. A. DiCarlo, V. Varano, V. Sansalone, and A. Tatone, Living Shell-Like Structures, in *Applied and Industrial Mathematics In Italy - II*, World Scientific, 2007.
    - First ideas concerning the modeling of saccular aneurysms.
3. M. Tringelová, P. Nardinocchi, L. Teresi, and A. DiCarlo, The cardiovascular system as a smart system, in *Topics on Mathematics for Smart Systems*, World Scientific, 2007.
    - Application to modeling of arterial walls.