MULTILAYERED BEAMS WITH DISTENSIBLE THICKNESS

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Placement and test velocity fields

Placement of each layer

$$\mathbf{x}_i : \mathcal{I} \to \mathcal{E}, \quad \mathbf{l}_i : \mathcal{I} \to \mathcal{V}. \tag{1}$$

Test velocity fields

$$\mathbf{w}_i : \mathcal{I} \to \mathcal{V}, \quad \mathbf{g}_i : \mathcal{I} \to \mathcal{V}, \tag{2}$$

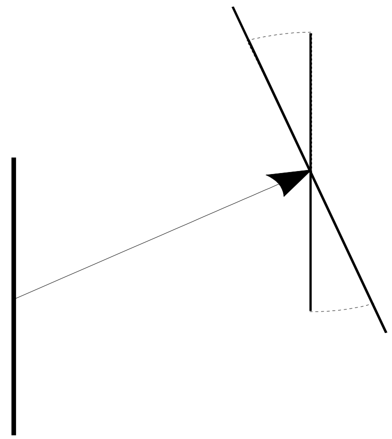
$$\mathbf{g}_i = \delta_i \, \mathbf{l}_i + \boldsymbol{\omega}_i \times \mathbf{l}_i, \tag{3}$$

$\mathbf{w}_i$ velocity,

δ_i thickness stretching,

$\boldsymbol{\omega}_i$ spin.

Affine deformations



Power balance

$$\mathcal{W}^{(ext)} + \mathcal{W}^{(in)} = 0. \tag{4}$$

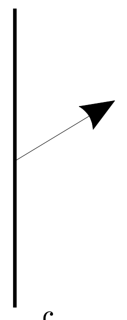
N layers

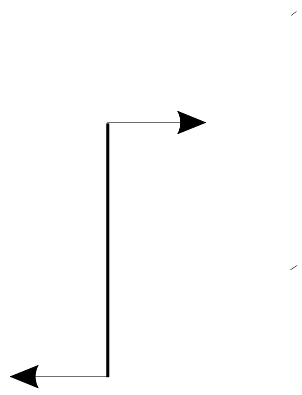
$$\mathcal{W}^{(ext)} := \sum_{i=1}^{N} \int_0^L \big(\mathbf{w}_i \cdot \mathbf{b}_i + \mathbf{g}_i \cdot \mathbf{c}_i\big) d\xi$$
$$+ \sum_{i=1}^{N} \big(\mathbf{w}_i(0) \cdot \mathsf{t}_i^{(0)} + \mathbf{w}_i(L) \cdot \mathsf{t}_i^{(L)} + \mathbf{g}_i(0) \cdot \mathsf{m}_i^{(0)} + \mathbf{g}_i(L) \cdot \mathsf{m}_i^{(L)}\big), \tag{5}$$

$$\mathcal{W}^{(in)} := -\sum_{i=1}^{N} \int_0^L \big(\mathbf{w}_i \cdot \mathfrak{b}_i + \mathbf{g}_i \cdot \mathfrak{c}_i + \mathbf{w}_i' \cdot \mathbf{s}_i + \mathbf{g}_i' \cdot \mathbf{m}_i\big) d\xi. \tag{6}$$

$\mathbf{b}_i$ bulk force, $\mathbf{c}_i$ bulk couple, $\mathbf{s}_i, \mathbf{m}_i$ 1-stress,
$\mathfrak{b}_i, \mathfrak{c}_i$ 0-stress, t_i boundary force, m_i boundary couple.

External interactions





Balance equations

Let us consider test velocity fields (2) without any bonding constraint. The balance equations can be derived from (4). By integration by parts

$$\int_0^L (\mathbf{w}_i' \cdot \mathbf{s}_i)\, d\xi = [\mathbf{w}_i \cdot \mathbf{s}_i]_0^L - \int_0^L (\mathbf{w}_i \cdot \mathbf{s}_i')\, d\xi, \tag{7}$$

$$\int_0^L (\mathbf{g}_i' \cdot \mathbf{m}_i)\, d\xi = [\mathbf{g}_i \cdot \mathbf{m}_i]_0^L - \int_0^L (\mathbf{g}_i \cdot \mathbf{m}_i')\, d\xi, \tag{8}$$

and substitution into (5) and (6), we get for each layer

$$\mathbf{b}_i - \mathfrak{b}_i + \mathbf{s}_i' = 0, \tag{9}$$

$$\mathbf{c}_i - \mathfrak{c}_i + \mathbf{m}_i' = 0, \tag{10}$$

On the boundary

$$\mathsf{t}_i^{(0)} + \mathbf{s}_i(0) = 0, \tag{11}$$

$$\mathsf{t}_i^{(L)} - \mathbf{s}_i(L) = 0, \tag{12}$$

$$\mathsf{m}_i^{(0)} + \mathbf{m}_i(0) = 0, \tag{13}$$

$$\mathsf{m}_i^{(L)} - \mathbf{m}_i(L) = 0. \tag{14}$$

The balance equation (9) and (10) can also be written as

$$\mathbf{b}_i - \mathfrak{b}_i + \mathbf{s}_i{}' = 0, \tag{15}$$

$$\mathbf{l}_i \cdot (\mathbf{c}_i - \mathfrak{c}_i + \mathbf{m}_i') = 0, \tag{16}$$

$$\mathbf{l}_i \times (\mathbf{c}_i - \mathfrak{c}_i + \mathbf{m}_i') = 0. \tag{17}$$

Material objectivity

The inner power be zero for any local rigid test velocity field

$$\sum_{i=1}^{N} \big(\mathbf{w}_i \cdot \mathfrak{b}_i + \mathbf{g}_i \cdot \mathfrak{c}_i + \mathbf{w}_i' \cdot \mathbf{s}_i + \mathbf{g}_i' \cdot \mathbf{m}_i\big) = 0. \tag{18}$$

rigid test velocity field

$$\mathbf{w}_i(\xi) = \bar{\mathbf{w}} + \bar{\boldsymbol{\omega}} \times \big(\mathbf{x}_i(\xi) - \mathbf{x}_0\big), \tag{19}$$

$$\mathbf{g}_i(\xi) = \bar{\boldsymbol{\omega}} \times \mathbf{l}_i(\xi). \tag{20}$$

The condition on the inner power density (18) is

$$\sum_{i=1}^{N} \big(\bar{\mathbf{w}} \cdot \mathfrak{b}_i + \bar{\boldsymbol{\omega}} \times \big(\mathbf{x}_i - \mathbf{x}_0\big) \cdot \mathfrak{b}_i + \bar{\boldsymbol{\omega}} \times \mathbf{l}_i \cdot \mathfrak{c}_i + \bar{\boldsymbol{\omega}} \times \mathbf{x}_i' \cdot \mathbf{s}_i + \bar{\boldsymbol{\omega}} \times \mathbf{l}_i' \cdot \mathbf{m}_i\big) = 0. \tag{21}$$

In order to be true for any $\bar{\boldsymbol{\omega}}$ and $\bar{\mathbf{w}}$, this condition implies

$$\sum_{i=1}^{N} \mathfrak{b}_i = 0, \tag{22}$$

$$\sum_{i=1}^{N} \big(\big(\mathbf{x}_i - \mathbf{x}_0\big) \times \mathfrak{b}_i + \mathbf{l}_i \times \mathfrak{c}_i + \mathbf{x}_i' \times \mathbf{s}_i + \mathbf{l}_i' \times \mathbf{m}_i\big) = 0. \tag{23}$$

0-Stress decomposition

Let us assume that the inner power is made up of two parts: one pertaining to each single layer and another describing the interaction between two layers

$$\begin{aligned} \mathcal{W}^{(in)} = &- \sum_{i=1}^{N} \int_0^L \left(\mathbf{w}_i \cdot \breve{\mathfrak{b}}_i + \mathbf{g}_i \cdot \breve{\mathfrak{c}}_i + \mathbf{w}_i' \cdot \mathbf{s}_i + \mathbf{g}_i' \cdot \mathbf{m}_i \right) d\xi \\ &- \sum_{i=1}^{N-1} \int_0^L \left((\mathbf{w}_{i+1}^- + \mathbf{w}_i^+) \cdot \mathfrak{s}_i + (\mathbf{w}_{i+1}^- - \mathbf{w}_i^+) \cdot \boldsymbol{\tau}_i \right) d\xi, \end{aligned} \tag{24}$$

where

$$\mathbf{w}_i^+ := \mathbf{w}_i + h_i\, \mathbf{g}_i, \tag{25}$$

$$\mathbf{w}_i^- := \mathbf{w}_i - h_i\, \mathbf{g}_i \tag{26}$$

are the upper and the lower surface velocity for each layer.

$\breve{\mathfrak{b}}_i$ dual to $\mathbf{w}_i$;

$\breve{\mathfrak{c}}_i$ dual to $\mathbf{g}_i$;

$\mathfrak{s}_i$ dual to the average velocity of two adjacent layers $i, i+1$;

$\boldsymbol{\tau}_i$ dual to the relative velocity of two adjacent layers $i, i+1$.

For a three layer beam the inner power is

$$
\begin{aligned}
&\sum_{i=1}^{3}\big(\mathbf{w}_i'\cdot\mathbf{s}_i+\mathbf{g}_i'\cdot\mathbf{m}_i\big)\\
&\qquad+\mathbf{w}_1\cdot\big(\breve{\mathfrak{b}}_1+\mathfrak{s}_1-\boldsymbol{\tau}_1\big)+\mathbf{g}_1\cdot\big(\breve{\mathfrak{c}}_1+h_1\,\mathfrak{s}_1-h_1\,\boldsymbol{\tau}_1\big)\\
&\qquad+\mathbf{w}_2\cdot\big(\breve{\mathfrak{b}}_2+\mathfrak{s}_1+\boldsymbol{\tau}_1+\mathfrak{s}_2-\boldsymbol{\tau}_2\big)+\mathbf{g}_2\cdot\big(\breve{\mathfrak{c}}_2-h_2\,\mathfrak{s}_1-h_2\,\boldsymbol{\tau}_1+h_2\,\mathfrak{s}_2-h_2\,\boldsymbol{\tau}_2\big)\\
&\qquad+\mathbf{w}_3\cdot\big(\breve{\mathfrak{b}}_3+\mathfrak{s}_2+\boldsymbol{\tau}_2\big)+\mathbf{g}_3\cdot\big(\breve{\mathfrak{c}}_3-h_3\,\mathfrak{s}_2-h_3\,\boldsymbol{\tau}_2\big).
\end{aligned}
\tag{27}
$$

By comparison with (6)

$$
\mathcal{W}^{(in)}:=-\sum_{i=1}^{N}\int_0^L\big(\mathbf{w}_i\cdot\mathfrak{b}_i+\mathbf{g}_i\cdot\mathfrak{c}_i+\mathbf{w}_i'\cdot\mathbf{s}_i+\mathbf{g}_i'\cdot\mathbf{m}_i\big)d\xi
$$

we get the following decomposition

$$\mathfrak{b}_1=\breve{\mathfrak{b}}_1+\mathfrak{s}_1-\boldsymbol{\tau}_1, \tag{28}$$

$$\mathfrak{b}_2=\breve{\mathfrak{b}}_2+\mathfrak{s}_1+\boldsymbol{\tau}_1+\mathfrak{s}_2-\boldsymbol{\tau}_2, \tag{29}$$

$$\mathfrak{b}_3=\breve{\mathfrak{b}}_3+\mathfrak{s}_2+\boldsymbol{\tau}_2, \tag{30}$$

$$\mathfrak{c}_1=\breve{\mathfrak{c}}_1+h_1\,\mathfrak{s}_1-h_1\,\boldsymbol{\tau}_1, \tag{31}$$

$$\mathfrak{c}_2=\breve{\mathfrak{c}}_2-h_2\,\mathfrak{s}_1-h_2\,\boldsymbol{\tau}_1+h_2\,\mathfrak{s}_2-h_2\,\boldsymbol{\tau}_2; \tag{32}$$

$$\mathfrak{c}_3=\breve{\mathfrak{c}}_3-h_3\,\mathfrak{s}_2-h_3\,\boldsymbol{\tau}_2. \tag{33}$$

Material objectivity (refined)

Each term of the decomposed inner power (24) be zero for any local rigid test velocity field. Let us consider the first part

$$\mathbf{w}_i \cdot \breve{\mathfrak{b}}_i + \mathbf{g}_i \cdot \breve{\mathfrak{c}}_i + \mathbf{w}_i' \cdot \mathbf{s}_i + \mathbf{g}_i' \cdot \mathbf{m}_i = 0. \tag{34}$$

By substitution of (19), (20) we get

$$\bar{\mathbf{w}} \cdot \breve{\mathfrak{b}}_i + \bar{\boldsymbol{\omega}} \times (\mathbf{x}_i - \mathbf{x}_0) \cdot \breve{\mathfrak{b}}_i + \bar{\boldsymbol{\omega}} \times \mathbf{l}_i \cdot \breve{\mathfrak{c}}_i + \bar{\boldsymbol{\omega}} \times \mathbf{x}_i' \cdot \mathbf{s}_i + \bar{\boldsymbol{\omega}} \times \mathbf{l}_i' \cdot \mathbf{m}_i = 0. \tag{35}$$

In order to be true for any $\bar{\mathbf{w}}$ and $\bar{\boldsymbol{\omega}}$, this condition implies

$$\breve{\mathfrak{b}}_i = 0, \tag{36}$$

$$(\mathbf{x}_i - \mathbf{x}_0) \times \breve{\mathfrak{b}}_i + \mathbf{l}_i \times \breve{\mathfrak{c}}_i + \mathbf{x}_i' \times \mathbf{s}_i. + \mathbf{l}_i' \times \mathbf{m}_i = 0, \tag{37}$$

The last one can also be written

$$\mathbf{l}_i \times \breve{\mathfrak{c}}_i + \mathbf{x}_i' \times \mathbf{s}_i + \mathbf{l}_i' \times \mathbf{m}_i = 0. \tag{38}$$

Let us consider the inner power due to the interaction of two layers. For any locally rigid test velocity field this is required to be zero as well

$$\big(\mathbf{w}_{i+1}^{-} + \mathbf{w}_{i}^{+}\big) \cdot \mathfrak{s}_i + \big(\mathbf{w}_{i+1}^{-} - \mathbf{w}_{i}^{+}\big) \cdot \boldsymbol{\tau}_i = 0. \tag{39}$$

Because of (25) and (26) we get

$$\mathbf{w}_i \cdot \big(\mathfrak{s}_i - \boldsymbol{\tau}_i\big) + h_i\, \mathbf{g}_i \cdot \big(\mathfrak{s}_i - \boldsymbol{\tau}_i\big) + \mathbf{w}_{i+1} \cdot \big(\mathfrak{s}_i + \boldsymbol{\tau}_i\big) - h_{i+1}\, \mathbf{g}_{i+1} \cdot \big(\mathfrak{s}_i + \boldsymbol{\tau}_i\big) = 0. \tag{40}$$

By substitution of (19), (20)

$$\begin{aligned} &\bar{\mathbf{w}} \cdot \big(\mathfrak{s}_i - \boldsymbol{\tau}_i\big) + \bar{\mathbf{w}} \cdot \big(\mathfrak{s}_i + \boldsymbol{\tau}_i\big) + \bar{\boldsymbol{\omega}} \times \big(\mathbf{x}_i - \mathbf{x}_0\big) \cdot \big(\mathfrak{s}_i - \boldsymbol{\tau}_i\big) + \bar{\boldsymbol{\omega}} \times \big(\mathbf{x}_{i+1} - \mathbf{x}_0\big) \cdot \big(\mathfrak{s}_i + \boldsymbol{\tau}_i\big) \\ &+ \bar{\boldsymbol{\omega}} \times \mathbf{l}_i \cdot h_i \big(\mathfrak{s}_i - \boldsymbol{\tau}_i\big) - \bar{\boldsymbol{\omega}} \times \mathbf{l}_{i+1} \cdot h_{i+1} \big(\mathfrak{s}_i + \boldsymbol{\tau}_i\big) = 0 \end{aligned} \tag{41}$$

which implies

$$\mathfrak{s}_i = 0, \tag{42}$$

and

$$\big(\mathbf{x}_{i+1} - \mathbf{x}_0\big) \times \boldsymbol{\tau}_i - \big(\mathbf{x}_i - \mathbf{x}_0\big) \times \boldsymbol{\tau}_i - \mathbf{l}_i \times h_i\, \boldsymbol{\tau}_i - \mathbf{l}_{i+1} \times h_{i+1}\, \boldsymbol{\tau}_i = 0. \tag{43}$$

The last condition can be transformed into

$$\big(\mathbf{x}_{i+1} - \mathbf{x}_i\big) \times \boldsymbol{\tau}_i - \big(h_{i+1}\, \mathbf{l}_{i+1} + h_i\, \mathbf{l}_i\big) \times \boldsymbol{\tau}_i = 0, \tag{44}$$

$$\big((\mathbf{x}_{i+1} - h_{i+1}\, \mathbf{l}_{i+1}) - (\mathbf{x}_i + h_i\, \mathbf{l}_i)\big) \times \boldsymbol{\tau}_i = 0, \tag{45}$$

$$\big(\mathbf{x}_{i+1}^{-} - \mathbf{x}_{i}^{+}\big) \times \boldsymbol{\tau}_i = 0. \tag{46}$$

Summary

Balance equations (15), (16), (17)

$$\mathbf{b}_i - \mathfrak{b}_i + \mathbf{s}_i{}' = 0,$$

$$\mathbf{l}_i \cdot (\mathbf{c}_i - \mathfrak{c}_i + \mathbf{m}_i') = 0,$$

$$\mathbf{l}_i \times (\mathbf{c}_i - \mathfrak{c}_i + \mathbf{m}_i') = 0.$$

Conditions derived from objectivity (22), (23), (36), (38), (42), (46),

$$\sum_{i=1}^{N} \mathfrak{b}_i = 0,$$

$$\sum_{i=1}^{N} \big(\mathbf{l}_i \times \mathfrak{c}_i + \mathbf{x}_i' \times \mathbf{s}_i + \mathbf{l}_i' \times \mathbf{m}_i\big) = 0,$$

$$\breve{\mathfrak{b}}_i = 0,$$

$$\mathfrak{s}_i = 0,$$

$$\big(\mathbf{x}_{i+1}^{-} - \mathbf{x}_i^{+}\big) \times \boldsymbol{\tau}_i = 0,$$

$$\mathbf{l}_i \times \breve{\mathfrak{c}}_i + \mathbf{x}_i' \times \mathbf{s}_i + \mathbf{l}_i' \times \mathbf{m}_i = 0.$$

By using the 0-stress decomposition given by (28), (29), (30), (31), (32), (33), into the balance equations (15), (16), (17), we get

$$\mathbf{b}_i - \left(\breve{\mathfrak{b}}_i + \mathfrak{s}_{i-1} + \mathfrak{s}_i + \boldsymbol{\tau}_{i-1} - \boldsymbol{\tau}_i\right) + \mathbf{s}_i' = 0, \tag{47}$$

$$\mathbf{l}_i \cdot \left(\mathbf{c}_i - \breve{\mathfrak{c}}_i + h_i\, \mathfrak{s}_{i-1} - h_i\, \mathfrak{s}_i + h_i\, \boldsymbol{\tau}_{i-1} + h_i\, \boldsymbol{\tau}_i + \mathbf{m}_i'\right) = 0, \tag{48}$$

$$\mathbf{l}_i \times \left(\mathbf{c}_i - \breve{\mathfrak{c}}_i + h_i\, \mathfrak{s}_{i-1} - h_i\, \mathfrak{s}_i + h_i\, \boldsymbol{\tau}_{i-1} + h_i\, \boldsymbol{\tau}_i + \mathbf{m}_i'\right) = 0. \tag{49}$$

The objectivity conditions (36), (38), (42), (46) are

$$\breve{\mathfrak{b}}_i = 0,$$

$$\mathfrak{s}_i = 0,$$

$$\left(\mathbf{x}_{i+1}^- - \mathbf{x}_i^+\right) \times \boldsymbol{\tau}_i = 0,$$

$$\mathbf{l}_i \times \breve{\mathfrak{c}}_i + \mathbf{x}_i' \times \mathbf{s}_i + \mathbf{l}_i' \times \mathbf{m}_i = 0.$$

The reduce form of the balance equations is

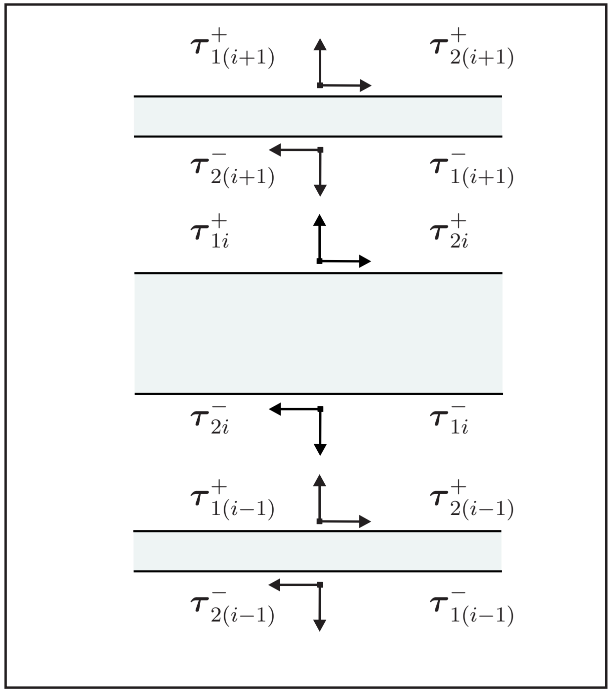
$$\mathbf{b}_i - \boldsymbol{\tau}_{i-1} + \boldsymbol{\tau}_i + \mathbf{s}_i' = 0, \tag{50}$$

$$\mathbf{l}_i \cdot \left(\mathbf{c}_i - \breve{\mathfrak{c}}_i + h_i\,(\boldsymbol{\tau}_{i-1} + \boldsymbol{\tau}_i) + \mathbf{m}_i'\right) = 0, \tag{51}$$

$$\mathbf{l}_i \times \mathbf{c}_i + \mathbf{x}_i' \times \mathbf{s}_i + \left(\mathbf{l}_i \times \mathbf{m}_i\right)' + h_i\, \mathbf{l}_i \times (\boldsymbol{\tau}_{i-1} + \boldsymbol{\tau}_i) = 0. \tag{52}$$

There is one condition still to be satisfied

$$\left(\mathbf{x}_{i+1}^- - \mathbf{x}_i^+\right) \times \boldsymbol{\tau}_i = 0.$$



By defining

$$\boldsymbol{\tau}_i^+ := \boldsymbol{\tau}_i,$$

$$\boldsymbol{\tau}_i^- := -\boldsymbol{\tau}_{i-1},$$

equations (50), (51), (52) can be written as

$$\mathbf{b}_i + (\boldsymbol{\tau}_i^+ + \boldsymbol{\tau}_i^-) + \mathbf{s}_i' = 0, \tag{53}$$

$$\mathbf{l}_i \cdot \big(\mathbf{c}_i - \breve{\mathfrak{c}}_i + h_i\,(\boldsymbol{\tau}_i^+ - \boldsymbol{\tau}_i^-) + \mathbf{m}_i'\big) = 0, \tag{54}$$

$$\mathbf{l}_i \times \mathbf{c}_i + \mathbf{x}_i' \times \mathbf{s}_i + \big(\mathbf{l}_i \times \mathbf{m}_i\big)' + h_i\,\mathbf{l}_i \times (\boldsymbol{\tau}_i^+ - \boldsymbol{\tau}_i^-) = 0. \tag{55}$$

Bonding constraints

$$\mathbf{x}_3^+ = \mathbf{x}_2 + h_2\,\mathbf{l}_2 + 2h_3\,\mathbf{l}_3, \tag{56}$$

$$\mathbf{x}_3 = \mathbf{x}_2 + h_2\,\mathbf{l}_2 + h_3\,\mathbf{l}_3, \tag{57}$$

$$\mathbf{x}_3^- = \mathbf{x}_2 + h_2\,\mathbf{l}_2 = \mathbf{x}_2^+, \tag{58}$$

$$\mathbf{x}_1^+ = \mathbf{x}_2 - h_2\,\mathbf{l}_2 = \mathbf{x}_2^-, \tag{59}$$

$$\mathbf{x}_1 = \mathbf{x}_2 - h_2\,\mathbf{l}_2 - h_1\,\mathbf{l}_1, \tag{60}$$

$$\mathbf{x}_1^- = \mathbf{x}_2 - h_2\,\mathbf{l}_2 - 2h_1\,\mathbf{l}_1. \tag{61}$$

$$\mathbf{w}_3^+ = \mathbf{w}_2 + h_2\,\mathbf{g}_2 + 2h_3\,\mathbf{g}_3, \tag{62}$$

$$\mathbf{w}_3 = \mathbf{w}_2 + h_2\,\mathbf{g}_2 + h_3\,\mathbf{g}_3, \tag{63}$$

$$\mathbf{w}_3^- = \mathbf{w}_2 + h_2\,\mathbf{g}_2 = \mathbf{w}_2^+, \tag{64}$$

$$\mathbf{w}_1^+ = \mathbf{w}_2 - h_2\,\mathbf{g}_2 = \mathbf{w}_2^-, \tag{65}$$

$$\mathbf{w}_1 = \mathbf{w}_2 - h_2\,\mathbf{g}_2 - h_1\,\mathbf{g}_1, \tag{66}$$

$$\mathbf{w}_1^- = \mathbf{w}_2 - h_2\,\mathbf{g}_2 - 2h_3\,\mathbf{g}_1. \tag{67}$$

Balace equations for bonded layers (68)

Power expressions (5)

$$\mathcal{W}^{(ext)} = \sum_{i=1}^{N} \int_0^L \big(\mathbf{w}_i \cdot \mathbf{b}_i + \mathbf{g}_i \cdot \mathbf{c}_i\big) d\xi$$
$$+ \sum_{i=1}^{N} \big(\mathbf{w}_i(0) \cdot \mathsf{t}_i^{(0)} + \mathbf{w}_i(L) \cdot \mathsf{t}_i^{(L)} + \mathbf{g}_i(0) \cdot \mathsf{m}_i^{(0)} + \mathbf{g}_i(L) \cdot \mathsf{m}_i^{(L)}\big),$$

and (24)

$$\mathcal{W}^{(in)} = -\sum_{i=1}^{N} \int_0^L \big(\mathbf{g}_i \cdot \breve{\mathfrak{c}}_i + \mathbf{w}_i' \cdot \mathbf{s}_i + \mathbf{g}_i' \cdot \mathbf{m}_i\big) d\xi.$$

where the 0-stress decomposition and the objectivity conditions (36) and (42) have been used

$$\breve{\mathfrak{b}}_i = 0,$$
$$\mathfrak{s}_i = 0,$$

together with the constraint

$$\mathbf{w}_{i+1}^{-} - \mathbf{w}_i^{+} = 0. \tag{69}$$

For a three layer beam the balance of power is

$$
\begin{aligned}
&\mathcal{W}^{(est)} + \mathcal{W}^{(in)} = \\
&\int_0^L \Big((\mathbf{w}_2 - h_2\,\mathbf{g}_2 - h_1\,\mathbf{g}_1) \cdot (\mathbf{b}_1 + \mathbf{s}_1') + \mathbf{g}_1 \cdot (\mathbf{c}_1 + \mathbf{m}_1' - \breve{\mathfrak{c}}_1) \Big) d\xi \\
&+ (\mathbf{w}_2(0) - h_2\,\mathbf{g}_2(0) - h_1\,\mathbf{g}_1(0)) \cdot (\mathsf{t}_1^{(0)} + \mathbf{s}_1(0)) + \mathbf{g}_1(0) \cdot (\mathsf{m}_1^{(0)} + \mathbf{m}_1(0)) \\
&+ (\mathbf{w}_2(L) - h_2\,\mathbf{g}_2(L) - h_1\,\mathbf{g}_1(L)) \cdot (\mathsf{t}_1^{(L)} - \mathbf{s}_1(L)) + \mathbf{g}_1(L) \cdot (\mathsf{m}_1^{(L)} - \mathbf{m}_1(L)) \\
&+ \int_0^L \Big(\mathbf{w}_2 \cdot (\mathbf{b}_2 + \mathbf{s}_2') + \mathbf{g}_2 \cdot (\mathbf{c}_2 + \mathbf{m}_2' - \breve{\mathfrak{c}}_2) \Big) d\xi \\
&+ \mathbf{w}_2(0) \cdot (\mathsf{t}_2^{(0)} + \mathbf{s}_2(0)) + \mathbf{g}_2(0) \cdot (\mathsf{m}_2^{(0)} + \mathbf{m}_2(0)) \\
&+ \mathbf{w}_2(L) \cdot (\mathsf{t}_2^{(L)} - \mathbf{s}_2(L)) + \mathbf{g}_2(L) \cdot (\mathsf{m}_2^{(L)} - \mathbf{m}_2(L)) \\
&+ \int_0^L \Big((\mathbf{w}_2 + h_2\,\mathbf{g}_2 + h_3\,\mathbf{g}_3) \cdot (\mathbf{b}_3 + \mathbf{s}_3') + \mathbf{g}_3 \cdot (\mathbf{c}_3 + \mathbf{m}_3' - \breve{\mathfrak{c}}_3) \Big) d\xi \\
&+ (\mathbf{w}_2(0) + h_2\,\mathbf{g}_2(0) + h_3\,\mathbf{g}_3(0)) \cdot (\mathsf{t}_3^{(0)} + \mathbf{s}_3(0)) + \mathbf{g}_3(0) \cdot (\mathsf{m}_3^{(0)} + \mathbf{m}_3(0)) \\
&+ (\mathbf{w}_2(L) + h_2\,\mathbf{g}_2(L) + h_3\,\mathbf{g}_3(L)) \cdot (\mathsf{t}_3^{(L)} - \mathbf{s}_3(L)) + \mathbf{g}_3(L) \cdot (\mathsf{m}_3^{(L)} - \mathbf{m}_3(L)) = 0
\end{aligned}
\tag{70}
$$

from which we get the balance equations

$\mathbf{w}_2(\xi)$

$$(\mathbf{b}_1 + \mathbf{s}_1') + (\mathbf{b}_2 + \mathbf{s}_2') + (\mathbf{b}_3 + \mathbf{s}_3') = 0, \tag{71}$$

$\mathbf{g}_1(\xi)$

$$\mathbf{c}_1 + \mathbf{m}_1' - \breve{\mathfrak{c}}_1 - h_1\,(\mathbf{b}_1 + \mathbf{s}_1') = 0, \tag{72}$$

$\mathbf{g}_2(\xi)$

$$\mathbf{c}_2 + \mathbf{m}_2' - \breve{\mathfrak{c}}_2 - h_2\,(\mathbf{b}_1 + \mathbf{s}_1') + h_2\,(\mathbf{b}_3 + \mathbf{s}_3') = 0, \tag{73}$$

$\mathbf{g}_3(\xi)$

$$\mathbf{c}_3 + \mathbf{m}_3' - \breve{\mathfrak{c}}_3 + h_3\,(\mathbf{b}_3 + \mathbf{s}_3') = 0. \tag{74}$$

and the boundary balance equations

$\mathbf{w}_2(0)$
$$\mathsf{t}_1^{(0)} + \mathbf{s}_1(0) + \mathsf{t}_2^{(0)} + \mathbf{s}_2(0) + \mathsf{t}_3^{(0)} + \mathbf{s}_3(0) = 0, \tag{75}$$

$\mathbf{g}_1(0)$
$$\mathsf{m}_1^{(0)} + \mathbf{m}_1(0) - h_1\,\mathsf{t}_1^{(0)} - h_1\,\mathbf{s}_1(0) = 0, \tag{76}$$

$\mathbf{g}_2(0)$
$$\mathsf{m}_2^{(0)} + \mathbf{m}_2(0) - h_2\,\mathsf{t}_1^{(0)} - h_2\,\mathbf{s}_1(0) + h_2\,\mathsf{t}_3^{(0)} + h_2\,\mathbf{s}_3(0) = 0, \tag{77}$$

$\mathbf{g}_3(0)$
$$\mathsf{m}_3^{(0)} + \mathbf{m}_3(0) + h_3\,\mathsf{t}_3^{(0)} + h_3\,\mathbf{s}_3(0) = 0, \tag{78}$$

$\mathbf{w}_2(L)$
$$\mathsf{t}_1^{(L)} - \mathbf{s}_1(L) + \mathsf{t}_2^{(L)} - \mathbf{s}_2(L) + \mathsf{t}_3^{(L)} - \mathbf{s}_3(L) = 0, \tag{79}$$

$\mathbf{g}_1(L)$
$$\mathsf{m}_1^{(L)} - \mathbf{m}_1(L) - h_1\,\mathsf{t}_1^{(L)} + h_1\,\mathbf{s}_1(L) = 0, \tag{80}$$

$\mathbf{g}_2(L)$
$$\mathsf{m}_2^{(L)} - \mathbf{m}_2(L) - h_2\,\mathsf{t}_1^{(L)} + h_2\,\mathbf{s}_1(L) + h_2\,\mathsf{t}_3^{(L)} - h_2\,\mathbf{s}_3(L) = 0, \tag{81}$$

$\mathbf{g}_3(L)$
$$\mathsf{m}_3^{(L)} - \mathbf{m}_3(L) + h_3\,\mathsf{t}_3^{(L)} - h_3\,\mathbf{s}_3(L) = 0. \tag{82}$$

As an alternative the balance equations can be written as

$$\sum_{i=1}^{3}(\mathbf{b}_i + \mathbf{s}'_i) = 0 \tag{83}$$

$$\mathbf{l}_1 \cdot \big(\mathbf{c}_1 - \breve{\mathfrak{c}}_1 + \mathbf{m}'_1 - h_1\,(\mathbf{b}_1 + \mathbf{s}'_1)\big) = 0 \tag{84}$$

$$\mathbf{l}_1 \times \big(\mathbf{c}_1 - \breve{\mathfrak{c}}_1 + \mathbf{m}'_1 - h_1\,(\mathbf{b}_1 + \mathbf{s}'_1)\big) = 0 \tag{85}$$

$$\mathbf{l}_2 \cdot \big(\mathbf{c}_2 - \breve{\mathfrak{c}}_2 + \mathbf{m}'_2 - h_2\,(\mathbf{b}_1 + \mathbf{s}'_1) + h_2\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0 \tag{86}$$

$$\mathbf{l}_2 \times \big(\mathbf{c}_2 - \breve{\mathfrak{c}}_2 + \mathbf{m}'_2 - h_2\,(\mathbf{b}_1 + \mathbf{s}'_1) + h_2\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0 \tag{87}$$

$$\mathbf{l}_3 \cdot \big(\mathbf{c}_3 - \breve{\mathfrak{c}}_3 + \mathbf{m}'_3 + h_3\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0 \tag{88}$$

$$\mathbf{l}_3 \times \big(\mathbf{c}_3 - \breve{\mathfrak{c}}_3 + \mathbf{m}'_3 + h_3\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0. \tag{89}$$

The objectivity condition still to be satisfied is (38)

$$\mathbf{l}_i \times \breve{\mathfrak{c}}_i + \mathbf{x}'_i \times \mathbf{s}_i + \mathbf{l}'_i \times \mathbf{m}_i = 0.$$

From that we get the following balance equations

$$\sum_{i=1}^{3} (\mathbf{b}_i + \mathbf{s}'_i) = 0 \tag{90}$$

$$\mathbf{l}_1 \cdot \big(\mathbf{c}_1 - \breve{\mathfrak{c}}_1 + \mathbf{m}'_1 - h_1\,(\mathbf{b}_1 + \mathbf{s}'_1)\big) = 0 \tag{91}$$

$$\mathbf{x}'_1 \times \mathbf{s}_1 + (\mathbf{l}_1 \times \mathbf{m}_1)' + \mathbf{l}_1 \times \big(\mathbf{c}_1 - h_1\,(\mathbf{b}_1 + \mathbf{s}'_1)\big) = 0 \tag{92}$$

$$\mathbf{l}_2 \cdot \big(\mathbf{c}_2 - \breve{\mathfrak{c}}_2 + \mathbf{m}'_2 - h_2\,(\mathbf{b}_1 + \mathbf{s}'_1) + h_2\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0 \tag{93}$$

$$\mathbf{x}'_2 \times \mathbf{s}_2 + (\mathbf{l}_2 \times \mathbf{m}_2)' + \mathbf{l}_2 \times \big(\mathbf{c}_2 - h_2\,(\mathbf{b}_1 + \mathbf{s}'_1) + h_2\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0 \tag{94}$$

$$\mathbf{l}_3 \cdot \big(\mathbf{c}_3 - \breve{\mathfrak{c}}_3 + \mathbf{m}'_3 + h_3\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0 \tag{95}$$

$$\mathbf{x}'_3 \times \mathbf{s}_3 + (\mathbf{l}_3 \times \mathbf{m}_3)' + \mathbf{l}_3 \times \big(\mathbf{c}_3 + h_3\,(\mathbf{b}_3 + \mathbf{s}'_3)\big) = 0, \tag{96}$$

together with the balance equations at the left boundary

$$\sum_{i=1}^{3} \big(\mathsf{t}_i^{(0)} + \mathbf{s}_i(0)\big) = 0 \tag{97}$$

$$\mathbf{l}_1 \cdot \big(- h_1\, \mathsf{t}_1^{(0)} + \mathsf{m}_1^{(0)} - h_1\, \mathbf{s}_1(0) + \mathbf{m}_1(0)\big) = 0 \tag{98}$$

$$\mathbf{l}_1 \times \big(- h_1\, \mathsf{t}_1^{(0)} + \mathsf{m}_1^{(0)} - h_1\, \mathbf{s}_1(0) + \mathbf{m}_1(0)\big) = 0 \tag{99}$$

$$\mathbf{l}_2 \cdot \big(- h_2\, \mathsf{t}_1^{(0)} + h_2\, \mathsf{t}_3^{(0)} + \mathsf{m}_2^{(0)} - h_2\, \mathbf{s}_1(0) + h_2\, \mathbf{s}_3(0) + \mathbf{m}_2(0)\big) = 0 \tag{100}$$

$$\mathbf{l}_2 \times \big(- h_2\, \mathsf{t}_1^{(0)} + h_2\, \mathsf{t}_3^{(0)} + \mathsf{m}_2^{(0)} - h_2\, \mathbf{s}_1(0) + h_2\, \mathbf{s}_3(0) + \mathbf{m}_2(0)\big) = 0 \tag{101}$$

$$\mathbf{l}_3 \cdot \big(h_3\, \mathsf{t}_3^{(0)} + \mathsf{m}_3^{(0)} + h_3\, \mathbf{s}_3(0) + \mathbf{m}_3(0)\big) = 0 \tag{102}$$

$$\mathbf{l}_3 \times \big(h_3\, \mathsf{t}_3^{(0)} + \mathsf{m}_3^{(0)} + h_3\, \mathbf{s}_3(0) + \mathbf{m}_3(0)\big) = 0; \tag{103}$$

and the balance equations at the right boundary

$$\sum_{i=1}^{3}\big(\mathsf{t}_i^{(L)}-\mathbf{s}_i(L)\big)=0 \tag{104}$$

$$\mathbf{l}_1\cdot\big(-h_1\,\mathsf{t}_1^{(L)}+\mathsf{m}_1^{(L)}+h_1\,\mathbf{s}_1(L)-\mathbf{m}_1(L)\big)=0 \tag{105}$$

$$\mathbf{l}_1\times\big(-h_1\,\mathsf{t}_1^{(L)}+\mathsf{m}_1^{(L)}+h_1\,\mathbf{s}_1(L)-\mathbf{m}_1(L)\big)=0 \tag{106}$$

$$\mathbf{l}_2\cdot\big(-h_2\,\mathsf{t}_1^{(L)}+h_2\,\mathsf{t}_3^{(L)}+\mathsf{m}_2^{(L)}+h_2\,\mathbf{s}_1(L)-h_2\,\mathbf{s}_3(L)-\mathbf{m}_2(L)\big)=0 \tag{107}$$

$$\mathbf{l}_2\times\big(-h_2\,\mathsf{t}_1^{(L)}+h_2\,\mathsf{t}_3^{(L)}+\mathsf{m}_2^{(L)}+h_2\,\mathbf{s}_1(L)-h_2\,\mathbf{s}_3(L)-\mathbf{m}_2(L)\big)=0 \tag{108}$$

$$\mathbf{l}_3\cdot\big(h_3\,\mathsf{t}_3^{(L)}+\mathsf{m}_3^{(L)}-h_3\,\mathbf{s}_3(L)-\mathbf{m}_3(L)\big)=0 \tag{109}$$

$$\mathbf{l}_3\times\big(h_3\,\mathsf{t}_3^{(L)}+\mathsf{m}_3^{(L)}-h_3\,\mathbf{s}_3(L)-\mathbf{m}_3(L)\big)=0. \tag{110}$$

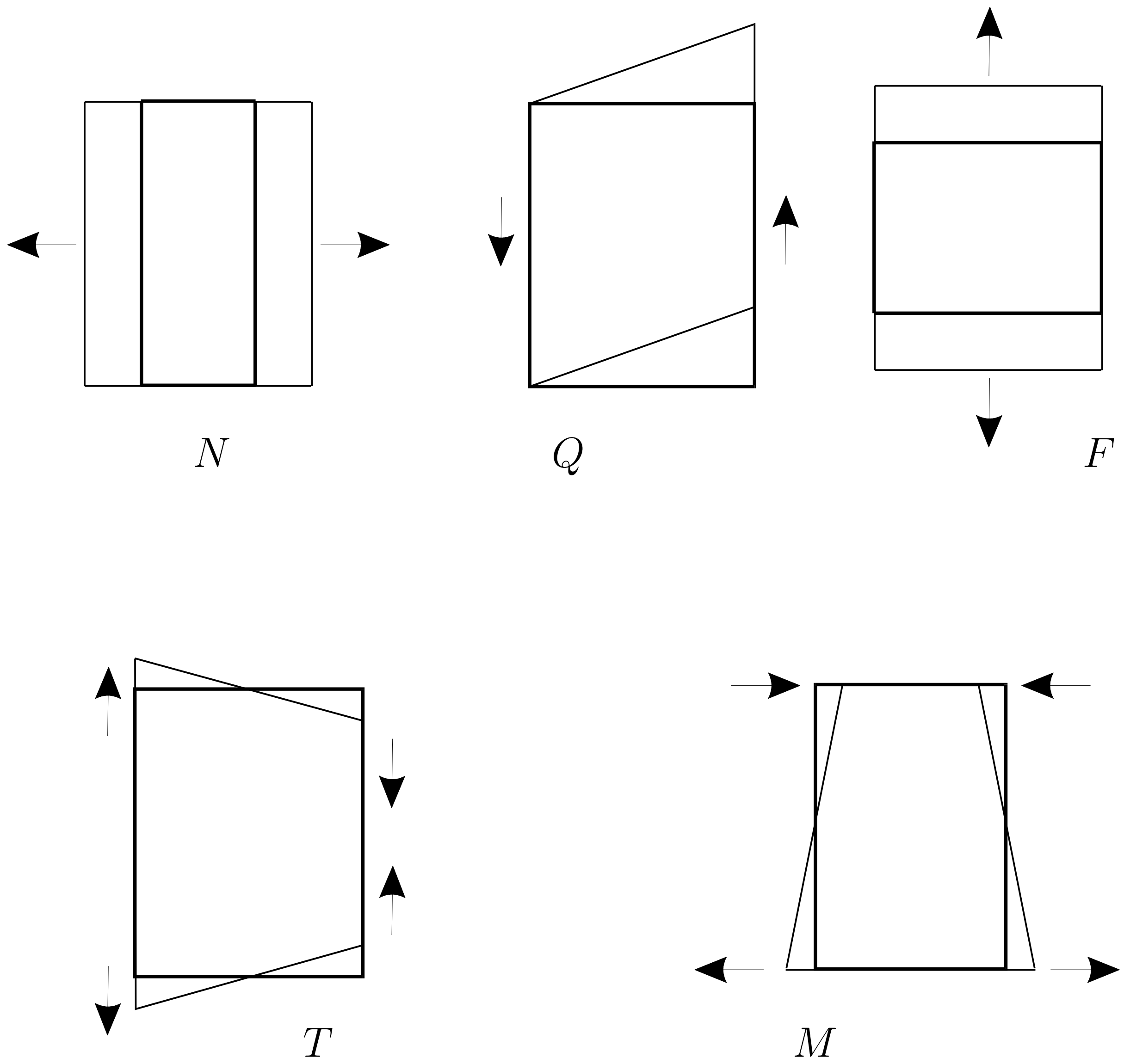
Scalar quantities

$$\mathcal{W}^{(est)} = \sum_{i=1}^{N} \int_0^L \big(\mathbf{w}_i \cdot \mathbf{b}_i + \mathbf{g}_i \cdot \mathbf{c}_i\big) d\xi + \sum_{i=1}^{N} \big(\mathbf{w}_i(0) \cdot \mathsf{t}_i^{(0)} + \mathbf{w}_i(L) \cdot \mathsf{t}_i^{(L)} + \mathbf{g}_i(0) \cdot \mathsf{m}_i^{(0)} + \mathbf{g}_i(L) \cdot \mathsf{m}_i^{(L)}\big)$$

$$\mathcal{W}^{(in)} = -\sum_{i=1}^{N} \int_0^L \big(\mathbf{g}_i \cdot \breve{\mathfrak{c}}_i + \mathbf{w}_i' \cdot \mathbf{s}_i + \mathbf{g}_i' \cdot \mathbf{m}_i\big) d\xi$$

$$\begin{aligned}
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathbf{s}_i &= (-\mathbf{n}_i) \cdot \mathbf{s}_i = -N_i(\xi), \\
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathbf{m}_i &= M_i(\xi), \\
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathsf{t}_i^{(0)} &= -N_i^{(0)}, \\
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathsf{t}_i^{(L)} &= -N_i^{(L)}, \\
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathsf{m}_i^{(0)} &= M_i^{(0)}, \\
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathsf{m}_i^{(L)} &= M_i^{(L)}, \\
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathbf{b}_i &= -b_{1i}(\xi), \\
\mathbf{e}_3 \cdot \mathbf{l}_i \times \mathbf{c}_i &= c_i(\xi),
\end{aligned}$$

$$\begin{aligned}
\mathbf{l}_i \cdot \mathbf{s}_i &= Q_i(\xi), \\
\mathbf{l}_i \cdot \mathbf{m}_i &= T_i(\xi), \\
\mathbf{l}_i \cdot \mathsf{t}_i^{(0)} &= Q_i^{(0)}, \\
\mathbf{l}_i \cdot \mathsf{t}_i^{(L)} &= Q_i^{(L)}, \\
\mathbf{l}_i \cdot \mathsf{m}_i^{(0)} &= T_i^{(0)}, \\
\mathbf{l}_i \cdot \mathsf{m}_i^{(L)} &= T_i^{(L)}, \\
\mathbf{l}_i \cdot \mathbf{b}_i &= b_{2i}(\xi), \\
\mathbf{l}_i \cdot \mathbf{c}_i &= P_i(\xi), \\
\mathbf{l}_i \cdot \mathfrak{c}_i &= F_i(\xi).
\end{aligned}$$


N
Q
F
T
M

Linearized scalar balance equations

From equations (90) – (96) we get

$$\sum_{i=1}^{3}\left(b_{2i}+Q_i'\right)=0, \tag{111}$$

$$\sum_{i=1}^{3}\left(b_{1i}+N_i'\right)=0, \tag{112}$$

$$P_1-F_1+T_1'-h_1b_{21}-h_1Q_1'=0, \tag{113}$$

$$P_2-F_2+T_2'-h_2b_{21}-h_2Q_1'+h_2b_{23}+h_2Q_3'=0, \tag{114}$$

$$P_3-F_3+T_3'+h_3b_{23}+h_3Q_3'=0, \tag{115}$$

$$c_1+Q_1+M_1'-h_1b_{11}-h_1N_1'=0, \tag{116}$$

$$c_2+Q_2+M_2'-h_2b_{11}+h_2N_1'+h_2b_{13}-h_2N_3'=0, \tag{117}$$

$$c_3+Q_3+M_3'-h_3b_{13}+h_3N_3'=0. \tag{118}$$

left boundary conditions

$$\sum_{i=1}^{3}(Q_i^{(0)} + Q_i(0)) = 0 \tag{119}$$

$$\sum_{i=1}^{3}(N_i^{(0)} + N_i(0)) = 0 \tag{120}$$

$$T_1^{(0)} - h_1 Q_1^{(0)} + T_1(0) - h_1 Q_1(0) = 0 \tag{121}$$

$$M_1^{(0)} + h_1 N_1^{(0)} + M_1(0) + h_1 N_1(0) = 0 \tag{122}$$

$$T_2^{(0)} - h_2 Q_1^{(0)} + h_2 Q_3^{(0)} + T_2(0) - h_2 Q_1(0) + h_2 Q_3(0) = 0 \tag{123}$$

$$M_2^{(0)} + h_2 N_1^{(0)} - h_2 N_3^{(0)} + M_2(0) + h_2 N_1(0) - h_2 N_3(0) = 0 \tag{124}$$

$$T_3^{(0)} + h_3 Q_3^{(0)} + T_3(0) + h_3 Q_3(0) = 0 \tag{125}$$

$$M_3^{(0)} - h_3 N_3^{(0)} + M_3(0) - h_3 N_3(0) = 0; \tag{126}$$

right boundary conditions

$$\sum_{i=1}^{3}(Q_i^{(L)} - Q_i(L)) = 0 \tag{127}$$

$$\sum_{i=1}^{3}(N_i^{(L)} - N_i(L)) = 0 \tag{128}$$

$$T_1^{(L)} - h_1 Q_1^{(L)} - T_1(L) + h_1 Q_1(L) = 0 \tag{129}$$

$$M_1^{(L)} + h_1 N_1^{(L)} - M_1(L) - h_1 N_1(L) = 0 \tag{130}$$

$$T_2^{(L)} - h_2 Q_1^{(L)} + h_2 Q_3^{(L)} - T_2(L) + h_2 Q_1(L) - h_2 Q_3(L) = 0 \tag{131}$$

$$M_2^{(L)} + h_2 N_1^{(L)} - h_2 N_3^{(L)} - M_2(L) - h_2 N_1(L) + h_2 N_3(L) = 0 \tag{132}$$

$$T_3^{(L)} + h_3 Q_3^{(L)} - T_3(L) - h_3 Q_3(L) = 0 \tag{133}$$

$$M_3^{(L)} - h_3 N_3^{(L)} - M_3(L) + h_3 N_3(L) = 0. \tag{134}$$

Reduced inner power (affine microstructure)

Inner power density

$$\mathbf{w} \cdot \mathfrak{b} + \mathbf{g} \cdot \mathfrak{c} + \mathbf{w}' \cdot \mathbf{s} + \dot{\mathbf{g}}' \cdot \mathbf{m} \tag{135}$$

Objectivity

$$\begin{aligned} \mathfrak{b} &= 0 \\ \mathbf{l} \times \mathfrak{c} &= -\mathbf{x}' \times \mathbf{s} - \mathbf{l}' \times \mathbf{m} \end{aligned} \tag{136}$$

By substituting

$$\mathbf{g} \cdot \mathfrak{c} = \big(\delta\, \mathbf{l} + \boldsymbol{\omega} \times \mathbf{l}\big) \cdot \mathfrak{c} = \delta\, \mathbf{l} \cdot \mathfrak{c} - \boldsymbol{\omega} \times \mathbf{x}' \cdot \mathbf{s} - \boldsymbol{\omega} \times \mathbf{l}' \cdot \mathbf{m} \tag{137}$$

$$\mathbf{g}' = \big(\delta\, \mathbf{l} + \boldsymbol{\omega} \times \mathbf{l}\big)' = \delta'\, \mathbf{l} + \delta\, \mathbf{l}' + \boldsymbol{\omega}' \times \mathbf{l} + \boldsymbol{\omega} \times \mathbf{l}', \tag{138}$$

we get the following expression for the inner power

$$\mathbf{g} \cdot \mathfrak{c} + \mathbf{w}' \cdot \mathbf{s} + \mathbf{g}' \cdot \mathbf{m} = \delta\, \big(\mathbf{l} \cdot \mathfrak{c} + \mathbf{l}' \cdot \mathbf{m}\big) + \big(\mathbf{w}' - \boldsymbol{\omega} \times \mathbf{x}'\big) \cdot \mathbf{s} + \delta'\, \mathbf{l} \cdot \mathbf{m} + \boldsymbol{\omega}' \cdot \mathbf{l} \times \mathbf{m}. \tag{139}$$

Linearizing near a natural state and using the components we arrive at

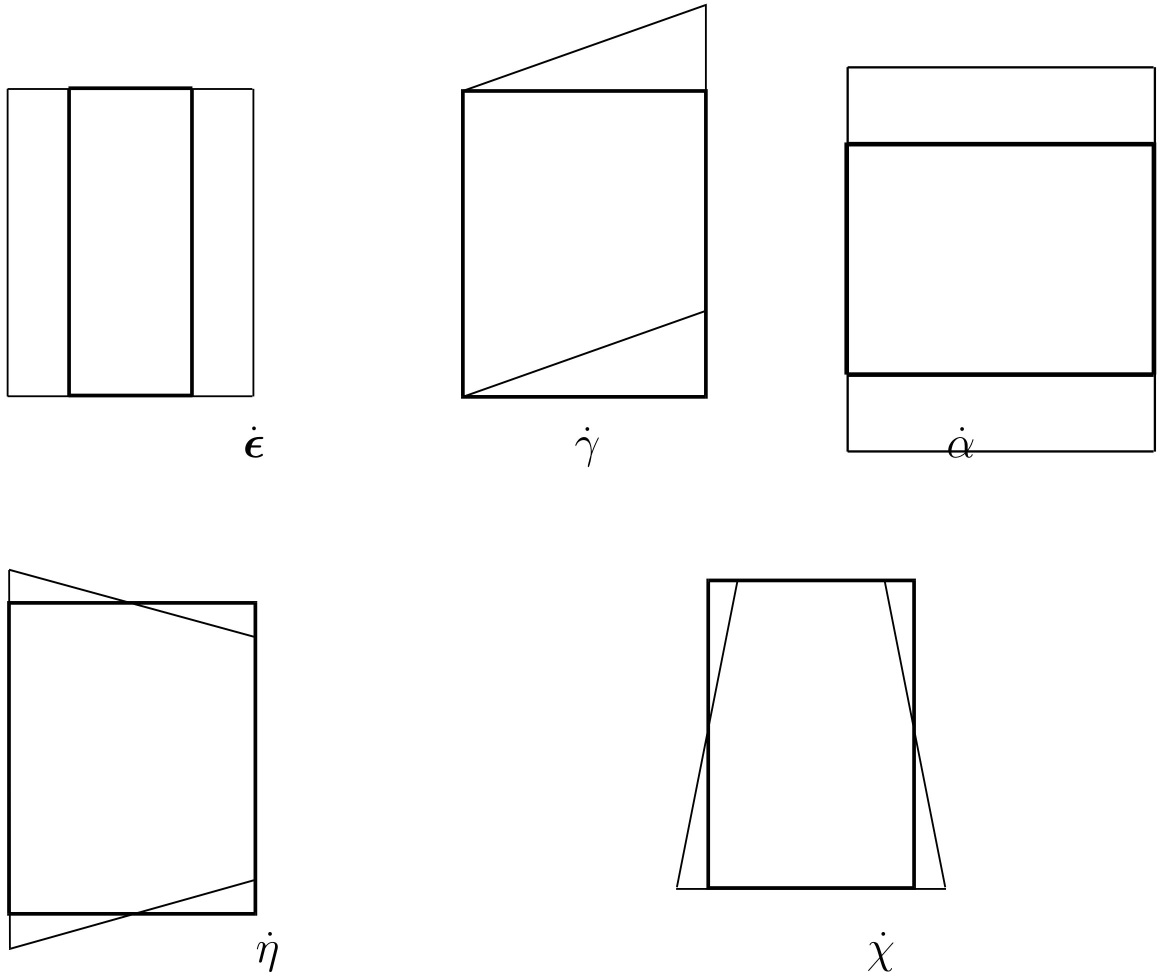
$$F\,\delta + N\,w_1' + Q\,(w_2' - \omega) + T\,\delta' + M\,\omega', \tag{140}$$

which can be interpreted as

$$F\,\dot{\alpha} + N\,\dot{\varepsilon} + Q\,\dot{\gamma} + T\,\dot{\eta} + M\,\dot{\chi}. \tag{141}$$

Streching

inner power: $N\dot{\epsilon} + Q\dot{\gamma} + F\dot{\alpha} + T\dot{\eta} + M\dot{\chi}$



2-Dimensional Cauchy continuum

Test velocity fields

$$\dot{u}_1(\xi,\zeta) = \dot{v}_1(\xi) - \dot{\theta}(\xi)\zeta, \tag{142}$$

$$\dot{u}_2(\xi,\zeta) = \dot{v}_2(\xi) + \dot{\alpha}(\xi)\zeta \tag{143}$$

Velocity gradient

$$\frac{\partial}{\partial\,\xi}\,\dot{u}_1(\xi,\zeta) = \dot{v}_1'(\xi) - \dot{\theta}'(\xi)\zeta, \tag{144}$$

$$\frac{\partial}{\partial\,\xi}\,\dot{u}_2(\xi,\zeta) = \dot{v}_2'(\xi) + \dot{\alpha}'(\xi)\zeta \tag{145}$$

$$\frac{\partial}{\partial\,\zeta}\,\dot{u}_1(\xi,\zeta) = -\dot{\theta}(\xi), \tag{146}$$

$$\frac{\partial}{\partial\,\zeta}\,\dot{u}_2(\xi,\zeta) = \dot{\alpha}(\xi) \tag{147}$$

Planar deformation

$$T\cdot G \;=\begin{pmatrix}\sigma_1 & \tau & 0\\ \tau & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial\xi}\,\dot{u}_1 & \frac{\partial}{\partial\zeta}\,\dot{u}_1 & 0\\ \frac{\partial}{\partial\xi}\,\dot{u}_2 & \frac{\partial}{\partial\zeta}\,\dot{u}_2 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{148}$$

$$T\cdot G \;= \sigma_1\big(\dot{v}_1'(\xi) - \zeta\dot{\theta}'(\xi)\big) + \tau\big(\dot{v}_2'(\xi) + \zeta\dot{\alpha}'(\xi) - \dot{\theta}(\xi)\big) + \sigma_2\dot{\alpha}(\xi) \tag{149}$$

$$
\begin{aligned}
-\mathcal{W}^{(in)} := \int_{-h}^{h} \big(T \cdot G\big)\, d\zeta \;&=\; \dot{v}_1' \int_{-h}^{h} \sigma_1\, d\zeta + \big(\dot{v}_2' - \dot{\theta}\big) \int_{-h}^{h} \tau\, d\zeta - \dot{\theta}' \int_{-h}^{h} \sigma_1 \zeta\, d\zeta \\
&\quad + \dot{\alpha}' \int_{-h}^{h} \tau \zeta\, d\zeta + \dot{\alpha} \int_{-h}^{h} \sigma_2\, d\zeta
\end{aligned}
\tag{150}
$$

$$
\begin{aligned}
-\mathcal{W}^{(in)} := \int_{-h}^{h} \big(T \cdot G\big)\, d\zeta \;&=\; \dot{\varepsilon} \int_{-h}^{h} \sigma_1\, d\zeta + \dot{\gamma} \int_{-h}^{h} \tau\, d\zeta - \dot{\chi} \int_{-h}^{h} \sigma_1 \zeta\, d\zeta \\
&\quad + \dot{\eta} \int_{-h}^{h} \tau \zeta\, d\zeta + \dot{\alpha} \int_{-h}^{h} \sigma_2\, d\zeta
\end{aligned}
\tag{151}
$$

Stress identification

Comparing the two expressions for the inner power we get

$$N\,\dot\varepsilon + Q\,\dot\gamma + M\,\dot\chi + T\,\dot\eta + F\,\dot\alpha \;=\; \dot\varepsilon \int_{-h}^{h} \sigma_1\,d\zeta + \dot\gamma \int_{-h}^{h} \tau\,d\zeta - \dot\chi \int_{-h}^{h} \sigma_1\zeta\,d\zeta + \dot\eta \int_{-h}^{h} \tau\zeta\,d\zeta + \dot\alpha \int_{-h}^{h} \sigma_2\,d\zeta \tag{152}$$

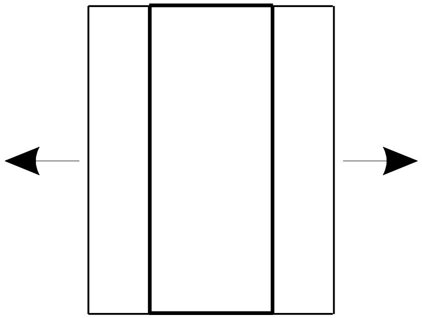
$$N = \int_{-h}^{h} \sigma_1\,d\zeta, \tag{153}$$

$$Q = \int_{-h}^{h} \tau\,d\zeta, \tag{154}$$

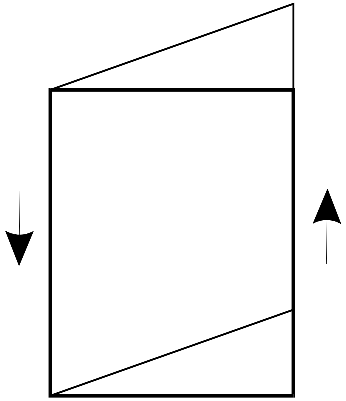
$$-M = \int_{-h}^{h} \sigma_1\zeta\,d\zeta, \tag{155}$$

$$T = \int_{-h}^{h} \tau\zeta\,d\zeta, \tag{156}$$

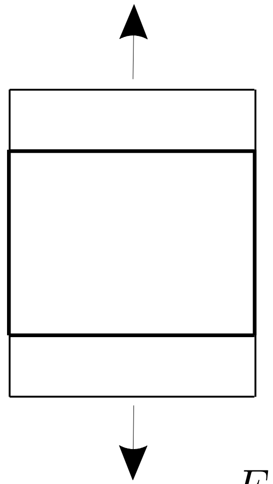
$$F = \int_{-h}^{h} \sigma_2\,d\zeta. \tag{157}$$



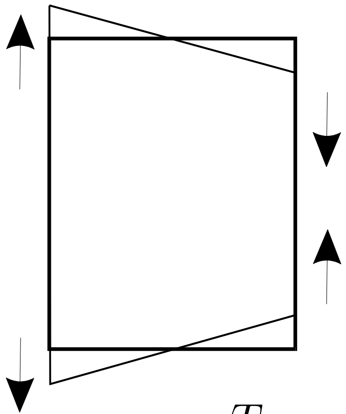
N



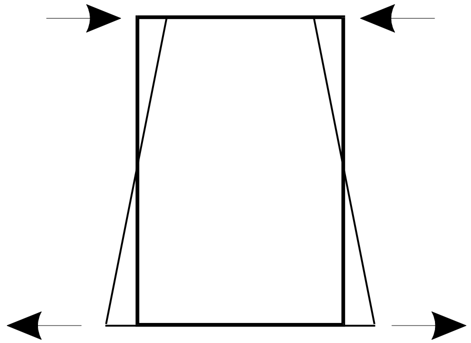
Q



F



T



M

Stress identification (planar deformations)

$$\mathbf{T} = \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \tag{158}$$

$$\begin{pmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{pmatrix} = \lambda(\varepsilon_{11} + \varepsilon_{22} + 0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\mu \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{159}$$

$$\sigma_1 = (\lambda + 2\mu)\,\varepsilon_{11} + \lambda\,\varepsilon_{22}, \tag{160}$$

$$\sigma_2 = \lambda\,\varepsilon_{11} + (\lambda + 2\mu)\,\varepsilon_{22}, \tag{161}$$

$$\sigma_3 = \lambda\,\varepsilon_{11} + \lambda\,\varepsilon_{22}, \tag{162}$$

$$\tau_{12} = 2\mu\varepsilon_{12}, \tag{163}$$

$$\tau_{13} = \tau_{23} = 0, \tag{164}$$

$$\tag{165}$$

$$T(\xi) = \int_{-h}^{h} \tau(\xi,\zeta)\zeta\, d\zeta = 2\mu \int_{-h}^{h} \varepsilon_{12}(\xi,\zeta)\zeta\, d\zeta \tag{166}$$

$$F(\xi) = \int_{-h}^{h} \sigma_2(\xi,\zeta)\, d\zeta = \lambda \int_{-h}^{h} \varepsilon_{11}(\xi,\zeta)\, d\zeta + (\lambda + 2\mu) \int_{-h}^{h} \varepsilon_{22}(\xi,\zeta)\, d\zeta \tag{167}$$

$$-M(\xi) = \int_{-h}^{h} \sigma_1(\xi,\zeta)\zeta\, d\zeta = (\lambda + 2\mu) \int_{-h}^{h} \varepsilon_{11}(\xi,\zeta)\zeta\, d\zeta + \lambda \int_{-h}^{h} \varepsilon_{22}(\xi,\zeta)\zeta\, d\zeta \tag{168}$$

$$N(\xi) = \int_{-h}^{h} \sigma_1(\xi,\zeta)\, d\zeta = (\lambda + 2\mu) \int_{-h}^{h} \varepsilon_{11}(\xi,\zeta)\, d\zeta + \lambda \int_{-h}^{h} \varepsilon_{22}(\xi,\zeta)\, d\zeta \tag{169}$$

$$Q(\xi) = \int_{-h}^{h} \tau(\xi,\zeta)\, d\zeta = 2\mu \int_{-h}^{h} \varepsilon_{12}(\xi,\zeta)\, d\zeta \tag{170}$$

Affine deformations

$$u_1(\xi,\zeta) = v_1(\xi) - \theta(\xi)\zeta, \tag{171}$$
$$u_2(\xi,\zeta) = v_2(\xi) + \alpha(\xi)\zeta \tag{172}$$

$$E := \operatorname{sym} \nabla u \tag{173}$$

$$\varepsilon_{11}(\xi,\zeta) = v_1'(\xi) - \theta'(\xi)\zeta = \varepsilon(\xi) - \chi(\xi)\zeta, \tag{174}$$
$$2\varepsilon_{12}(\xi,\zeta) = v_2'(\xi) - \theta(\xi) + \alpha'(\xi)\zeta = \gamma(\xi) + \eta(\xi)\zeta, \tag{175}$$
$$\varepsilon_{22}(\xi,\zeta) = \alpha(\xi). \tag{176}$$

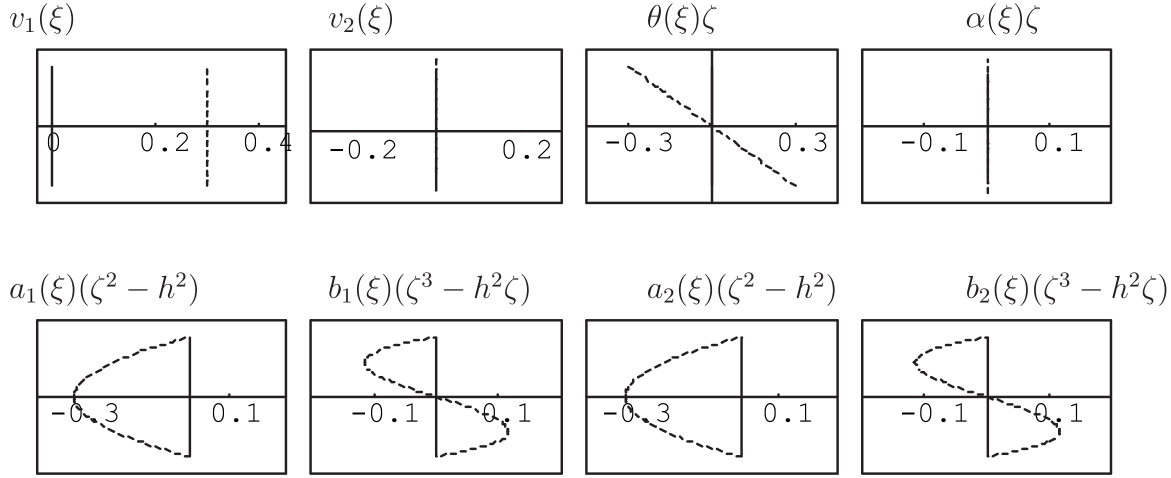
Affine constitutive identification

$$T(\xi) = \frac{2}{3}\mu h^3\,\eta(\xi), \tag{177}$$
$$F(\xi) = 2\lambda h\,\varepsilon(\xi) + 2(\lambda+2\mu)h\,\alpha(\xi), \tag{178}$$
$$M(\xi) = \frac{2}{3}(\lambda+2\mu)h^3\,\chi(\xi), \tag{179}$$
$$N(\xi) = 2h(\lambda+2\mu)\,\varepsilon(\xi) + 2\lambda h\,\alpha(\xi), \tag{180}$$
$$Q(\xi) = 2\mu h\,\gamma(\xi) \tag{181}$$

Cubic deformations

$$u_1(\xi,\zeta) = v_1(\xi) - \theta(\xi)\zeta + a_1(\xi)(\zeta^2 - h^2) + b_1(\xi)(\zeta^3 - h^2\zeta), \tag{182}$$

$$u_2(\xi,\zeta) = v_2(\xi) + \alpha(\xi)\zeta + a_2(\xi)(\zeta^2 - h^2) + b_2(\xi)(\zeta^3 - h^2\zeta) \tag{183}$$



$$\varepsilon_{11}(\xi,\zeta) = v_1'(\xi) - \theta'(\xi)\zeta + a_1'(\xi)(\zeta^2 - h^2) + b_1'(\xi)(\zeta^3 - h^2\zeta) \tag{184}$$

$$\begin{aligned} 2\varepsilon_{12}(\xi,\zeta) &= v_2'(\xi) - \theta(\xi) + \alpha'(\xi)\zeta + 2a_1(\xi)\zeta + b_1(\xi)(3\zeta^2 - h^2) \\ &\quad + a_2'(\xi)(\zeta^2 - h^2) + b_2'(\xi)(\zeta^3 - h^2\zeta) \end{aligned} \tag{185}$$

$$\varepsilon_{22}(\xi,\zeta) = \alpha(\xi) + 2a_2(\xi)\zeta + b_2(\xi)(3\zeta^2 - h^2) \tag{186}$$

Mixed constitutive identification

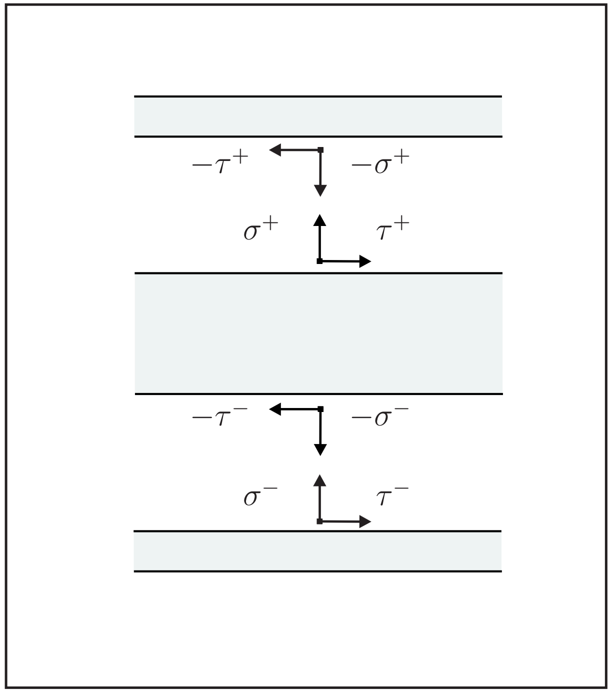
$$T(\xi) = \frac{2}{3}\mu h^3(\eta(\xi) + 2a_1(\xi)) - \frac{4}{15}\mu h^5\, b_2'(\xi), \tag{187}$$

$$F(\xi) = 2\lambda h\, \varepsilon(\xi) - \frac{4}{3}\lambda h^3\, a_1'(\xi) + 2h(\lambda + 2\mu)\, \alpha(\xi), \tag{188}$$

$$M(\xi) = \frac{2}{3}h^3(\lambda + 2\mu)\, \chi(\xi) + \frac{4}{15}h^5(\lambda + 2\mu)\, b_1'(\xi) - \frac{4}{3}\lambda h^3\, a_2(\xi), \tag{189}$$

$$N(\xi) = 2h(\lambda + 2\mu)\, \varepsilon(\xi) - \frac{4}{3}h^3(\lambda + 2\mu)\, a_1'(\xi) + 2\lambda h\, \alpha(\xi), \tag{190}$$

$$Q(\xi) = 2\mu h\, \gamma(\xi) - \frac{4}{3}\mu h^3\, a_2'(\xi) \tag{191}$$



Coefficients a_1, a_2, b_1, b_2 can be expressed in terms of the interlaminar stress. Let us introduce the following "boundary balance equations"

$$\sigma_+(\xi) = \sigma_2(\xi, h), \tag{192}$$

$$\sigma_-(\xi) = \sigma_2(\xi, -h), \tag{193}$$

$$\tau_+(\xi) = \tau_{12}(\xi, h), \tag{194}$$

$$\tau_-(\xi) = \tau_{12}(\xi, -h) \tag{195}$$

As $\sigma_2(\xi,h)$, e $\tau_{12}(\xi,h)$ are given as functions of ε_{11}, ε_{12}, by substituting (184), (185), (186) for ε_{11}, ε_{22}, ε_{12} we get

$$\begin{aligned}\sigma_+(\xi) = 2\,\mu\,\big(2\,h\,a_2(\xi) + \alpha(\xi) + 2\,h^2\,b_2(\xi)\big) + \lambda\,\big(2\,h\,a_2(\xi) + \alpha(\xi)\\ + 2\,h^2\,b_2(\xi) - h\,\chi(\xi) + \varepsilon(\xi)\big),\end{aligned} \tag{196}$$

$$\begin{aligned}\sigma_-(\xi) = 2\,\mu\,\big(-2\,h\,a_2(\xi) + \alpha(\xi) + 2\,h^2\,b_2(\xi)\big) + \lambda\,\big(-2\,h\,a_2(\xi)\\ + \alpha(\xi) + 2\,h^2\,b_2(\xi) + h\,\chi(\xi) + \varepsilon(\xi)\big),\end{aligned} \tag{197}$$

$$\tau_+(\xi) = \mu\,\big(2\,h\,a_1(\xi) + 2\,h^2\,b_1(\xi) + \gamma(\xi) + h\,\eta(\xi)\big), \tag{198}$$

$$\tau_-(\xi) = \mu\,\big(-2\,h\,a_1(\xi) + 2\,h^2\,b_1(\xi) + \gamma(\xi) - h\,\eta(\xi)\big), \tag{199}$$

from which we can obtain the following expressions

$$a_1(\xi) = -\frac{1}{2}\eta(\xi) + \frac{1}{4h\mu}\big(\tau_+(\xi) - \tau_-(\xi)\big), \tag{200}$$

$$a_2(\xi) = \frac{\lambda}{2(\lambda+2\mu)}\chi(\xi) + \frac{1}{4h(\lambda+2\mu)}\big(\sigma_+(\xi) - \sigma_-(\xi)\big), \tag{201}$$

$$b_1(\xi) = -\frac{1}{2h^2}\gamma(\xi) + \frac{1}{4h^2\mu}\big(\tau_+(\xi) + \tau_-(\xi)\big), \tag{202}$$

$$b_2(\xi) = -\frac{\lambda}{2h^2(\lambda+2\mu)}\varepsilon(\xi) - \frac{1}{2h^2}\alpha(\xi) + \frac{1}{4h^2(\lambda+2\mu)}\big(\sigma_+(\xi)+,\sigma_-(\xi)\big). \tag{203}$$

$$T(\xi) = \frac{2\lambda\mu h^3}{15(\lambda+2\mu)}\,\varepsilon'(\xi) + \frac{2}{15}\mu h^3\,\eta(\xi) - \frac{\mu h^3}{15(\lambda+2\mu)}(\sigma'_+(\xi)+\sigma'_-(\xi)) + \frac{1}{3}h^3(\tau_+(\xi)-\tau_-(\xi)) \quad (204)$$

$$F(\xi) = 2h\lambda\,\varepsilon(\xi) + 2h(\lambda+2\mu)\,\alpha(\xi) + \frac{2}{3}\lambda h^3\,\eta'(\xi) - \frac{\lambda h^2}{3\mu}(\tau'_+(\xi) - \tau'_-(\xi)) \quad (205)$$

$$\begin{aligned} M(\xi) = {} & \frac{8\mu h^3(\lambda+\mu)}{3(\lambda+2\mu)}\,\chi(\xi) - \frac{2}{15}h^3(\lambda+2\mu)\,\gamma'(\xi) \\ & - \frac{\lambda h^2}{3(\lambda+2\mu)}(\sigma_+(\xi) - \sigma_-(\xi)) + \frac{h^3}{15\mu}(\lambda+2\mu)(\tau'_+(\xi) + \tau'_-(\xi)) \end{aligned} \quad (206)$$

$$N(\xi) = 2h(\lambda+2\mu)\,\varepsilon(\xi) + 2h\lambda\,\alpha(\xi) + \frac{2}{3}h^3(\lambda+2\mu)\,\eta'(\xi) - \frac{h^2}{3\mu}(\lambda+2\mu)(\tau'_+(\xi) - \tau'_-(\xi)) \quad (207)$$

$$Q(\xi) = 2h\mu\,\gamma(\xi) - \frac{2\lambda\mu h^3}{3(\lambda+2\mu)}\,\chi'(\xi) - \frac{\mu h^2}{3(\lambda+2\mu)}(\sigma'_+(\xi) - \sigma'_-(\xi)) \quad (208)$$

Model with cubic microstructure

Placement

$$\begin{aligned} \mathbf{x}_i &: \mathcal{I} \to \mathcal{E}, & \mathbf{l}_i &: \mathcal{I} \to \mathcal{V}, \\ \mathbf{a}_i &: \mathcal{I} \to \mathbb{R}^2, & \mathbf{b}_i &: \mathcal{I} \to \mathbb{R}^2. \end{aligned} \tag{209}$$

Velocity field

$$\begin{aligned} \dot{\mathbf{x}}_i &: \mathcal{I} \to \mathcal{V}, & \dot{\mathbf{l}}_i &: \mathcal{I} \to \mathcal{V}, \\ \dot{\mathbf{a}}_i &: \mathcal{I} \to \mathbb{R}^2, & \dot{\mathbf{b}}_i &: \mathcal{I} \to \mathbb{R}^2. \end{aligned} \tag{210}$$

$$\mathcal{W}^{(ext)} + \mathcal{W}^{(in)} = 0. \tag{211}$$

$$\begin{aligned}\mathcal{W}^{(ext)} := &\sum_{i=1}^{N} \int_0^L \big(\dot{\mathbf{x}}_i \cdot \mathbf{b}_i + \dot{\mathbf{l}} \cdot \mathbf{c}_i\big) d\xi \\ &+ \sum_{i=1}^{N} \Big(\dot{\mathbf{x}}_i(0) \cdot \mathsf{t}_i^{(0)} + \dot{\mathbf{x}}_i(L) \cdot \mathsf{t}_i^{(L)} + \dot{\mathbf{l}}_i(0) \cdot \mathsf{m}_i^{(0)} + \dot{\mathbf{l}}_i(L) \cdot \mathsf{m}_i^{(L)} \\ &\qquad + \dot{\mathbf{a}}_i(0) \cdot \mathsf{A}_i^{(0)} + \dot{\mathbf{a}}_i(L) \cdot \mathsf{A}_i^{(L)} + \dot{\mathbf{b}}_i(0) \cdot \mathsf{B}_i^{(0)} + \dot{\mathbf{b}}_i(L) \cdot \mathsf{B}_i^{(L)}\Big),\end{aligned} \tag{212}$$

.

$$\mathcal{W}^{(in)} := -\sum_{i=1}^{N} \int_0^L \big(\dot{\mathbf{x}}_i \cdot \mathfrak{b}_i + \dot{\mathbf{l}} \cdot \mathfrak{c}_i + \dot{\mathbf{x}}_i' \cdot \mathbf{s}_i + \dot{\mathbf{l}}_i' \cdot \mathbf{m}_i\big) + \dot{\mathbf{a}} \cdot \mathcal{A} + \dot{\mathbf{b}} \cdot \mathcal{B} + \dot{\mathbf{a}}' \cdot A + \dot{\mathbf{b}}' \cdot B) d\xi. \tag{213}$$

$\mathbf{b}_i$	bulk force,
$\mathbf{c}_i$	bulk couple,
$\mathbf{s}_i, \mathbf{m}_i, A_i, B_i$	1-stress,
$\mathfrak{b}_i, \mathfrak{c}_i, \mathcal{A}_i, \mathcal{B}_i$	0-stress,
t_i	boundary force,
m_i	boundary couple,
$\mathsf{A}_i, \mathsf{B}_i$	other boundary force descriptors.

Balance equations (cubic microstructure)

$$\mathbf{b}_i - \mathfrak{b}_i + \mathbf{s}'_i = 0, \tag{214}$$

$$\mathbf{c}_i - \mathfrak{c}_i + \mathbf{m}'_i = 0, \tag{215}$$

$$-\mathcal{A}_i + A'_i = 0, \tag{216}$$

$$-\mathcal{B}_i + B'_i = 0, \tag{217}$$

$$\mathsf{t}_i^{(0)} + \mathbf{s}_i(0) = 0, \tag{218}$$

$$\mathsf{t}_i^{(L)} - \mathbf{s}_i(L) = 0, \tag{219}$$

$$\mathsf{m}_i^{(0)} + \mathbf{m}_i(0) = 0, \tag{220}$$

$$\mathsf{m}_i^{(L)} - \mathbf{m}_i(L) = 0, \tag{221}$$

$$\mathsf{A}_i^{(0)} + A_i(0) = 0, \tag{222}$$

$$\mathsf{A}_i^{(L)} - A_i(L) = 0, \tag{223}$$

$$\mathsf{B}_i^{(0)} + B_i(0) = 0, \tag{224}$$

$$\mathsf{B}_i^{(L)} - B_i(L) = 0. \tag{225}$$