# A One-dimensional Model for Multilayered Planar Beams

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## ABSTRACT

A one-dimensional model for multilayered planar beams is defined by joining one-dimensional beam models *à la* Cosserat together. An interlaminar stress arises naturally from the expression of the inner working, and its singular part at the boundary as well. Local and boundary balance equations are derived from an assumed expression for the working.

**Keywords:** multilayered beams, laminated composites, interlaminar stress

## 1 LAYERWISE CONTINUUM

Let us consider a planar beam made up of several layers. Instead of using a Cauchy continuum and assuming an interpolation of the displacement field over the cross section, as in the *layerwise theory* of Reddy,[4] we assume that each layer can be modeled as a Cosserat one-dimensional continuum, following an idea of Epstein and Glockner.[3] The main reason for this choice is that in this way the interlaminar stress enters directly into the model.

A shape is described for each layer by the two regular functions

$$(\mathbf{x}^{(i)} : \mathcal{I} \to \mathcal{E}, \quad \mathbf{d}^{(i)} : \mathcal{I} \to \mathcal{V}). \tag{1}$$

The first one is a parametrization of the axis, the second one assigns to each point of the axis the attitude of the cross section; $\mathcal{I}$ is the real interval $[0, L]$, $\mathcal{E}$ is the Euclidean space of dimension 2, $\mathcal{V}$ is the corresponding two-dimensional vector space, which we regard as a subspace of a three-dimensional space $\mathcal{U}$. Assuming $\|\mathbf{d}^{(i)}\| = 1$, a velocity field is described by the functions

$$(\mathbf{v}^{(i)} : \mathcal{I} \to \mathcal{V}, \quad \boldsymbol{\omega}^{(i)} : \mathcal{I} \to \mathcal{V}^{\perp}). \tag{2}$$

The *working* $W$ is assumed to be the sum of an *inner working* $W_{in}$ and of an *outer working* $W_{out}$ which, for a beam made up of $N$ layers, are assumed to have the following representations

$$W_{out} := \sum_{i=1}^{N} \int_0^L (\mathbf{b}^{(i)} \cdot \mathbf{v}^{(i)} + \mathbf{c}^{(i)} \cdot \boldsymbol{\omega}^{(i)}) ds + \sum_{i=1}^{N} \left( \mathbf{t}_0^{(i)} \cdot \mathbf{v}^{(i)}(0) + \mathbf{t}_L^{(i)} \cdot \mathbf{v}^{(i)}(L) + \mathbf{m}_0^{(i)} \cdot \boldsymbol{\omega}^{(i)}(0) + \mathbf{m}_L^{(i)} \cdot \boldsymbol{\omega}^{(i)}(L) \right), \tag{3}$$

$$W_{in} := -\sum_{i=1}^{N}\int_0^L (\mathbf{k}^{(i)}\cdot\mathbf{v}^{(i)} + \mathbf{g}^{(i)}\cdot\boldsymbol{\omega}^{(i)} + \mathbf{t}^{(i)}\cdot\mathbf{v}^{(i)\prime} + \mathbf{m}^{(i)}\cdot\boldsymbol{\omega}^{(i)\prime})ds$$
$$+\sum_{i=1}^{N-1}\int_0^L \boldsymbol{\tau}^{(i)}\cdot(\mathbf{v}_-^{(i+1)} - \mathbf{v}_+^{(i)})ds + \sum_{i=1}^{N-1}\Big(\boldsymbol{\tau}_0^{(i)}\cdot(\mathbf{v}_-^{(i+1)}(0) - \mathbf{v}_+^{(i)}(0)) + \boldsymbol{\tau}_L^{(i)}\cdot(\mathbf{v}_-^{(i+1)}(L) - \mathbf{v}_+^{(i)}(L))\Big). \quad (4)$$

These expressions define the bulk interaction $(\mathbf{b}^{(i)}, \mathbf{c}^{(i)})$ and the contact interaction at the left boundary $(\mathbf{t}_0^{(i)}, \mathbf{m}_0^{(i)})$ and at the right boundary $(\mathbf{t}_L^{(i)}, \mathbf{m}_L^{(i)})$. We denote by $\mathbf{v}_-^{(i)}$ and $\mathbf{v}_+^{(i)}$ the velocity corresponding to

$$\mathbf{x}_-^{(i)} := \mathbf{x}^{(i)} - \frac{1}{2}h^{(i)}\mathbf{d}^{(i)}, \quad \mathbf{x}_+^{(i)} := \mathbf{x}^{(i)} + \frac{1}{2}h^{(i)}\mathbf{d}^{(i)} \quad (5)$$

to be seen as the positions of the points at the bottom and the top boundary of the $i$-th layer of thickness $h^{(i)}$. Through the expression of $W_{in}$ *the interlaminar stress* $\boldsymbol{\tau}^{(i)}$ is defined in a natural way, just as the stress $\mathbf{t}^{(i)}$ and the couple stress $\mathbf{m}^{(i)}$ are. Note that a singular part of $\boldsymbol{\tau}^{(i)}$ at the left boundary $\boldsymbol{\tau}_0^{(i)}$ and another one at the right boundary $\boldsymbol{\tau}_L^{(i)}$ have been assumed explicitly.

Following the axiomatic scheme proposed by Di Carlo,[2] let us assume that for any velocity field be

$$W_{out} + W_{in} = 0, \quad (6)$$

and for any rigid velocity field be

$$W_{in} = 0. \quad (7)$$

From the balance of working axiom (6) we obtain the local balance equations

$$\mathbf{t}^{(i)\prime} + (\boldsymbol{\tau}^{(i)} - \boldsymbol{\tau}^{(i-1)}) - \mathbf{k}^{(i)} + \mathbf{b}^{(i)} = 0, \quad (8)$$
$$\mathbf{m}^{(i)\prime} - \mathbf{g}^{(i)} + \frac{1}{2}h^{(i)}\mathbf{d}^{(i)}\times(\boldsymbol{\tau}^{(i)} + \boldsymbol{\tau}^{(i-1)}) + \mathbf{c}^{(i)} = 0, \quad (9)$$

and the boundary conditions

$$\mathbf{t}^{(i)}(0) = -\mathbf{t}_0^{(i)} - (\boldsymbol{\tau}_0^{(i)} - \boldsymbol{\tau}_0^{(i-1)}), \quad (10)$$
$$\mathbf{t}^{(i)}(L) = \mathbf{t}_L^{(i)} + (\boldsymbol{\tau}_L^{(i)} - \boldsymbol{\tau}_L^{(i-1)}), \quad (11)$$
$$\mathbf{m}^{(i)}(0) = -\mathbf{m}_0^{(i)} - \frac{1}{2}h^{(i)}\mathbf{d}^{(i)}(0)\times(\boldsymbol{\tau}_0^{(i)} + \boldsymbol{\tau}_0^{(i-1)}), \quad (12)$$
$$\mathbf{m}^{(i)}(L) = \mathbf{m}_L^{(i)} + \frac{1}{2}h^{(i)}\mathbf{d}^{(i)}(L)\times(\boldsymbol{\tau}_L^{(i)} + \boldsymbol{\tau}_L^{(i-1)}). \quad (13)$$

Let us consider the $i$-th layer as a sub-body of the multilayered beam. The outer and the inner workings get the following expressions

$$W_{out}^{(i)} := \int_0^L (\tilde{\mathbf{b}}^{(i)}\cdot\mathbf{v}^{(i)} + \tilde{\mathbf{c}}^{(i)}\cdot\boldsymbol{\omega}^{(i)})ds + \tilde{\mathbf{t}}_0^{(i)}\cdot\mathbf{v}^{(i)}(0) + \tilde{\mathbf{t}}_L^{(i)}\cdot\mathbf{v}^{(i)}(L) + \tilde{\mathbf{m}}_0^{(i)}\cdot\boldsymbol{\omega}^{(i)}(0) + \tilde{\mathbf{m}}_L^{(i)}\cdot\boldsymbol{\omega}^{(i)}(L), \quad (14)$$

$$W_{in}^{(i)} := -\int_0^L (\mathbf{k}^{(i)}\cdot\mathbf{v}^{(i)} + \mathbf{g}^{(i)}\cdot\boldsymbol{\omega}^{(i)} + \mathbf{t}^{(i)}\cdot\mathbf{v}^{(i)\prime} + \mathbf{m}^{(i)}\cdot\boldsymbol{\omega}^{(i)\prime})ds, \quad (15)$$

where $(\tilde{\mathbf{b}}^{(i)}, \tilde{\mathbf{c}}^{(i)})$, $(\tilde{\mathbf{t}}_0^{(i)}, \tilde{\mathbf{m}}_0^{(i)})$, $(\tilde{\mathbf{t}}_L^{(i)}, \tilde{\mathbf{m}}_L^{(i)})$ are the interactions with both the external world and the other layers. From the balance of working axiom (6) we obtain the following local balance equations

$$\mathbf{t}^{(i)\prime} - \mathbf{k}^{(i)} + \tilde{\mathbf{b}}^{(i)} = 0, \quad (16)$$
$$\mathbf{m}^{(i)\prime} - \mathbf{g}^{(i)} + \tilde{\mathbf{c}}^{(i)} = 0, \quad (17)$$

together with the boundary conditions

$$\mathbf{t}^{(i)}(0) = -\tilde{\mathbf{t}}_0^{(i)}, \qquad \mathbf{t}^{(i)}(L) = \tilde{\mathbf{t}}_L^{(i)}, \tag{18}$$
$$\mathbf{m}^{(i)}(0) = -\tilde{\mathbf{m}}_0^{(i)}, \qquad \mathbf{m}^{(i)}(L) = \tilde{\mathbf{m}}_L^{(i)}. \tag{19}$$

From the invariance axiom (7) we obtain the constitutive conditions

$$\mathbf{k}^{(i)} = 0, \tag{20}$$
$$\mathbf{g}^{(i)} = -\mathbf{x}^{(i)\prime} \times \mathbf{t}^{(i)}, \tag{21}$$
$$(\mathbf{x}_+^{(i)} - \mathbf{x}_-^{(i+1)}) \times \boldsymbol{\tau}^{(i)} = 0, \tag{22}$$

and the conditions for the singular part of $\boldsymbol{\tau}$ in the same form as (22) at the boundary.

Corresponding equations can be derived for any sub-body made up of contiguous layers. By these equations and equations (8) and (9), the interactions on the $i$-th layer turn out to be the sum of the interactions with the external world, defined through (3), and the interaction with the contiguous layers (those applied at the top boundary are denoted by '+', those applied at the bottom boundary are denoted by '−') as follows

$$\tilde{\mathbf{b}}^{(i)} = \mathbf{b}^{(i)} + \mathbf{b}_+^{(i)} + \mathbf{b}_-^{(i)}, \qquad \tilde{\mathbf{c}}^{(i)} = \mathbf{c}^{(i)} + \mathbf{c}_+^{(i)} + \mathbf{c}_-^{(i)}, \tag{23}$$
$$\mathbf{b}_+^{(i)} = \boldsymbol{\tau}^{(i)}, \qquad \mathbf{c}_+^{(i)} = \frac{1}{2} h^{(i)} \mathbf{d}^{(i)} \times \boldsymbol{\tau}^{(i)}, \tag{24}$$
$$\mathbf{b}_-^{(i)} = -\boldsymbol{\tau}^{(i-1)}, \qquad \mathbf{c}_-^{(i)} = \frac{1}{2} h^{(i)} \mathbf{d}^{(i)} \times \boldsymbol{\tau}^{(i-1)}. \tag{25}$$

Such a decomposition applies also to the left and right boundary interactions in (18) and (19)

$$\tilde{\mathbf{t}}_0^{(i)} = \mathbf{t}_0^{(i)} + \mathbf{t}_{0+}^{(i)} + \mathbf{t}_{0-}^{(i)}, \qquad \tilde{\mathbf{t}}_L^{(i)} = \mathbf{t}_L^{(i)} + \mathbf{t}_{L+}^{(i)} + \mathbf{t}_{L-}^{(i)}, \tag{26}$$
$$\tilde{\mathbf{m}}_0^{(i)} = \mathbf{m}_0^{(i)} + \mathbf{m}_{0+}^{(i)} + \mathbf{m}_{0-}^{(i)}, \qquad \tilde{\mathbf{m}}_L^{(i)} = \mathbf{m}_L^{(i)} + \mathbf{m}_{L+}^{(i)} + \mathbf{m}_{L-}^{(i)}, \tag{27}$$
$$\mathbf{t}_{0+}^{(i)} = \boldsymbol{\tau}_0^{(i)}, \qquad \mathbf{t}_{L+}^{(i)} = \boldsymbol{\tau}_L^{(i)}, \tag{28}$$
$$\mathbf{t}_{0-}^{(i)} = -\boldsymbol{\tau}_0^{(i-1)}, \qquad \mathbf{t}_{L-}^{(i)} = -\boldsymbol{\tau}_L^{(i-1)}, \tag{29}$$
$$\mathbf{m}_{0+}^{(i)} = \frac{1}{2} h^{(i)} \mathbf{d}^{(i)} \times \boldsymbol{\tau}_0^{(i)}, \qquad \mathbf{m}_{L+}^{(i)} = \frac{1}{2} h^{(i)} \mathbf{d}^{(i)} \times \boldsymbol{\tau}_L^{(i)}, \tag{30}$$
$$\mathbf{m}_{0-}^{(i)} = \frac{1}{2} h^{(i)} \mathbf{d}^{(i)} \times \boldsymbol{\tau}_0^{(i-1)}, \qquad \mathbf{m}_{L-}^{(i)} = \frac{1}{2} h^{(i)} \mathbf{d}^{(i)} \times \boldsymbol{\tau}_L^{(i-1)}. \tag{31}$$

Note that the interactions between two continguous layers are described by opposite forces which can be seen as contact forces applied at the top and bottom boundary of each layer. These contact forces correspond to the interlaminar stress $\boldsymbol{\tau}^{(i)}$, which must satisfy the condition (22). Further, it is interesting to note that $\boldsymbol{\tau}^{(i)}$ plays the same role as $\mathbf{t}^{(i)}$ in equations (8) and (9), making them look like those of a two-dimensional continuum.

It is worth noting here that up to now no bonding constraint between the layers has been assumed. So the interaction should be characterized by a constituive prescription for $\boldsymbol{\tau}^{(i)}$.

Let us assume now that the layers are perfectly bonded together so that

$$\mathbf{x}_+^{(i)} = \mathbf{x}_-^{(i+1)}. \tag{32}$$

As a consequence any velocity field is such that

$$\mathbf{v}_+^{(i)} - \mathbf{v}_-^{(i+1)} = 0. \tag{33}$$

Hence the interlaminar stress is merely reactive, as the corresponding term in the expression of the working (4) is zero. Through (32) and (5) every parametrization $\mathbf{x}^{(i)}$ can be expressed in terms of any of them, say $\mathbf{x}^{(r)}$, as follows

$$\mathbf{x}^{(i)} = \begin{cases} \mathbf{x}^{(r)} + \dfrac{1}{2}h^{(r)}\mathbf{d}^{(r)} + h^{(r+1)}\mathbf{d}^{(r+1)} + \cdots + \dfrac{1}{2}h^{(i)}\mathbf{d}^{(i)} & i > r, \\ \mathbf{x}^{(r)} - \dfrac{1}{2}h^{(r)}\mathbf{d}^{(r)} - h^{(r-1)}\mathbf{d}^{(r-1)} - \cdots - \dfrac{1}{2}h^{(i)}\mathbf{d}^{(i)} & i < r. \end{cases} \tag{34}$$

By substituting the corresponding expressions for the velocity field $\mathbf{v}^{(i)}$ into (3) and (4), the balance equations turn out to be

$$\sum_{l=1}^{N} (\mathbf{t}^{(l)\prime} + \mathbf{b}^{(l)}) = 0, \tag{35}$$

$$\mathbf{m}^{(i)\prime} - \mathbf{g}^{(i)} + \frac{h^{(i)}}{2}\mathbf{d}^{(i)} \times \big( \sum_{l=i+1}^{N} (\mathbf{t}^{(l)\prime} + \mathbf{b}^{(l)}) - \sum_{l=1}^{i-1} (\mathbf{t}^{(l)\prime} + \mathbf{b}^{(l)})\big) + \mathbf{c}^{(i)} = 0, \tag{36}$$

while the boundary conditions are

$$\sum_{l=1}^{N} (\mathbf{t}^{(l)}(0) + \mathbf{t}_0^{(l)}) = 0, \quad \sum_{l=1}^{N} (\mathbf{t}^{(l)}(L) - \mathbf{t}_L^{(l)}) = 0, \tag{37}$$

$$\mathbf{m}^{(i)}(0) = -\mathbf{m}_0^{(i)} - \frac{h^{(i)}}{2}\mathbf{d}^{(i)}(0) \times \big( \sum_{l=i+1}^{N} (\mathbf{t}^{(l)}(0) + \mathbf{t}_0^{(l)}) - \sum_{l=1}^{i-1} (\mathbf{t}^{(l)}(0) + \mathbf{t}_0^{(l)})\big), \tag{38}$$

$$\mathbf{m}^{(i)}(L) = \mathbf{m}_L^{(i)} - \frac{h^{(i)}}{2}\mathbf{d}^{(i)}(L) \times \big( \sum_{l=i+1}^{N} (\mathbf{t}^{(l)}(L) - \mathbf{t}_L^{(l)}) - \sum_{l=1}^{i-1} (\mathbf{t}^{(l)}(L) - \mathbf{t}_L^{(l)})\big). \tag{39}$$

From equations (8), (10) and (11) we can obtain at last the following expressions for the interlaminar stress

$$\boldsymbol{\tau}^{(i)} = -\sum_{l=1}^{i} (\mathbf{t}^{(l)\prime} + \mathbf{b}^{(l)}) = \sum_{l=i+1}^{N} (\mathbf{t}^{(l)\prime} + \mathbf{b}^{(l)}), \tag{40}$$

$$\boldsymbol{\tau}_0^{(i)} = -\sum_{l=1}^{i} (\mathbf{t}^{(l)}(0) + \mathbf{t}_0^{(l)}) = \sum_{l=i+1}^{N} (\mathbf{t}^{(l)}(0) + \mathbf{t}_0^{(l)}), \tag{41}$$

$$\boldsymbol{\tau}_L^{(i)} = -\sum_{l=1}^{i} (-\mathbf{t}^{(l)}(L) + \mathbf{t}_0^{(l)}) = \sum_{l=i+1}^{N} (-\mathbf{t}^{(l)}(L) + \mathbf{t}_0^{(l)}). \tag{42}$$

## 2 STRAIN MEASURES AND CONSTITUTIVE RELATIONS

For each layer the axial strain $\epsilon^{(i)}$ and the shear strain $\gamma^{(i)}$ are defined, as in Antman,[1] by setting

$$\mathbf{x}^{(i)\prime} = (1 + \epsilon^{(i)})\mathbf{n}^{(i)} + \gamma^{(i)}\mathbf{d}^{(i)}, \tag{43}$$

where $\mathbf{n}^{(i)}$ is a unit vector orthogonal to $\mathbf{d}^{(i)}$. The attitude of the cross sections can be described, with respect to a reference shape, by an angle $\theta^{(i)}$ such that

$$\mathbf{d}^{(i)} = -\sin\theta^{(i)}\mathbf{n}_{\mathbf{r}}^{(i)} + \cos\theta^{(i)}\mathbf{d}_{\mathbf{r}}^{(i)}, \tag{44}$$

while the axis can be described by a displacement vector

$$\mathbf{u}^{(i)} := \mathbf{x}^{(i)} - \mathbf{x}_{\mathbf{r}}^{(i)} = u_1^{(i)}\mathbf{n}_{\mathbf{r}}^{(i)} + u_2^{(i)}\mathbf{d}_{\mathbf{r}}^{(i)}, \tag{45}$$

where $\mathbf{x}_\mathbf{r}^{(i)}$ defines the axis in the reference shape, and $\mathbf{d}_\mathbf{r}^{(i)}$ the attitude of the cross section, such that $\mathbf{n}_\mathbf{r}^{(i)} := \mathbf{x}_\mathbf{r}^{(i)\prime}$ is the corresponding unit orthogonal vector. Defining the curvature $k^{(i)}$ by setting

$$\mathbf{d}^{(i)\prime} = -k^{(i)}\mathbf{n}^{(i)}, \tag{46}$$

the change of curvature turns out to be

$$\mu^{(i)} := k^{(i)} - k_\mathbf{r}^{(i)} = \theta^{(i)\prime}, \tag{47}$$

while the axial and shear strains are related to $u_1^{(i)}$, $u_2^{(i)}$ and $\theta^{(i)}$ through the following expressions

$$\epsilon^{(i)} = (1 + u_1^{(i)\prime} - k_\mathbf{r}^{(i)} u_2^{(i)}) \cos\theta^{(i)} + (u_2^{(i)\prime} + k_\mathbf{r}^{(i)} u_1^{(i)}) \sin\theta^{(i)} - 1, \tag{48}$$

$$\gamma^{(i)} = -(1 + u_1^{(i)\prime} - k_\mathbf{r}^{(i)} u_2^{(i)}) \sin\theta^{(i)} + (u_2^{(i)\prime} + k_\mathbf{r}^{(i)} u_1^{(i)}) \cos\theta^{(i)}. \tag{49}$$

Defining the scalar components of the stress with respect to the orthonormal basis $\{\mathbf{n}^{(i)}, \mathbf{d}^{(i)}, \mathbf{n}^{(i)} \times \mathbf{d}^{(i)}\}$ by the expressions

$$\mathbf{t}^{(i)} = N^{(i)}\mathbf{n}^{(i)} + T^{(i)}\mathbf{d}^{(i)}, \tag{50}$$

$$\mathbf{m}^{(i)} = M^{(i)}\mathbf{n}^{(i)} \times \mathbf{d}^{(i)}, \tag{51}$$

the constitutive relations for an elastic material can be given the general form

$$N^{(i)} = \hat{N}^{(i)}(\epsilon^{(i)}, \gamma^{(i)}, \mu^{(i)}), \tag{52}$$

$$T^{(i)} = \hat{T}^{(i)}(\epsilon^{(i)}, \gamma^{(i)}, \mu^{(i)}), \tag{53}$$

$$M^{(i)} = \hat{M}^{(i)}(\epsilon^{(i)}, \gamma^{(i)}, \mu^{(i)}). \tag{54}$$

If the layers are not perfectly bonded together, then a constitutive relation for the interlaminar stress should be given in the form

$$\boldsymbol{\tau}^{(i)} = \hat{\boldsymbol{\tau}}^{(i)}(\mathbf{x}_-^{(i+1)} - \mathbf{x}_+^{(i)}), \tag{55}$$

where $\hat{\boldsymbol{\tau}}^{(i)}$ has to satisfy the constitutive condition (22).

# 3 SAMPLE PROBLEMS

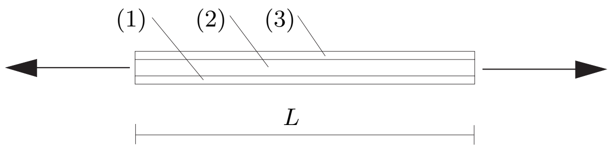
## Case (1)

Let us consider a straight beam made up of three perfectly bonded layers, with an axial force applied at the ends of the middle layer, as in Fig. 1. Denoting by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ a fixed orthonormal basis, let us assume in the reference shape $\mathbf{n}^{(i)} = \mathbf{e}_1$ and $\mathbf{d}^{(i)} = \mathbf{e}_2$. The solution to the linear problem has been computed starting from the linearized version of equations (35), (36) and of the corresponding boundary conditions. Linear constitutive relations have been assigned in the usual form

$$N^{(i)} = (EA)^{(i)}\varepsilon^{(i)}, \quad T^{(i)} = (GA)^{(i)}\gamma^{(i)}, \quad M^{(i)} = (EI)^{(i)}\mu^{(i)}. \tag{56}$$

The middle layer is an aluminum sheet and the external layers are made of a piezoelectric material. The stiffness coefficients are those corresponding to the thickness of the three layers and to the elastic moduli given in Fig. 1. The length is $L = 1\,\mathrm{in}$, the applied force is $1\,\mathrm{lb}$.

The values of the components of $\mathbf{t}^{(i)}$ and $\boldsymbol{\tau}^{(i)}$ along the axis, near the right end, where $s = L$, are shown in Fig. 2. The last two graphs show clearly the rapid variation of the regular part of the interlaminar stress in a




$h^{(1)} = h^{(3)} = 5 \times 10^{-3}$ in, $h^{(2)} = 15 \times 10^{-3}$ in
$E^{(1)} = E^{(3)} = 1.03 \times 10^{7}$ psi, $E^{(2)} = 9.14 \times 10^{6}$ psi
$G^{(1)} = G^{(3)} = 0.386 \times 10^{7}$ psi, $G^{(2)} = 3.57 \times 10^{6}$ psi

Figure 1: A three layer beam.

narrow region near the boundary, while the first three show the corresponding variation of the normal and shear stress in the external layers. Note that by (42) and (13) is

$$\boldsymbol{\tau}_L^{(2)} = -\mathbf{t}^{(3)}(L), \tag{57}$$

$$\frac{1}{2} h^{(3)} \mathbf{d}^{(3)} \times \boldsymbol{\tau}_L^{(2)} = \mathbf{m}^{(3)}(L). \tag{58}$$

The singular part of the interlaminar stress amounts exactly to the jump of the stress at the boundary and this causes the moment, in the external layers, to be different from zero.

## Case (2)

Let us consider the same beam with the axial unit force applied at the ends of the bottom layer. Note in Fig. 3 that, near the ends, the shear stress is different from zero and in Fig. 4 that the moment in the upper layer changes its sign. Fig. 5 shows the regular part of the interlaminar stress.

## Case (3)

As another case let us consider the same beam loaded by two opposite unit couples at the ends, made up each of two opposite forces applied at the external layers. Fig. 6 shows the stress and Fig. 7 shows the regular part of the interlaminar stress.

## Case (1) extended

The solution to the first case can be extended to a three layer beam, whose middle layer is longer than the other ones (see Fig. 8). A further extension can be considered allowing the upper and lower layers to cover the whole length of the middle layer, still assuming them bonded only for the original length. Under the action of the axial force the external layers will bend at the ends of the bonded region, because of a jump of the shear strain reflecting a jump of the shear stress.

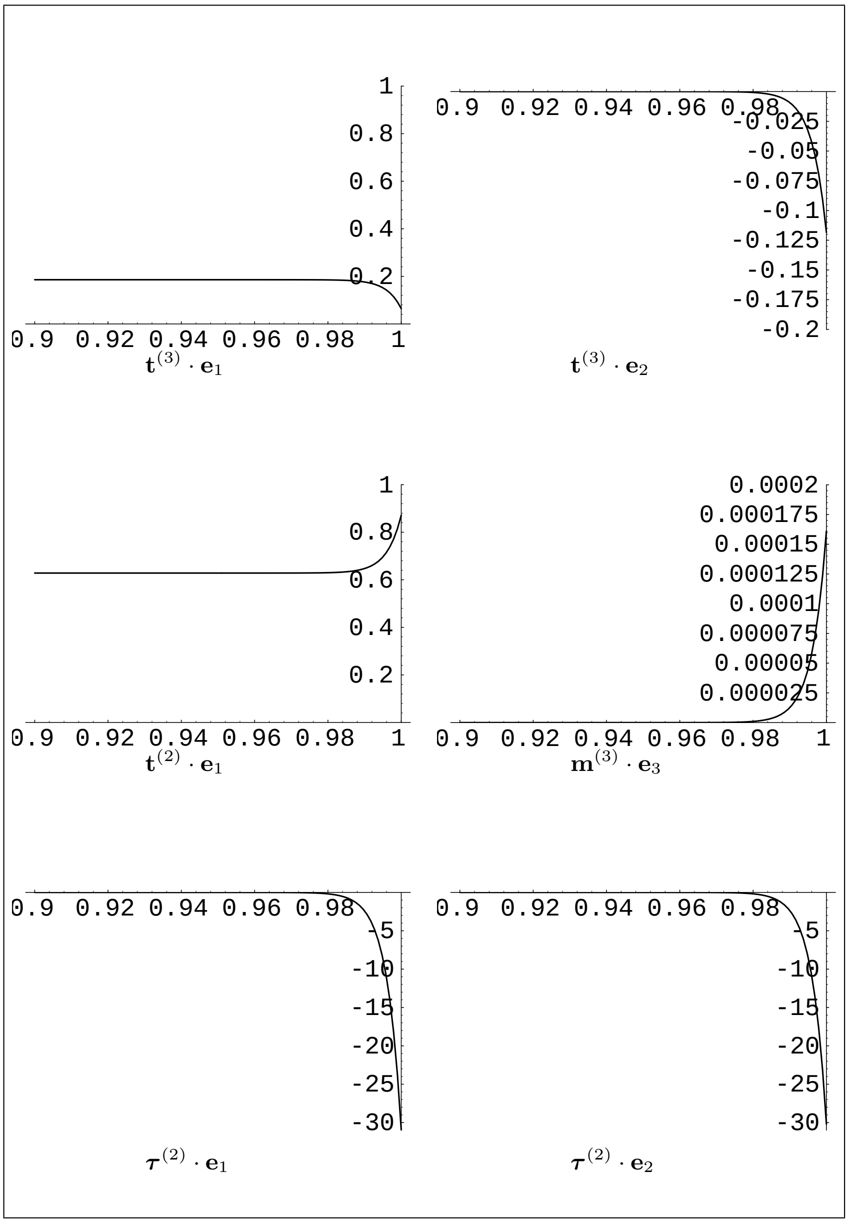



Figure 2: Case (1): Stress and interlaminar stress near $s = L$.

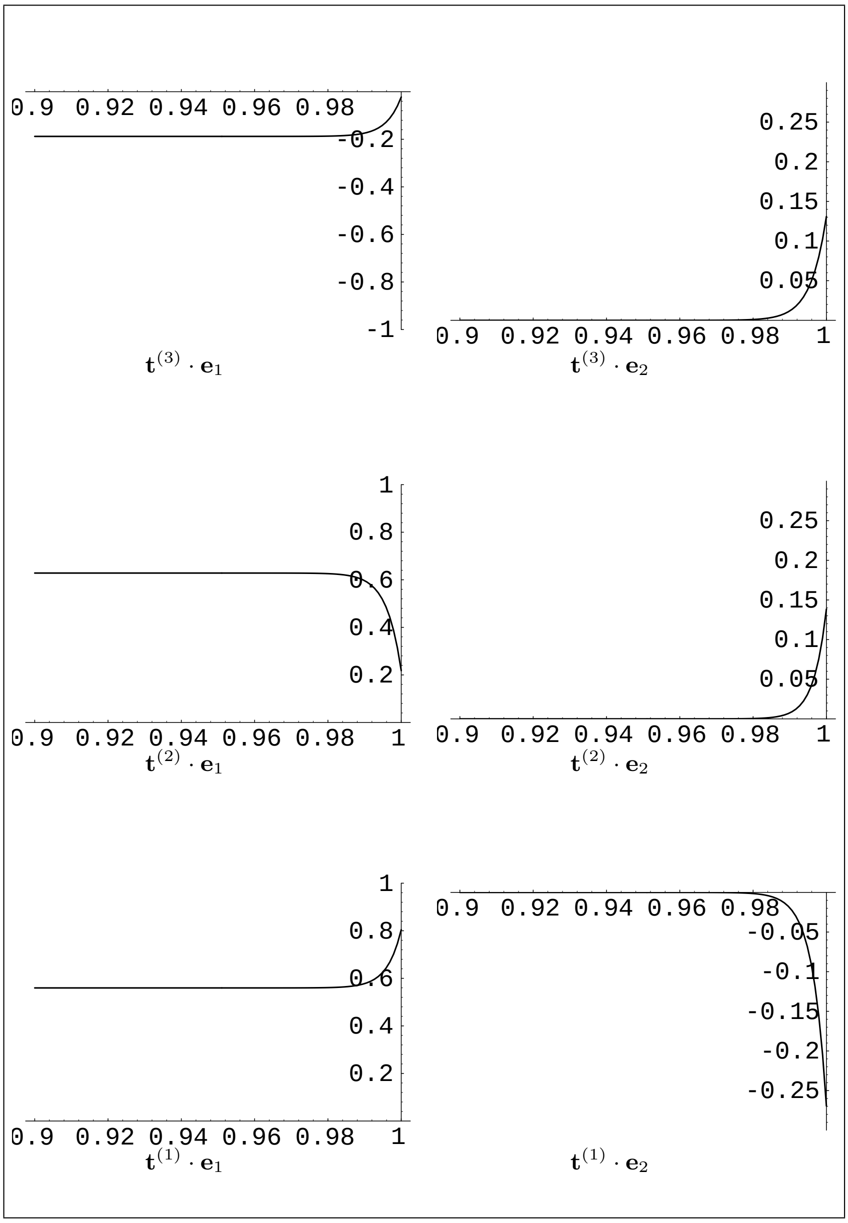



Figure 3: Case (2): Stress near $s = L$.

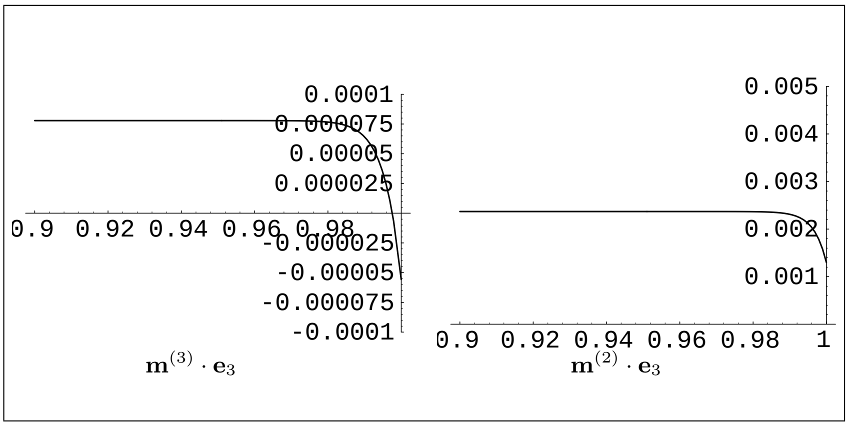



Figure 4: Case (2): Moments near $s = L$.

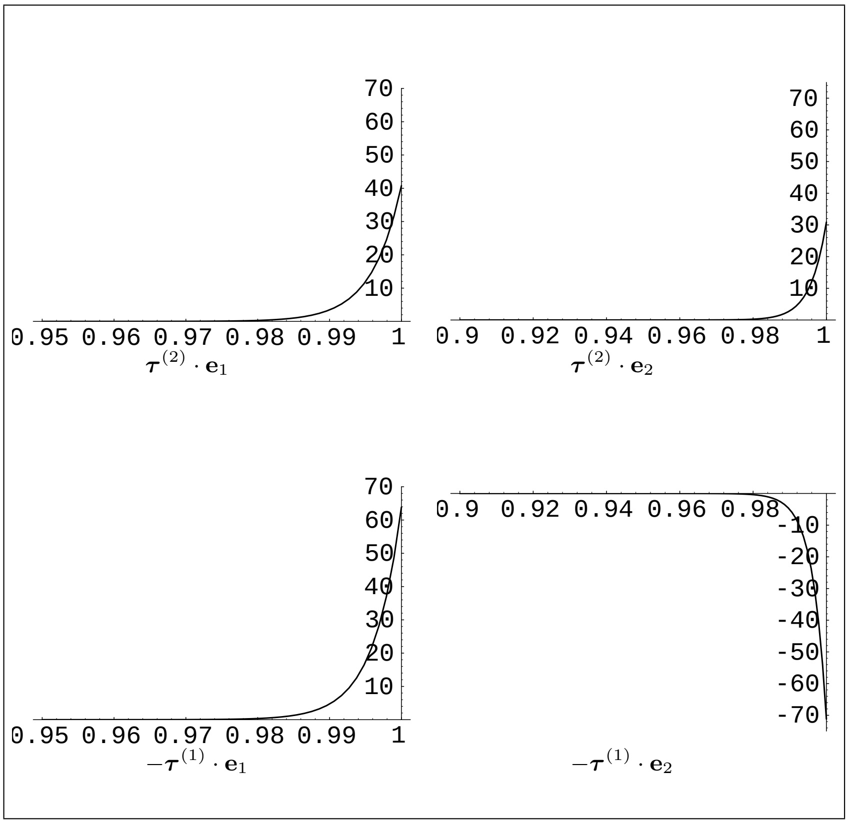



Figure 5: Case (2): Interlaminar stress near $s = L$.

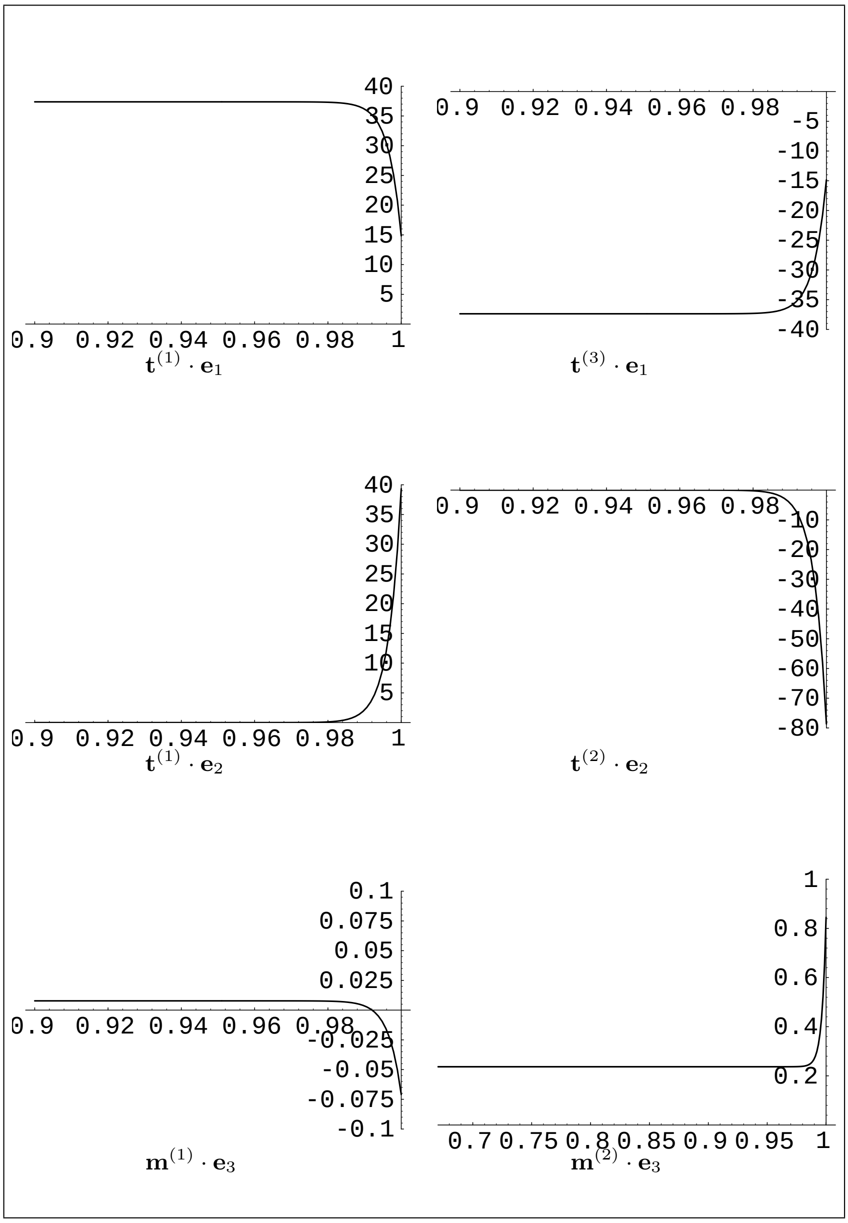



Figure 6: Case (3): Stress near $s = L$.

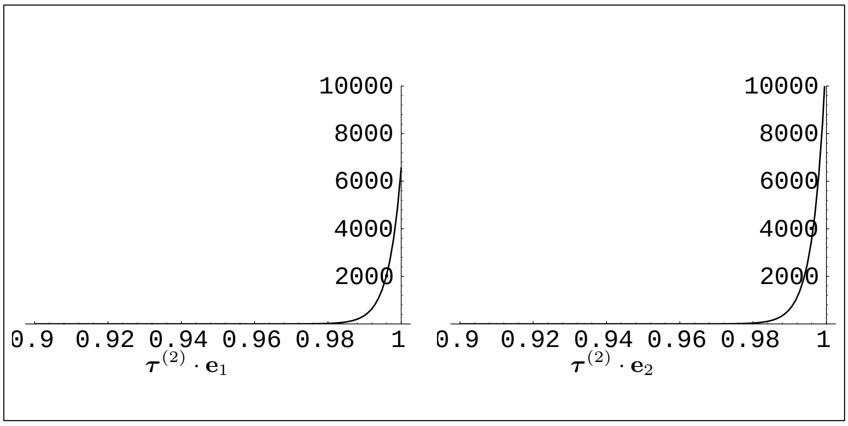



Figure 7: Case (3): Interlaminar stress near $s = L$.

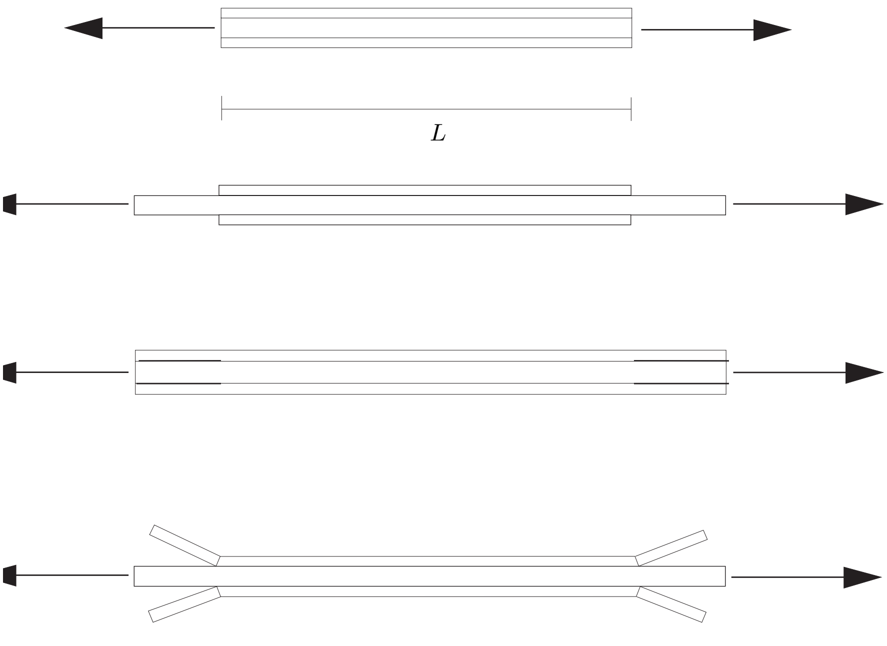



Figure 8: Case (1) extended.

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