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CONTINUUM MODELLING OF A BEAM-LIKE LATTICED TRUSS: IDENTIFICATION OF THE CONSTITUTIVE FUNCTIONS FOR THE CONTACT AND INERTIAL ACTIONS

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SOMMARIO. La definizione di un modello di continuo atto a descrivere sinteticamente il comportamento meccanico di una struttura modulare viene riguardata come identificazione (dei parametri costitutivi) di un modello "sommario" a partire dalla specificazione di un modello "più fine". Il modello fine descrive in dettaglio una trave reticolare piana, il cui modulo è costituito da una coppia di diaframmi rigidi connessi da aste rettilinee elastiche; sia i diaframmi che le aste sono dotati di massa; il modello sommario è un continuo monodimensionale dotato di struttura euclidea. Un risultato interessante è che il valore della densità dell'azione di inerzia in un punto dipende non solo dal valore della accelerazione in quel punto – come si assume nei modelli usuali di continuo – ma anche dalle sue derivate prima e seconda, rispetto alla coordinata materiale.

SUMMARY. The derivation of a continuum model apt to give a compendious description of the mechanical behaviour of a latticed structure is envisaged here as a procedure leading to the identification (of the constitutive parameters) of a "coarse" model starting from a prescribed "finer" one. The fine model considered describes a planar modular beam whose module is made up of a pair of rigid diaphragms connected by straight elastic bars; diaphragms and bars have both mass. The coarse model is a one-dimensional continuum endowed with Euclidean structure. An interesting result is that the value of the density of the inertial actions at a point depends not only on the value of the acceleration at that same point – as is usually taken for granted in conventional continuum models – but also of its first and second derivatives with respect to the material coordinate.

1. INTRODUCTION

The description of the mechanical behaviour of large latticed structures by continuum models is the subject of many technical contributions (see for instance [1, 2, 3, 4]). Although different points of view have been adopted, the interest is usually confined to essentially linear and unsystematic theories. The aim of the present paper is to take a first step toward a more general and rational approach. The derivation of a continuum model apt to give a compendious description of the mechanical behaviour of a latticed structure is seen here as a procedure leading to the identification of the constitutive parameters of a "coarse" model starting from a prescribed "finer" one (1). The fine model considered is a planar modular beam whose module is made up of a pair of rigid diaphragms connected by a number of straight elastic bars. Diaphragms and bars have both mass. The coarse model is a one-dimensional continuum endowed with Euclidean structure, that is – roughly speaking – a one-dimensional manifold of rigid bodies. A relation between the two models is established by assigning a map from the local placements of the coarse model to the local placements of a module of the fine model. Then, the coarse constitutive functions are identified by assuming that for any pair of corresponding motions the power expended by the actions prescribed within the fine model equals the power expended by the corresponding actions within the coarse model. This procedure is here applied to characterize the coarse inertial and contact (elastic) dynamical actions. The properties so inherited by the coarse model from the fine one are by no means trivial. In particular, the inertial actions possess a singular part (concentrated at the boundary of smooth subbodies), and its regular part has a density whose value at a point depends not only on the value of the acceleration at that same point – as is usually taken for granted in conventional continuum models – but also of its first and second derivatives (with respect to the material coordinate).

2. THE COARSE MODEL: A ONE-DIMENSIONAL CONTINUUM ENDOWED WITH EUCLIDEAN STRUCTURE

Let $\mathscr{B}$ be a body whose motion is defined as a function

$$\chi : \mathscr{B} \times \mathbb{R} \to \mathscr{E} \times \mathscr{V} \qquad (2.1)$$

that takes each body-point $\mathcal{O} \in \mathscr{B}$ and time $t \in \mathbb{R}$ into a place $\mathrm{x} \in \mathscr{E}$ and a unit vector (*director*) $\mathrm{d} \in \mathscr{V}$, where $\mathbb{R}$ is the field of real numbers, $\mathscr{E}$ is a two-dimensional Euclidean manifold and $\mathscr{V}$ its translation space (2).

We suppose the section $\chi(\mathcal{O}, \cdot)$ to be a smooth curve for any $\mathcal{O} \in \mathscr{B}$; moreover, for any $t \in \mathbb{R}$ the section $\chi(\cdot, t)$ – that is, the *placement* at time t – is supposed to have the following properties: the natural projection of its image

(1) We borrow here the terminology established by Muncaster in [5]. Our point of view is not fully conforming to his, however.

(2) We shall name *Lin* the algebra of linear endomorphisms of $\mathscr{V}$; *Orth* (thought of as a submanifold of *Lin*) the orthogonal group; *Skw* $\subset$ *Lin* the subspace of skew-symmetric endomorphisms of $\mathscr{V}$.

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$\chi(\mathscr{B}, t)$ on $\mathscr{E}$ is a (piecewise) smooth one-dimensional submanifold $\mathscr{C}_t$ (called the *axis*), and $\mathbf{d}$ is given by a (piecewise) smooth vector field on $\mathscr{C}_t$. With each (regular) place $\mathbf{x} \in \mathscr{C}_t$ will then be smoothly associated an orthonormal vector basis $(\mathbf{d}_1, \mathbf{d}_2)$ of $\mathscr{V}$ such that

$$\mathbf{d}_2 \equiv \mathbf{d} \tag{2.2}$$

We select a placement $\kappa : \mathscr{B} \to \mathscr{E} \times \mathscr{V}$ as a reference placement, and let $\mathscr{C}_\kappa$ be the corresponding reference axis, $\mathbf{D}$ the reference director field and $(\mathbf{D}_1, \mathbf{D}_2)$, with $\mathbf{D}_2 \equiv \mathbf{D}$, the correspondent orthonormal basis. After introducing the arc length parameter s induced on $\mathscr{C}_\kappa$ by the metric of $\mathscr{E}$, all fields defined on $\mathscr{C}_\kappa$ or $\mathscr{C}_t$ will be described by functions of this parameter. In particular, we shall name $\mathbf{X}$ the parametrization of $\mathscr{C}_\kappa$ in terms of s and $\mathbf{x}$ the corresponding parametrization of $\mathscr{C}_t$ (in the sense that, for each value of s, $\mathbf{x}(s)$ is the place presently occupied by the body-point whose reference place is $\mathbf{X}(s)$). It is assumed that such a parametrization exists at each $t \in \mathbb{R}$, and depends smoothly on time.

The *rotation* $\mathbf{R}$ is defined as the unique element of *Lin* such that

$$\begin{aligned} \mathbf{d}_1 &= \mathbf{R}\mathbf{D}_1 \\ \mathbf{d}_2 &= \mathbf{R}\mathbf{D}_2 \end{aligned} \tag{2.3}$$

while the *velocity* is the pair $(\mathbf{w}, \mathbf{W})$

$$\begin{aligned} \mathbf{w} &:= \dot{\mathbf{x}} \\ \mathbf{W} &:= \dot{\mathbf{R}}\mathbf{R}^T \end{aligned} \tag{2.4}$$

a dot denoting differentiation with respect to time. Notice that, due to the orthonormality of both the reference and the present bases, $\mathbf{R} \in Orth$ and $\mathbf{W} \in Skw$.

It is convenient to introduce the *right strain* $(\mathbf{u}, \mathbf{U})$ (as opposite to the *left strain* $(\mathbf{v}, \mathbf{V})$: see Appendix A)

$$\begin{aligned} \mathbf{u} &:= \mathbf{R}^T\mathbf{x}' - \mathbf{X}' \\ \mathbf{U} &:= \mathbf{R}^T\mathbf{R}' \end{aligned} \tag{2.5}$$

the prime denoting differentiation with respect to s.

The dynamical actions entering the model are described by *contact force* and *couple* fields $(\mathbf{t}, \mathbf{T}) \in \mathscr{V} \times Skw$, and by densities of body force and couple $(\mathbf{b}, \mathbf{B}) \in \mathscr{V} \times Skw$: inertial forces and couples are included in $(\mathbf{b}, \mathbf{B})$. The mechanical *power* on the part of $\mathscr{B}$ corresponding to the interval $[s_0, s_1]$ is hence

$$\mathscr{W} := \int_{s_0}^{s_1} \left(\mathbf{b} \cdot \mathbf{w} - \frac{1}{2}\mathbf{B} \cdot \mathbf{W}\right) ds + \left[\mathbf{t} \cdot \mathbf{w} - \frac{1}{2}\mathbf{T} \cdot \mathbf{W}\right]_{s_0}^{s_1} \tag{2.6}$$

By assuming the power and both forces and couples to be frame-indifferent (3), (2.6) implies the following balance equations

$$\begin{aligned} \mathbf{t}' + \mathbf{b} &= \mathbf{o} \\ \mathbf{T}' + \mathbf{x}' \wedge \mathbf{t} + \mathbf{B} &= \mathbf{O} \end{aligned} \tag{2.7}$$

The power formula resulting from (2.6) and (2.7) reads

$$\begin{aligned} \mathscr{W} &= \int_{s_0}^{s_1} w\, ds \\ w &:= \mathbf{t} \cdot (\mathbf{w}' - \mathbf{W}\mathbf{x}') - \frac{1}{2}\mathbf{T} \cdot \mathbf{W}' \end{aligned} \tag{2.8}$$

the density w being called the *stress power.*

Using (2.4) and (2.5), we can conveniently transform formula (2.8_2) into

$$w = \mathbf{s} \cdot \dot{\mathbf{u}} - \frac{1}{2} S \cdot \dot{U} \tag{2.9}$$

where

$$\mathbf{s} := \mathbf{R}^T\mathbf{t}, \ \ \mathbf{S} := \mathbf{R}^T\mathbf{T}\mathbf{R} \tag{2.10}$$

It is worth noticing that nothing has hitherto been specified about the relationship between the dynamical actions described by $(\mathbf{t}, \mathbf{T})$ and $(\mathbf{b}, \mathbf{B})$ and the motion of $\mathscr{B}$. The identification of constitutive relations for contact and inertial actions induced by an underlying finer model (such as the one described in Sec. 3) is in fact the aim of the ensuing analysis (see Sec. 4).

3. THE FINE MODEL: A PLANAR MODULAR BEAM

We consider here a planar *modular* beam whose module is composed of a pair of flat rigid diaphragms connected by a number of elastic bars constrained to remain straight. By "modular", we mean that there exists a placement such that both geometric and mechanical properties are periodic with respect to it; in the following we shall choose a periodic reference placement. The placement of each diaphragm is completely specified by the place $\mathbf{p}_o \in \mathscr{E}$ of one of its points and a (unit) vector $\mathbf{d} \in \mathscr{V}$. It can therefore be parametrized in terms of a parameter y as follows:

$$\mathbf{p}(y) = \mathbf{p}_o + y\mathbf{d}, \quad y \in [y_0, y_1] \tag{3.1}$$

We shall label with a superscript minus (−) and plus (+) respectively, two consecutive diaphragms belonging to a given module, and with a subscript b the typical bar of a module. The minus and plus end points of a bar being respectively fixed in the minus and plus diaphragms, their places $\mathbf{p}_b^-, \mathbf{p}_b^+$ are given by (see Fig. 1)

$$\begin{aligned} \mathbf{p}_b^- &= \mathbf{p}_o^- + y_b^-\mathbf{d}^- \\ \mathbf{p}_b^+ &= \mathbf{p}_o^+ + y_b^+\mathbf{d}^+ \end{aligned} \tag{3.2}$$

A placement of a module is hence completely determined by a 4-tuple $(\mathbf{p}_o^-, \mathbf{d}^-, \mathbf{p}_o^+, \mathbf{d}^+)$. We shall refer in particular to the place $\mathbf{p}_o^- + 1/2(\mathbf{p}_o^+ - \mathbf{p}_o^-)$ as the *centre* of the module.

We shall also select a reference placement $(\mathbf{P}_o^-, \mathbf{D}^-; \mathbf{P}_o^+, \mathbf{D}^+)$, and call L the *reference length* of the module, defined as $L := \| \mathbf{P}_o^+ - \mathbf{P}_o^- \|$, assuming $L > 0$.

(3) A *change of frame* is assumed here to be such that the velocity $(\mathbf{w}^*, \mathbf{W}^*)$ in a second framing is related to the velocity in the first framing by the formula

$$\mathbf{w}(s,t) - \mathbf{Q}(t)^T\mathbf{w}^*(s,t) = \mathbf{w}_o(t) - \mathbf{Q}(t)^T\dot{\mathbf{Q}}(t)(\mathbf{x}(s,t) - \mathbf{x}_o)$$

$$\mathbf{W}(s,t) - \mathbf{Q}(t)^T\mathbf{W}^*(s,t)\,\mathbf{Q}(t) = -\mathbf{Q}(t)^T\dot{\mathbf{Q}}(t)$$

where $\mathbf{x}_o \in \mathscr{E}$, $\mathbf{w}_o(t) \in \mathscr{V}$, $\mathbf{Q}(t) \in Orth.$

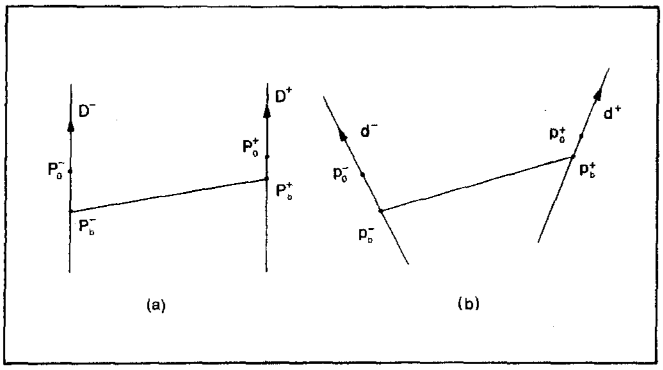

Fig. 1. Reference (a) and present (b) shape of two consecutive diaphragms and of a generic bar connecting them.

We want to consider here only a model in which all kinematical quantities are related to diaphragms. To obtain such a model, we describe each bar by the one-dimensional continuum defined in the previous section, subjected to the following constraints:

(a) shear and flexural undeformability

$$U_b = O, \quad u_b = \epsilon X'_b \tag{3.3}$$

(b) constant axial strain

$$\epsilon' = 0 \tag{3.4}$$

The bar being assumed to be straight in each one of its placements (the reference one included) the constraints (3.3), (3.4) imply that

$$x'_b = (p_b^+ - p_b^-)/\ell_b \tag{3.5}$$

$$X'_b = R_b^T (p_b^+ - p_b^-)/\| p_b^+ - p_b^- \| \tag{3.6}$$

where $\ell_b := \| P_b^+ - P_b^- \|$ is the reference length of the bar.

Under the aforementioned assumptions, the expression (2.9) for the stress power reduces to

$$w_b = n \cdot \dot{u}_b \tag{3.7}$$

where

$$n := \sigma X'_b \tag{3.8}$$

is the determined part of the contact force s introduced in (2.10).

In this context, the most general elastic constitutive relation is obtained by assigning, for all $s \in [0, \ell_b]$, the scalar function

$$\hat{\sigma}(\cdot\,; s) : \epsilon \mapsto \sigma \tag{3.9}$$

In conclusion, the power expended in a bar is given by

$$\mathscr{W}_b = \frac{\bar{\sigma}_b(\epsilon)}{\| p_b^+ - p_b^- \|} (p_b^+ - p_b^-) \cdot (p_b^+ - p_b^-)^{\circ} = \frac{\bar{\sigma}_b(\epsilon)}{\| p_b^+ - p_b^- \|} (p_b^+ - p_b^-) \cdot (p_b^+ - p_b^-)^{\circ} \tag{3.10}$$

where

$$\epsilon := \frac{1}{\ell_b} \| p_b^+ - p_b^- \| - 1 \tag{3.11}$$

$$\bar{\sigma}_b(\epsilon) := \frac{1}{\ell_b} \int_0^{\ell_b} \hat{\sigma}(\epsilon; s)\, ds$$

and the *co-rotational* time rate $\mathring{r}_b$ of a vector-valued function of time r_b is defined as

$$\mathring{r}_b = \dot{r}_b - W_b r_b \tag{3.12}$$

The power of the inertial forces acting on a bar will be assumed to be, in an inertial frame,

$$\mathscr{P}_b^{in} := -\int_0^{\ell_b} \rho_b \ddot{x}_b \cdot \dot{x}_b\, ds \tag{3.13}$$

where ρ_b denotes the mass density in the reference placement. According to (3.5), the placement of a bar is described by

$$x_b(s) = p_b^- + \frac{s}{\ell_b}(p_b^+ - p_b^-) \tag{3.14}$$

The power (3.13) can hence be given the expression

$$\begin{aligned} \mathscr{P}_b^{in} = &-(m_b^{--} \ddot{p}_b^- + m_b^{-+} \ddot{p}_b^+) \cdot \dot{p}_b^- \\ &-(m_b^{+-} \ddot{p}_b^- + m_b^{++} \ddot{p}_b^+) \cdot \dot{p}_b^+ \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} m_b^{--} &:= \int_0^{\ell_b} \rho_b(s) \left(1 - \frac{s}{\ell_b}\right)^2 ds \\ m_b^{-+} = m_b^{+-} &:= \int_0^{\ell_b} \rho_b(s) \frac{s}{\ell_b} \left(1 - \frac{s}{\ell_b}\right) ds \\ m_b^{++} &:= \int_0^{\ell_b} \rho_b(s) \left(\frac{s}{\ell_b}\right)^2 ds \end{aligned} \tag{3.16}$$

The expressions (3.10) for $\mathscr{W}_b$ and (3.15) for $\mathscr{P}_b^{in}$ are given in terms of the motion of the end points of a bar, but – after substitution of (3.2) – both $\mathscr{W}_b$ and $\mathscr{P}_b^{in}$ will depend on the motion of the diaphragms.

The power of the inertial forces acting on a diaphragm is assumed to be, in an inertial frame, (compare assumption (3.13) for a bar)

$$\mathscr{P}_d^{in} := -\int_{y_0}^{y_1} \rho_d \ddot{p} \cdot \dot{p}\, dy \tag{3.17}$$

where ρ_d denotes the mass density and p is given by (3.1).

Because each diaphragm belongs to two successive modules, we shall ascribe to each module one half of the power (3.17). Accordingly, the power of the inertial forces acting on the left and right diaphragms of a module is, respectively,

$$\begin{aligned} \mathscr{P}_-^{in} &= \frac{1}{2}[-(m_0 \ddot{p}_o^- + m_1 \ddot{d}^-) \cdot \dot{p}_o^- - (m_1 \ddot{p}_o^- + m_2 \ddot{d}^-) \cdot \dot{d}^-] \\ \mathscr{P}_+^{in} &= \frac{1}{2}[-(m_0 \ddot{p}_o^+ + m_1 \ddot{d}^+) \cdot \dot{p}_o^+ - (m_1 \ddot{p}_o^+ + m_2 \ddot{d}^+) \cdot \dot{d}^+] \end{aligned} \tag{3.18}$$

where

$$m_0 := \int_{y_0}^{y_1} \rho_d(y) dy.$$

$$m_1 := \int_{y_0}^{y_1} \rho_d(y) y dy \tag{3.19}$$

$$m_2 := \int_{y_0}^{y_1} \rho_d(y) y^2 dy$$

4. IDENTIFICATION OF THE CONSTITUTIVE PARAMETERS OF THE COARSE MODEL

We consider the local placement of a material neighbourhood of a (regular) point s of the coarse continuum to be sufficiently described (to the purpose of evaluating its mechanical response) by the values $\mathbf{x}(s)$, $\mathbf{d}(s)$, $\mathbf{x}'(s)$, $\mathbf{d}'(s)$. Hence we map this coarse local placement into a placement of a module of the fine model by

$$\begin{aligned}
\mathbf{p}_o^- &= \mathbf{x}(s) - \frac{L}{2}\mathbf{x}'(s) \\
\mathbf{p}_o^+ &= \mathbf{x}(s) + \frac{L}{2}\mathbf{x}'(s) \\
\mathbf{d}^- &= \mathbf{d}(s) - \frac{L}{2}\mathbf{d}'(s) \\
\mathbf{d}^+ &= \mathbf{d}(s) + \frac{L}{2}\mathbf{d}'(s)
\end{aligned} \tag{4.1}$$

assuming the module centered in $\mathbf{x}(s)$ and denoting by L the reference length of the module.

The next step is to characterize the dynamical actions in the coarse model so that they correspond – in a sense we now make precise – to those acting on the fine one. To this end we assign to the stress power at s of the continuum coarse model the value

$$w(s) = \frac{1}{L}\sum_b \mathscr{W}_b \tag{4.2}$$

where $\mathscr{W}_b$ is the power expended (3.10), expressed in terms of $\mathbf{x}$, $\mathbf{d}$ and their derivatives, after substitution of (3.2) and (4.1). By comparing the above expression for w with the general one (2.9), we infer the constitutive functions induced on the coarse model for the contact actions

$$\begin{aligned}
\mathbf{s}(s) &= \sum_b \mathbf{s}_b ; \quad \mathbf{s}_b := \mathbf{R}(s)^T \mathbf{R}_b(\overline{\sigma}_b \mathbf{X}'_b) \\
\mathbf{S}(s) &= \sum_b \mathbf{S}_b ; \quad \mathbf{S}_b := \frac{1}{2}(y_b^- + y_b^+)\,\mathbf{D}(s) \wedge \mathbf{s}_b
\end{aligned} \tag{4.3}$$

It is apparent that $\mathbf{s}_b$ is but the average contact force of bar b pulled back to the reference placement by $\mathbf{R}(s)$, while $\mathbf{S}_b$ is the corresponding moment with respect to the centre of the module. However, one should pay attention to the fact that in (4.3$_1$) the bar force $\mathbf{R}_b(\overline{\sigma}_b \mathbf{X}'_b)$ is intended to be given in terms of the places $\mathbf{p}_b^-$, $\mathbf{p}_b^+$ through (3.6) and (3.11); those places, in turn, are intended to be expressed in terms of the coarse placement through (3.2) and (4.1). The details of the deduction of (4.3), which is straightforward but less trivial than its outcome suggests, are given in Appendix B. Note that (4.3) defines a *simple* material and that the principle of material frame indifference is trivially satisfied.

Let us consider now the external dynamical actions. While the identification of the contact actions has been performed by using the expression of the stress power, the identification of the applied and the inertial actions will rely on the expressions of their power, in the two models and at corresponding motions. As the applied forces are problem dependent, we shall consider inertial actions only. For any part of our modular beam whose length, in the reference shape, be ℓ, let us assume that, in any motion, the power of the inertial actions for the coarse model to be

$$\mathscr{P}^{in} = \int_{s_0}^{s_1} \frac{1}{L}\left(\sum_b \mathscr{P}_b^{in} + \mathscr{P}_-^{in} + \mathscr{P}_+^{in}\right) ds \tag{4.5}$$

where $s_1 - s_0 = \ell$ and $\mathscr{P}_b^{in}$, $\mathscr{P}_-^{in}$, $\mathscr{P}_+^{in}$ are given in terms of $\mathbf{x}$, $\mathbf{d}$, and their derivatives, by (3.15) and (3.18) through substitution of (3.2), (4.1).

Expression (4.5), as can easily be seen, contains not only the velocity field $(\mathbf{w}, \mathbf{W})$, but also some derivatives with respecto to s. Therefore it cannot directly be regarded as the mechanical power of the inertial actions pertaining to the coarse model. Those actions will be identified as the dynamical fields whose power, evaluated via (2.6), is equal to that given by (4.5) for *any* velocity field $(\mathbf{w}, \mathbf{W})$, at least of class C^2.

To this end the two expressions for the power are required to have the same value for *all* the velocity fields, at least C^2, such that: a) vanish at s_0 and at s_1; or b) vanish anywhere but in $[s_0, s_0 + \epsilon) \subset [s_0, s_1]$; or c) vanish anywhere but in $(s_1 - \epsilon, s_1] \subset [s_0, s_1]$, for any positive ϵ.

The inertial action so identified turns out to be the sum of two parts

$$\int_{s_0}^{s_1} (\mathbf{b}^{in}, \mathbf{B}^{in})\, ds = \int_{s_0}^{s_1} (\hat{\mathbf{b}}, \hat{\mathbf{B}})\, ds + (\breve{\mathbf{b}}, \breve{\mathbf{B}}) \tag{4.6}$$

where $(\breve{\mathbf{b}}, \breve{\mathbf{B}})$ is a discrete measure concentrated in $\{s_0, s_1\}$ with weights $(-\overline{\mathbf{b}}_0, -\overline{\mathbf{B}}_0)$ and $(\overline{\mathbf{b}}_1, \overline{\mathbf{B}}_1)$, respectively. The densities $\hat{\mathbf{b}}$, $\hat{\mathbf{B}}$ are given by

$$\hat{\mathbf{b}} := \frac{1}{L}\left[-\left(\sum_b (m_b^{--} + m_b^{+-} + m_b^{-+} + m_b^{++}) + m_0\right)\ddot{\mathbf{x}} + \frac{L^2}{4}\left(\sum_b (m_b^{--} - m_b^{+-} - m_b^{-+} + m_b^{++}) + m_0\right)\ddot{\mathbf{x}}''\right.$$

$$-\left(\sum_b ((m_b^{--}+m_b^{+-})y_b^- + (m_b^{-+}+m_b^{++})y_b^+) + m_1\right)\ddot{\mathbf{d}}$$

$$+L\sum_b (m_b^{+-}y_b^- - m_b^{-+}y_b^+)\,\ddot{\mathbf{d}}'$$

$$+\frac{L^2}{4}\left(\sum_b ((m_b^{--}-m_b^{+-})y_b^- - (m_b^{-+}-m_b^{++})y_b^+) + m_1\right)\ddot{\mathbf{d}}'' \qquad (4.7_1)$$

$$\hat{\mathbf{B}} := \frac{1}{L}\,\mathbf{d}\wedge$$

$$\left[-\left(\sum_b (y_b^-(m_b^{--}+m_b^{-+}) + y_b^+(m_b^{+-}+m_b^{++})) + m_1\right)\ddot{\mathbf{x}}\right.$$

$$+L\sum_b (y_b^+ m_b^{+-} - y_b^- m_b^{-+})\,\ddot{\mathbf{x}}'$$

$$+\frac{L^2}{4}\left(\sum_b (y_b^-(m_b^{--}-m_b^{-+}) - y_b^+(m_b^{+-}-m_b^{++})) + m_1\right)\ddot{\mathbf{x}}''$$

$$-\left(\sum_b (y_b^-(m_b^{--}y_b^- + m_b^{-+}y_b^+) + y_b^+(m_b^{+-}y_b^- + m_b^{++}y_b^+)) + m_2\right)\ddot{\mathbf{d}}$$

$$\left.+\frac{L^2}{4}\left(\sum_b (y_b^-(m_b^{--}y_b^- - m_b^{-+}y_b^+) - y_b^+(m_b^{+-}y_b^- - m_b^{++}y_b^+)) + m_2\right)\ddot{\mathbf{d}}''\right] \qquad (4.7_2)$$

The weights of the discrete measure $(\breve{\mathbf{b}}, \breve{\mathbf{B}})$ are given, for $k = 0$ and $k = 1$, by the expressions

$$\bar{\mathbf{b}}_k := \frac{1}{2}\left[-\left(\sum_b (-m_b^{--}+m_b^{++})\right)\ddot{\mathbf{x}}(s_k)\right.$$

$$-\frac{L}{2}\left(\sum_b (m_b^{--}-m_b^{+-}-m_b^{-+}+m_b^{++}) + m_0\right)\ddot{\mathbf{x}}'(s_k)$$

$$+\left(\sum_b ((m_b^{--}-m_b^{+-})y_b^- + (m_b^{-+}-m_b^{++})y_b^+)\right)\ddot{\mathbf{d}}(s_k)$$

$$\left.-\frac{L}{2}\left(\sum_b ((m_b^{--}-m_b^{+-})y_b^- - (m_b^{-+}-m_b^{++})y_b^+) + m_1\right)\ddot{\mathbf{d}}'(s_k)\right] \qquad (4.8_1)$$

$$\bar{\mathbf{B}}_k := \frac{1}{2}\,\mathbf{d}(s_k)\wedge$$

$$\left[-\left(\sum_b (-y_b^-(m_b^{--}+m_b^{-+}) + y_b^+(m_b^{+-}+m_b^{++}))\right)\ddot{\mathbf{x}}(s_k)\right.$$

$$-\frac{L}{2}\left(\sum_b (y_b^-(m_b^{--}-m_b^{-+}) - y_b^+(m_b^{+-}-m_b^{++})) + m_1\right)\ddot{\mathbf{x}}'(s_k)$$

$$+\left(\sum_b (y_b^-(m_b^{--}y_b^- + m_b^{-+}y_b^+) - y_b^+(m_b^{+-}y_b^- + m_b^{++}y_b^+))\right)\ddot{\mathbf{d}}(s_k)$$

$$\left.-\frac{L}{2}\left(\sum_b (y_b^-(m_b^{--}y_b^- - m_b^{-+}y_b^+) - y_b^+(m_b^{+-}y_b^- - m_b^{++}y_b^+)) + m_2\right)\ddot{\mathbf{d}}'(s_k)\right] \qquad (4.8_2)$$

Notice that the constitutive prescription obtained for the inertial actions in the coarse model is far from trivial, despite the seemingly innocuous assumptions (4.1), (3.13) and (3.17), (3.14). In particular, the densities of these actions are not pointwise related to the value of the acceleration but also to the value of its first and second derivatives with respect to the material coordinate.

It should be stressed that the above constitutive prescriptions are intended to be given in an *inertial frame* and that is why they turn out not to be frame-indifferent. This is not so surprising as no axiom like the principle of material frame indifference has been introduced for body actions. Note that this does not contradicts the requirement for the inertial forces and couple to be frame indifferent, those being the *values* of constitutive functions.

Further information on the deduction of (4.7) and (4.8) is given in Appendix C.

5. CONCLUDING REMARKS

A procedure for the identification of mechanical response in a continuum model of a latticed module beam has been formally introduced. The identification consists in comparing two different abstract models and constructing a map which carries a constitutive function from one model to the other.

An interesting result, which is a consequence of some simple assumptions usually left implicit in a heuristic approach, is the form of the identified inertial actions. It may be conjectured that such a peculiar form could be important when considering phenomena such as the propagation of pulses with (relatively) short wavelength.

As a final remark, we note that boundary conditions should also be properly identified starting from the fine model: the constitutive prescription so obtained for the interactions with the external world through the boundaries could differ in interesting ways from more conventional assumptions.

APPENDIX A: RIGHT AND LEFT STRAINS

We call *right* strain the pair $(\mathbf{u}, \mathbf{U})$ defined by (2.5), and *left* strain the pair $(\mathbf{v}, \mathbf{V})$ defined by

$$\mathrm{v} := \mathrm{x}' - \mathrm{R}\mathrm{X}'$$
$$\mathrm{V} := \mathrm{R}'\mathrm{R}^T \tag{A.1}$$

This terminology is motivated by analogy with the role played by the right and left stretch tensors in the standard continuum theory. In fact, the right (left) strain stems from a comparison between quantities associated with the reference (present) placement and the corresponding quantities on the present (reference) placement pulled back (pushed forward) by **R**.

In the papers by Antman on the theory of rods and its applications (see e.g. [6]) only left "strains" are taken into consideration; the usefulness of introducing right "strains" has been suggested to us by a paper of Capriz [7].

From a strictly kinematical point of view, the two kinds of strains are obviously equivalent, because

$$\mathrm{v} = \mathrm{R}\mathrm{u}$$
$$\mathrm{V} = \mathrm{R}\mathrm{U}\mathrm{R}^T \tag{A.2}$$

so that (**v**, **V**) vanishes if and only if so does (**u**, **U**) – which is a necessary and sufficient condition for the transplacement (**u**, **R**) to be rigid. The difference between them emerges when a referential expression for the stress power (2.8) is sought: in fact, using (2.4) and (A.1) instead of (2.4) and (2.5), one gets

$$w = \mathrm{t} \cdot \mathring{\mathrm{v}} - \frac{1}{2} \mathrm{T} \cdot \mathring{\mathrm{F}} \tag{A.3}$$

instead of (2.9); the *co-rotational* time rates appearing above are defined as

$$\mathring{\mathrm{v}} := \dot{\mathrm{v}} - \mathrm{W}\mathrm{v}$$
$$\mathring{\mathrm{V}} := \dot{\mathrm{V}} - [\mathrm{W}, \mathrm{V}]$$

where $[\mathrm{W}, \mathrm{V}] := \mathrm{WV} - \mathrm{VW}$ is the commutator of W and V.

APPENDIX B: IDENTIFICATION OF THE CONTACT ACTIONS

The power expended in a bar is expressed by (3.10) as

$$\mathscr{W}_b = \frac{\overline{\sigma}_b(\epsilon)}{\| \mathrm{p}_b^+ - \mathrm{p}_b^- \|} (\mathrm{p}_b^+ - \mathrm{p}_b^-) \cdot [(\dot{\mathrm{p}}_b^+ - \dot{\mathrm{p}}_b^-) - \mathrm{W}_b(\mathrm{p}_b^+ - \mathrm{p}_b^-)] \tag{B.1}$$

Because $\mathrm{W}_b \in Skw$, it follows that

$$(\mathrm{p}_b^+ - \mathrm{p}_b^-) \cdot \mathrm{W}_b (\mathrm{p}_b^+ - \mathrm{p}_b^-) = 0 \tag{B.2}$$

But this holds true even if W_b is replaced by a skew tensor whatsoever. As we are relating a placement of the module to the tangent of a coarse placement at a point s, it is natural to substitute W(s) for W_b in (B.1), and write

$$\mathscr{W}_b = \mathrm{R}_b(\overline{\sigma}_b(\epsilon)\mathrm{X}_b') \cdot [(\dot{\mathrm{p}}_b^+ - \dot{\mathrm{p}}_b^-) - \mathrm{W}(s)(\mathrm{p}_b^+ - \mathrm{p}_b^-)] \tag{B.3}$$

where use is also made of (3.6).

By substituting (3.2), (4.1) and (2.5) within (B.3), one obtains the expression

$$\mathscr{W}_b = \mathrm{R}_b(\overline{\sigma}_b(\epsilon)\,\mathrm{X}_b') \cdot (\dot{\mathrm{x}}_b' - \mathrm{W}(s)\,\mathrm{x}_b')$$

$$= \mathrm{R}_b(\overline{\sigma}_b(\epsilon)\,\mathrm{X}_b') \cdot \left[L\mathrm{R}(s)\,\dot{\mathrm{e}}(s) + \frac{L}{2}\,(y_b^+ - y_b^-)\mathrm{W}'(s)\,\mathrm{d}(s) \right]$$

which, after substitution into (4.2) and comparison with (2.9), yields the final formulae (4.3), as given in the text. It is hardly worth mentioning that the expounded line of reasoning – while expedient to get a concise and orderly deduction – does not imply any further assumption: one could also get (4.3) starting from the "simplified" statement (3.10_2), though in a less perspicuous way.

APPENDIX C: IDENTIFICATION OF THE INERTIAL ACTIONS

By substituting, as stated in Sec. 4, expressions (3.15), (3.18) and then (3.2) and (4.1) into (4.5), an expression of the power of inertial actions is obtained which contains derivatives of the velocity field with respect to s. In order to get an expression matching with (4.4), the differentiation operating on the velocity needs to be removed through an integration by parts; this leads to

$$\mathscr{P}^{in} = \int_{s_0}^{s_1} \left(\hat{\mathrm{b}} \cdot \mathrm{w} - \frac{1}{2} \hat{\mathrm{B}} \cdot \mathrm{W} \right) ds + \left[\breve{\mathrm{b}} \cdot \mathrm{w} - \frac{1}{2} \breve{\mathrm{B}} \cdot \mathrm{W} \right]_{s_0}^{s_1} \tag{C.1}$$

Intermediate expressions for $\hat{\mathrm{b}}$, $\hat{\mathrm{B}}$, $\overline{\mathrm{b}}_k$, $\overline{\mathrm{B}}_k$, ($k = 0, 1$) (analogous to the ones given in the text for the contact actions (4.3)) are as follows

$$\hat{\mathrm{b}} := -\frac{1}{L} \sum_b \Big[(m_b^{--} + m_b^{+-})\,\ddot{\mathrm{p}}_b^- + (m_b^{-+} + m_b^{++})\ddot{\mathrm{p}}_b^+$$
$$- \frac{L}{2}(m_b^{+-} - m_b^{--})(\ddot{\mathrm{p}}_b^-)' - \frac{L}{2}(m_b^{++} - m_b^{-+})(\ddot{\mathrm{p}}_b^+)' \Big]$$
$$+ \frac{1}{2L} \Big[m_0(\ddot{\mathrm{p}}_o^+ + \ddot{\mathrm{p}}_o^-) - \frac{L}{2} m_0 (\ddot{\mathrm{p}}_o^+ - \ddot{\mathrm{p}}_o^-)'$$
$$+ m_1(\ddot{\mathrm{d}}^+ + \ddot{\mathrm{d}}^-) - \frac{L}{2} m_1 (\ddot{\mathrm{d}}^+ - \ddot{\mathrm{d}}^-)' \Big] \tag{C.2$_1$}$$

$$\hat{\mathrm{B}} := -\frac{1}{L} \mathrm{d} \wedge \sum_b \Big[(y_b^- m_b^{--} + y_b^+ m_b^{+-})\ddot{\mathrm{p}}_b^-$$
$$+ (y_b^- m_b^{-+} + y_b^+ m_b^{++})\ddot{\mathrm{p}}_b^+ - \frac{L}{2}(y_b^+ m_b^{+-} - y_b^- m_b^{--})\,(\ddot{\mathrm{p}}_b^-)'$$
$$- \frac{L}{2}(y_b^+ m_b^{++} - y_b^- m_b^{-+})(\ddot{\mathrm{p}}_b^+)' \Big]$$
$$+ \frac{1}{2L} \mathrm{d} \wedge \Big[m_1(\ddot{\mathrm{p}}_o^+ + \ddot{\mathrm{p}}_o^-) - \frac{L}{2} m_1 (\ddot{\mathrm{p}}_o^+ - \ddot{\mathrm{p}}_o^-)'$$
$$+ m_2(\ddot{\mathrm{d}}^+ + \ddot{\mathrm{d}}^-) - \frac{L}{2} m_2\,(\ddot{\mathrm{d}}^+ - \ddot{\mathrm{d}}^-)' \Big] \tag{C.2$_2$}$$

$$\overline{\mathrm{b}}_k := -\frac{1}{2} \sum_b [(m_b^{+-} - m_b^{--})\ddot{\mathrm{p}}_b^- + (m_b^{++} - m_b^{-+})\ddot{\mathrm{p}}_b^+]$$

$$+\frac{1}{4}\,[m_0(\ddot{\mathbf{p}}_o^+ - \ddot{\mathbf{p}}_o^-) + m_1(\ddot{\mathbf{d}}^+ - \ddot{\mathbf{d}}^-)], \qquad k = 0, 1 \quad (C.3_1)$$

$$\overline{\mathbf{B}}_k := -\frac{1}{2}\,\mathbf{d}(s_k) \wedge \sum_b \,[(y_b^+ m_b^{+\,-} - y_b^- m_b^{-\,-})\,\ddot{\mathbf{p}}_b$$

$$+ (y_b^+ m_b^{+\,+} - y_b^- m_b^{-\,+})\ddot{\mathbf{p}}_b^+] + \frac{1}{4}\,\mathbf{d}(s_k) \wedge [m_1(\ddot{\mathbf{p}}_o^+ - \ddot{\mathbf{p}}_o^-)$$

$$+ m_2(\ddot{\mathbf{d}}^+ - \ddot{\mathbf{d}}^-)], \qquad k = 0, 1 \qquad (C.3_2)$$

The final formulae (4.7), (4.8) given in the text are obtained by expressing explicitly $\mathbf{p}_b^-$, $\mathbf{p}_b^+$, $\mathbf{p}_o^-$, $\mathbf{p}_o^+$, $\mathbf{d}^-$, $\mathbf{d}^+$ in terms of **x**, **d**.

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REFERENCES

[1] NOOR A.K., ANDERSEN C.M., *Analysis of beam-like lattice trusses*, Comp. Meth. Appl. Mech. Engng,, Vol. 2, 1979, pp. 53-70.

[2] ABRATE S., SUN C.T., *Dynamic analysis of geometrically nonlinear truss structures*, Comp. & Structures, Vol. 17, 1983, pp. 491-497.

[3] ABRATE S., *Continuum modeling of latticed structures*, Shock and Vibration Digest, Vol. 17, 1985, pp. 15-21.

[4] ATLURI S.N., AMOS A.K., *Large Space Structures: Dynamics and Control*, Springer Verlag, 1988.

[5] MUNCASTER R.G., *Invariant manifolds in mechanics I: The general construction of coarse theories from fine theories*, Arch. Rational Mech. Anal., Vol. 84, 1983, pp. 353-373.

[6] ANTMANN S.S., *Qualitative theory of the ordinary differential equations of nonlinear elasticity*, in Mechanics Today, S. Nemat Nasser ed., Vol. 1, Pergamon Press, 1974.

[7] CAPRIZ G., *A contribution to the theory of rods*, Riv. Mat. Univ. Parma, Vol. 7, 1981, pp. 489-506.

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