J. Symbolic Computation (1985) **1**, 317–321

Using Symbolic Computation in Buckling Analysis

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(*Received* 5 *February* 1985)

1. Introduction

Asymptotic buckling analysis of elastic structures can be considered a well established procedure (Budiansky, 1974). It consists in bifurcation analysis of a system of one-parameter differential equations: balance, compatibility and constitutive equations.

The aim of this work was to experiment with the use of automatic symbolic computation in the asymptotic bifurcation analysis of elastic beams. To this end an application of the system REDUCE (Hearn, 1971) has been devised in particular for (a) the generation of the formal perturbation equations and (b) the construction of a procedure for solving a specific problem. Further details can be found in Rizzi & Tatone (1985).

A first assessment of the use of symbolic computation systems in structural mechanics can be found in Noor & Andersen (1979), while another application to the solution of perturbation problems is in Noor & Balch (1984).

2. Bifurcation Analysis

Let us consider a body acted upon by a conservative system of forces depending on a parameter λ. The balance, constitutive and compatibility equations can be set in the form

$$\begin{aligned} \langle \sigma, e'(u)\delta u\rangle - p(\lambda)f'(u)\delta u = 0 \quad \forall\ \delta u \\ \sigma = s(\varepsilon) \\ \varepsilon = e(u) \end{aligned} \tag{1}$$

where u, σ, ε represent respectively the displacement, stress and strain field and a prime denotes differentiation of any function with respect to its own argument. The first of equations (1) is the variational form of the balance equation, the bilinear form $\langle . , . \rangle$ denoting "virtual work".

The solution of the nonlinear equations (1) consists of finding the fields u, σ, ε once the potential energy $f(u)$ of the external forces and the parameter function $p(\lambda)$ have been given.

These fields will be dependent on the parameter λ, hence the solutions will be described by a curve in a differential manifold, the points of which represent all admissible fields u, σ, ε.

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Bifurcation analysis consists in looking for a bifurcating branch while moving along a known regular branch.

Let us denote the known branch by $u^f(\lambda)$, $\sigma^f(\lambda)$, $\varepsilon^f(\lambda)$, and the bifurcating branch by

$$\begin{aligned} u(t) &= u^f(\lambda(t)) + v(t) \\ \sigma(t) &= \sigma^f(\lambda(t)) + \gamma(t) \\ \varepsilon(t) &= \varepsilon^f(\lambda(t)) + \tau(t) \end{aligned} \tag{2}$$

where t is an abscissa along this new branch, which is zero at the bifurcation point.

Here we are interested in a description of the bifurcating branch in a neighbourhood of the bifurcation point, hence in calculating the series expansion

$$u(t) = u_0 + \dot{u}_0 t + 1/2 \ddot{u}_0 t^2 + o(t^2) \tag{3}$$

together with the corresponding expansions for $\sigma(t)$, $\varepsilon(t)$, $\lambda(t)$, the dot denoting differentiation with respect to t.

To this end a perturbation analysis of equations (1) is to be carried out.

First order perturbation equations are as follows

$$\begin{aligned} \langle \sigma, e''\dot{v}\, \delta u \rangle + \langle \dot{\tau}, e'\, \delta u \rangle - pf''\dot{v}\, \delta u &= 0 \\ \dot{\tau} &= s'\dot{\gamma} \\ \dot{\gamma} &= e'\dot{v} \end{aligned} \tag{4}$$

where every function is evaluated along the basic solution branch.

Second and third order perturbation equations are more and more lengthy and we do not give them here.

By imposing the condition of orthogonality, with respect to the first order solution, on the known term of the second order equations we obtain the equation

$$\begin{aligned} &\langle s''\dot{\gamma}^2, \dot{\gamma} \rangle + 3\langle \dot{\tau}, e''\dot{v}^2 \rangle + \langle \sigma, e'''\dot{v}^3 \rangle - pf'''\dot{v}^3 + \\ &\quad 2\dot{\lambda}(\langle s''\hat{\varepsilon}, \dot{\gamma}^2 \rangle + 2\langle \dot{\tau}, e''\hat{u}\dot{v} \rangle + \langle \hat{\sigma}, e''\dot{v}^2 \rangle + \\ &\qquad \langle \sigma, e'''\hat{u}\dot{v}^2 \rangle - \hat{p}f''\dot{v}^2 - pf'''\hat{u}\dot{v}^2) = 0 \end{aligned} \tag{5}$$

from which the expression for $\dot{\lambda}_0$ can be derived. The caret denotes here differentiation with respect to λ.

The corresponding orthogonality condition on the third order equations gives the expression for $\ddot{\lambda}_0$.

The solution procedure consists in solving the linear differential equations of each order and then substituting the solution into the equations of higher order.

What we have examined until now are perturbation equations written in an abstract form for a generic continuum governed by equations (1). Let us examine now how to specialize these equations to a one-dimensional continuum beam model and how such a process can be performed by some symbolic computation procedures.

3. The Beam Model

A beam immersed in a two-dimensional euclidean space is characterized by a displacement field, a strain field and contact force and couple fields which can be represented respectively by the scalar fields $u(s)$, $v(s)$, $\theta(s)$; $\varepsilon(s)$, $\gamma(s)$, $\mu(s)$; $N(s)$, $T(s)$, $M(s)$.

The symbol s denotes an abscissa along the reference beam axis.

Hence the symbols u, ε, σ appearing in equations (2) have, in this context, the following meaning†

$$\begin{aligned} u &= [u(s)\ v(s)\ \theta(s)] \\ \varepsilon &= [\varepsilon(s)\ \gamma(s)\ \mu(s)] \\ \sigma &= [N(s)\ T(s)\ M(s)] \end{aligned} \tag{6}$$

while the function $e(u)$ takes the specific form

$$\begin{aligned} \varepsilon &= u' - (k+\theta')v + \cos\theta - 1 \\ \gamma &= v' + (k+\theta')u - \sin\theta \\ \mu &= \theta' \end{aligned} \tag{7}$$

where k is the curvature in the reference beam axis and the prime denotes differentiation with respect to s.

Every term in equations (4) and (5), such as $\langle\sigma, e''\dot{v}\,\delta u\rangle$, will get the more specific meaning of the integral of the matrix product of vectors like those in equations (6) over the beam axis.

4. The Symbolic Procedure

The Exec 1100 version of REDUCE (Griss, 1979) has been used on the UNIVAC 1100/80 of the University "La Sapienza" and on the UNIVAC 1100/60 of the University of L'Aquila.

The feature of the main modules for generating the perturbation equations will be outlined here.

As clearly appears looking at the equations in the abstract form (4) and (5), what is needed first are the Gateaux differentials of the function $e(u)$ as defined in equations (7). Our module GATEAUX performs the differentiation process up to a specified order along different directions.

This module acts on identifiers which stand for ε, γ, μ. Binding these identifiers to the expressions (7) we obtain the differentials peculiar to our beam model. On the other hand a different model, i.e. expressions different from (7), can be handled as well by the same module.

It is worth noting here that the expressions (7) contains the *unknown* functions u, v, θ.

Once every term between brackets in equation $(4)_1$ has been generated, the corresponding terms in the differential balance equations are to be derived. In order to do this we have to construct the adjoint of the differential operators $e'(u)$ and $e''(u)\dot{v}$. These operators are obtained, together with the boundary conditions defining their domain, by successive integration by parts. Our module GREEN, which takes on this task, acts on an identifier which is expected to point to an expression in terms of u, v, θ, δu, δv, $\delta\theta$ and their derivatives with respect to the abscissa s. If this identifier is bound to the expression on the left side of $(4)_1$, or any part of it, and ε, γ, μ are bound to the expressions (7), the module GREEN produces the corresponding terms of the three differential balance equations, as well as the terms in the natural boundary conditions, we are looking for.

† Following the usual notation some letters happen to be used both in the abstract and in the specific expressions with slightly different meanings. The reader is warned not to confuse them.

This module makes use of the standard differentiation function DF together with the procedure COEFF for isolating the coefficients of the derivatives of δu, δv, $\delta\theta$.

Before moving on it is worthwhile to comment on the role played by the output facilities of a symbolic computation system.

In this work we faced some difficulties in dealing with symbols for denoting formal derivatives (i.e. derivatives of unknown functions). In fact taking successive derivatives of expressions like that of $e(u)$ leads to the generation of new symbols denoting derivatives of u, v, θ with respect to the abscissa s, the force parameter λ, and the perturbation parameter t. While it is easy to conceive symbols when using pencil and paper by attaching dots, carets and primes to a letter, to build up identifiers in a symbolic computation language for representing the same things is a very different matter. Using the expression DF(F, s, t) for denoting formal derivatives of a function F is unsatisfactory. Readable and meaningful expressions can be obtained by defining, as we did, the function F as an OPERATOR and using the notation $F(n, m)$ for denoting the partial derivative of F of order n with respect to the abscissa s and of order m with respect to the perturbation parameter t. In this way the differentiation consists only in increasing the indices n and m. Nevertheless the expressions are far less readable than in usual handwriting.

5. An Application

As an application of the general procedure for curved beams, the bifurcation analysis of a clamped circular arch under hydrostatic pressure has been carried out.

The constitutive equations $(4)_2$ are assumed to be

$$N = A\varepsilon; \qquad T = G\gamma; \qquad M = B\mu. \tag{8}$$

The system of external forces is defined by $p(\lambda) = \lambda$ and by the potential energy function

$$f(u) = \int_{\mathscr{C}} [(k+\theta')(u^2+v^2)/2 - v\cos\theta - u\sin\theta - u'v]\,\mathrm{d}s \tag{9}$$

Once $f(u)$ has been given the expression above its Gateaux differentials are generated. Using again the module GREEN the perturbation equations, as well as the expressions for λ_0 and $\dot{\lambda}_0$, can now be completed.

This is the end of the first major step of the whole procedure.

The second step consists of the successive solutions of the three systems, each of three ordinary linear differential equations, and of the evaluation of the expressions for λ_0 and $\dot{\lambda}_0$.

The first order system of differential equations defines an eigenvalue problem. The characteristic equation is generated, together with the solution $\dot{u}_0$, $\dot{v}_0$, $\dot{\theta}_0$, by using an interactive procedure which exploits some peculiarities of the problem at hand. An implicit expression for λ_0 is obtained as well. This turns out to be a zero of a polynomial of degree four. The calculation of the roots of this polynomial is postponed till a numerical evaluation of the solution is to be performed.

The second order system, which has the same homogeneous part as the first order system, is solved by testing some trial functions for a particular solution, thus obtaining the functions $\ddot{u}_0$, $\ddot{v}_0$, $\ddot{\theta}_0$. Also in this step the expressions for some constants appearing in the functions above are not given explicitly. They happen to be solution to a system of four algebraic linear equations. An attempt to find an explicit solution exceeded the reasonable cpu time limit. As there is no particular reason to have explicit expressions for them, their calculation is left to the final numerical evaluation.

The expressions for λ_0 and $\ddot{\lambda}_0$ are then constructed just by performing a definite integration after the substitution of the solutions $\dot{u}_0$, $\dot{v}_0$, $\dot{\theta}_0$ and $\ddot{u}_0$, $\ddot{v}_0$, $\ddot{\theta}_0$ into the general expressions. In order to integrate these expressions, which contain only simple trigonometric functions, a careful substitution technique has been used.

6. Conclusions

As a final result we have obtained a parametric solution, so retaining the possibility of dealing with a variety of cases simply performing numerical substitutions. For instance we can evaluate the solutions corresponding to different values of the arch length and the elastic constants A, G and B.

An interesting problem is how to perform numerical evaluations. The expressions for $\ddot{\lambda}_0$, for instance, turns out to be very long and, in some cases, very sensitive to rounding errors. Arranging the arithmetic operations in order to avoid disastrous cancellation errors would be a crucial point. An algorithm performing numerical evaluations in floating point representation, but able to follow the "best" path, would be a valuable tool.

As a final remark on our experience, we can say that the use of a symbolic computation system has given evidence of how even a cumbersome analysis, as was carried out, can be reduced to defining simple procedures, easy to check, which take on the task of constructing the perturbation equations and solving them. Nevertheless we should be aware of the fact that the second task cannot possibly be performed in general for problems more complex than that just examined. But even in such cases the perturbation equations, which can always be generated using the procedures above, could be the starting point for a numerical approach leading to an approximate solution.

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