Nonstandard Models for Thin-Walled Beams With a View to Applications

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A direct theory of a one-dimensional structured continuum is introduced in order to study the postbuckling behavior of thin-walled beams. A simply supported beam bent by end couples is analyzed showing that, in the case of nonsymmetric cross sections, lateral buckling gives rise to imperfection sensitivity. Then an axially loaded beam is studied taking also into account the interaction between torsional and flexural buckling. The results obtained prove that in this case imperfection sensitivity, though slighter than in the previous case, arises also for symmetric cross sections.

1 Introduction

In a previous work (Tatone and Rizzi, 1991) the writers have shown how some relevant aspects of the mechanical behavior of thin-walled beams can be described by means of a one-dimensional continuum. This continuum is endowed with a "minimal" local structure introduced *ab initio,* that is without any resort to other continuum theories. It can easily be seen that the classical (linear) theories of Wagner (1929), Kappus (1937) and Vlasov (1961), can be recovered by introducing suitable assumptions to the proposed theory and, in particular, by imposing an internal constraint.

The aim of this work is to analyze, making use of the cited theory, two cases of bifurcation of initially straight beams that can be considered exemplary in many aspects.

The first case regards a simply supported beam free to warp and loaded at the end sections by conservative bending couples. The buckling analysis performed will show that, while for the buckling load fair assumptions lead to the classical result of Timoshenko (1910), beams with nonsymmetric sections can be *imperfection sensitive,* as a consequence of the coupling between torsion, warping, and bending.

The second case consists of a cantilever beam, with warping restrained at the clamped end section, compressed by a force applied at the free end, showing both flexural and torsional buckling. By exploiting the exact kinematics of the model, the post-buckling analysis has been carried out within the framework of the asymptotic bifurcation theory. The results show that *also* beams with symmetric cross sections can show slight imperfection sensitivity, as a consequence of the interaction of the buckling modes.

It must be pointed out that all the results have been obtained in closed form, despite the complexity of the problems involved.

As an example a numerical computation is performed for the first case, just to show how to use the given general expressions to get their values corresponding to a particular cross section. Following this example one could compute those values for classes of cross sections of different shapes and get results useful from a technical point of view.

2 One-Dimensional Continuum Model

The body we call "beam" can be thought of as a continuous array of "sections" along a smooth curve (the beam axis). We will consider only motions of the body in which the beam axis remains a smooth curve while each section can undergo a rigid motion plus a warping. We will give an exact description of the motion, apart from the warping that will be taken into account by means of a single scalar parameter.

Let us consider a reference configuration for the beam, and denote by $\mathcal{C}_o$ the corresponding *beam axis.* Then we will call $X: s \mapsto X(s) \in \mathcal{E}$ a parametrization of $\mathcal{C}_o$, where $s \in [0, l]$ is a curve length parameter and $\mathcal{E}$ is a three-dimensional Euclidean space[1].

Let the reference configuration be the one assumed by the beam in the given motion at time $t = 0$. The configuration at time t can be described by a parametrization $x(\cdot, t): s \mapsto x(s, t) \in \mathcal{E}$ of the beam axis $\mathcal{C}_t$, such that $x(s, t)$ is the place at time t corresponding to the place $X(s)$ at time $t = 0$, and for each s by the proper orthogonal tensor $\mathbf{R}(s, t)$ and a scalar $\alpha(s, t)$.

In our interpretation $\mathbf{R}(s, t)$ and $\alpha(s, t)$ describe the rotation and the warping, with respect to the reference configuration, of the beam cross section at $x(s, t)$. The functions x, $\mathbf{R}$, α will be assumed to be sufficiently smooth. The tensor $\mathbf{R}(s, t)$ is regarded as a linear transformation over the three-dimensional vector space V, defined as the translation space of $\mathcal{E}$[1].

The velocity is defined by

$$\mathbf{v} := \dot{\mathbf{x}}, \quad \mathbf{W} := \dot{\mathbf{R}}\mathbf{R}^T, \quad a := \dot{\alpha}, \tag{1}$$

where a dot denotes differentiation with respect to time, and $\mathbf{W}$ is a skew symmetric tensor. A suitable definition of strain measures, suggested by a similar definition in Capriz (1981), is given as follows:

$$\mathbf{e} := \mathbf{R}^T\mathbf{x}' - \mathbf{X}', \quad \mathbf{E} := \mathbf{R}^T\mathbf{R}', \quad \beta := \alpha', \tag{2}$$

where a prime denotes differentiation with respect to s and $\mathbf{E}$ is a skew symmetric tensor.

Note that $\mathbf{W}$ and $\mathbf{E}$ express a rate of change of $\mathbf{R}$ with respect to a scalar parameter (though such a parameter is not the same). Note, further, the different role played by $\mathbf{R}^T$ in the definitions of $\mathbf{W}$ and $\mathbf{E}$. A geometric interpretation of such a difference (which is helpful to motivate the above definitions) can be given looking at rotations as elements of a Lie group as in Simo and Vu Quoc (1986, 1988). From the mechanical point of view it is worth noting that the strain measures in (2) turn out to be constant in a change of frame.

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[1] For a definition of Euclidean space and of the corresponding translation space see, e.g., Bowen and Wang (1976).

Corresponding to each point $\mathbf{x}(s, t)$, the contact actions are characterized by a vector $\mathbf{t}$, (the contact force) a skew-symmetric tensor $\mathbf{T}$ (the contact couple) and a scalar quantity ϖ, which in this context will be given the meaning of *bimoment,* while the body actions are characterized by a vector $\mathbf{b}$, a skew-symmetric tensor $\mathbf{B}$, and a scalar quantity η. Such a characterization of the actions results from the following expression of the mechanical power:

$$W := \int_0^l (\mathbf{b}\cdot\mathbf{v} + \mathbf{B}\cdot\mathbf{W} + \eta a)dS + [\mathbf{t}\cdot\mathbf{v} + \mathbf{T}\cdot\mathbf{W} + \varpi a]_0^l, \quad (3)$$

where the inner product of skew-symmetric tensors is defined as $\mathbf{B}\cdot\mathbf{W} = \frac{1}{2}\,tr(\mathbf{B}\mathbf{W}^T)$.

The conditions for the power to be objective are expressed by the following balance equations:

$$\mathbf{t}' + \mathbf{b} = \mathbf{0},$$
$$\mathbf{T}' + \mathbf{B} - \mathbf{x}' \wedge \mathbf{t} = \mathbf{0}. \quad (4)$$

Substituting these relations into the expression (3) we obtain the stress power formula

$$W = \int_0^l (\mathbf{s}\cdot\dot{\mathbf{e}} + \mathbf{S}\cdot\dot{\mathbf{E}} + \pi a + \varpi a')ds, \quad (5)$$

where $\mathbf{s} := \mathbf{R}^T\mathbf{t}$, $\mathbf{S} := \mathbf{R}^T\mathbf{T}\mathbf{R}$, $\dot{\mathbf{e}} = \mathbf{v}' - \mathbf{W}\mathbf{x}'$, $\dot{\mathbf{E}} = \mathbf{W}'$,

$$\text{and } \pi = \eta + \varpi'. \quad (6)$$

It is worth noting that this expression is a special case of (4.17) in (Green and Laws, 1966).

By choosing an orthonormal basis $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3)$ of V, let us consider a beam reference configuration with a straight axis and orthogonal plane cross sections. If we assume $\mathbf{D}_1 = \mathbf{X}'$, we can define the components of $\mathbf{s}$, $\mathbf{S}$, $\mathbf{E}$, which will be used in the sequel, through the following relations

$$\mathbf{s} = Q_1\mathbf{D}_1 + Q_2\mathbf{D}_2 + Q_3\mathbf{D}_3,$$
$$\mathbf{S} = M_1\mathbf{D}_2 \wedge \mathbf{D}_3 + M_2\mathbf{D}_3 \wedge \mathbf{D}_1 + M_3\mathbf{D}_1 \wedge \mathbf{D}_2,$$
$$\mathbf{E} = \mu_1\mathbf{D}_2 \wedge \mathbf{D}_3 + \mu_2\mathbf{D}_3 \wedge \mathbf{D}_1 + \mu_3\mathbf{D}_1 \wedge \mathbf{D}_2. \quad (7)$$

The components introduced above can be easily recognized as those usually adopted in the engineering theory of beams. In particular Q_1 is the normal component of the stress while Q_2 and Q_3 are the shear components; M_1 is the twisting component of the stress couple and M_2 and M_3 are the bending ones; finally μ_1 is the torsional deformation measure and μ_2 and μ_3 the bending ones.

It also turns out to be useful to define the vector $\mathbf{u} := \mathbf{x} - \mathbf{X}$ and its components v_i with respect to the basis

$$\mathbf{d}_i := \mathbf{R}\mathbf{D}_i, \quad i = 1, 2, 3. \quad (8)$$

A parametrization of the rotation group is given by decomposing each rotation into the product of three rotations: the first one, of amplitude θ_1, about $\mathbf{D}_1$, the second one, of amplitude θ_2, about the vector obtained by applying the first rotation to $\mathbf{D}_2$, the third one, of amplitude θ_3, about the vector obtained by applying the previous rotations to $\mathbf{D}_3$.

Assuming that the material is elastic and homogeneous, it can be proved that because of the given definitions for the strain measures, the most general constitutive relations for the contact actions are of the form

$$S = \hat{S}(\mathbf{e}, \mathbf{E}, \alpha, \beta), \quad (9)$$

where S stands for one of the $\mathbf{s}$, $\mathbf{S}$, π, ϖ.

If we want to cast into our model the property that, as shown by experimental observations, the warping is an effect of the other strain measures, it seems reasonable to assume an internal constraint of the form

$$\alpha = \hat{\alpha}(\mathbf{e}, \mathbf{E}), \quad (10)$$

with $\hat{\alpha}(\mathbf{o}, \mathbf{O}) = 0$. This constraint, for any objective scalar function $\hat{\alpha}$, turns out to be objective, any change of frame leaving both $\mathbf{e}$ and $\mathbf{E}$ unchanged. By limiting ourselves to consider a linear constraint between warping and torsion, we put (10) in the form

$$\alpha = k\mu_1, \quad (k \in \mathbb{R}) \Rightarrow \beta = k\mu_1'. \quad (11)$$

Such a choice, despite its simplicity, proves to be effective for the cases we are interested in studying.

It can be useful to give some motivations about the assumptions on warping in (11). To this end the reference to theories in which one-dimensional beam models are derived from two or three-dimensional continua is the right tool to use.

In a recent work Simo and Vu Quoc (1991), dealing with the derivation of a one-dimensional beam model from a three-dimensional Cauchy continuum, assumed that for each cross section of the beam, the out-of-plane component of the displacement field (that we call w) was related to the warping deformation measure (α in this paper) by the following relationship

$$w = f\alpha, \quad (12)$$

f being the Saint-Venant warping function. Denoting by D the beam cross section, it follows immediately that $\alpha = \gamma \int_D w$, where $\gamma = (\int_D f)^{-1}$.

Such a result furnishes one of the possible kinematical meaning for the warping deformation measure α which results to be a weighted integral over the beam cross section of the out-of-plane displacement.

Coming then to the constraint (11) with the expression (12) in mind, one obtains $w = kf\mu_1$ that, in turn, can be seen as a generalization of the Saint-Venant solution to the uniform torsion problem.

Incidentally, it must be said that the constraint (11) is the one underlying the Vlasov's model.

Further, we will assume the material to be shear-indeformable and require

$$\mathbf{e} = \epsilon\,\mathbf{X}', \quad (13)$$

ϵ being the axial extensional deformation measure.

By the axiom of determinism for constrained materials—which states that the stress is determined by the motion only to within an arbitrary additive part which does no work in any motion compatible with the constraint—the indeterminate part of the stress corresponding to the above constraints, which will be denoted by a subscript v, must satisfy the following scalar relations

$$Q_{1v} = 0,\ M_{1v} = -k\pi_v,\ M_{2v} = 0,\ M_{3v} = 0,\ \varpi_v = 0, \quad (14)$$

while Q_{2v}, Q_{3v}, as well as π_v, can take any value.

The constitutive functions, consistent with all the material assumptions made till now, will be assumed to be

$$Q_1 = a\epsilon + \tfrac{1}{2}p\mu_1^2, \quad Q_2 = Q_{2v}, \quad Q_3 = Q_{3v},$$
$$M_1 = (c + q_2\mu_2 + q_3\mu_3 + p\epsilon + r\beta)\mu_1 + M_{1v},$$
$$M_2 = b_2\mu_2 + \tfrac{1}{2}q_2\mu_1^2, \quad M_3 = b_3\mu_3 + \tfrac{1}{2}q_3\mu_1^2,$$
$$\varpi = h\beta + \tfrac{1}{2}r\mu_1^2, \quad \pi = \pi_v. \quad (15)$$

If we also assume $\eta = 0$, from (6) and $(15)_8$ follows that

$\pi_v = \varpi'$. Then, by virtue of (11), the relations $(15)_4$, $(15)_7$, and $(15)_8$ finally become

$$M_1 = (c + q_2\mu_2 + q_3\mu_3 + p\epsilon)\mu_1 - k^2 h\mu_1'',$$

$$\varpi = kh\mu_1' + \tfrac{1}{2}r\mu_1^2,$$

$$\pi = kh\mu_1'' + r\mu_1\mu_1'. \tag{16}$$

Although (15) and (16) are simple polinomial expressions, some comments must be done in order to better understand their mechanical meaning. It is clear that the coefficients a, b, c, stand for the axial, flexural, and torsional stiffnesses, while kh is the warping stiffness. The coefficient p accounts for the coupling between the torsion and the extension, and is responsible for the Poynting effect in the one-dimensional model. In fact it is a classical result in nonlinear elasticity that the axial force must depend nonlinearly on the torsion (see Truesdell and Noll, 1965). The same effect leads to the relationships between flexural couples and torsion which is accounted for by the coefficients q. Finally, the assumption that the bimoment depends nonlinearly on the torsion has been introduced here in view of the results shown by Møllmann (1986).

By transforming the strain-displacement relations (2) and the balance Eq. (4) into their scalar form, one obtains, by means of (15) and (16), a set of nonlinear field equations that, supplied with the boundary conditions, gives rise to a nonlinear boundary value problem.

3 Bifurcation Analysis

The nonlinear boundary value problems we are going to face are of the form

$$F(u, \lambda) = 0. \tag{17}$$

A solution (u, λ) consists of a set of functions describing a configuration, and a scalar parameter λ affecting the applied actions, such that the balance equations (4), together with the appropriate boundary conditions, are satisfied.

The graph of the solutions (u, λ) can be thought of as being made of regular branches. Usually a branch $(u^f(\tau), \lambda^f(\tau))$ crossing the point $(0, 0)$, where $u = 0$ denotes the reference configuration, is easily computed and one is interested in finding a bifurcation point on it. Denoting by $(u^b(t), \lambda^b(t))$ an intersecting branch, assume the parametrization of the two branches be such that $u^b(0) = u^f(\tau_c)$, $\lambda^b(0) = \lambda^f(\tau_c) = \lambda_c$, and define the new set of functions

$$v := u^b - u^f, \tag{18}$$

made up with the differences of the corresponding functions in u^b and u^f.

As v inherits the parametrization from the intersecting branches, by assuming there exists a diffeomorphism relating the parameters t and τ, we can give the following series expansions near $t = 0$

$$v_t = \tilde{v}_c t + \tfrac{1}{2}\tilde{\tilde{v}}_c t^2 + o(t^2),$$

$$\lambda_t = \lambda_c + \tilde{\lambda}_c t + \tfrac{1}{2}\tilde{\tilde{\lambda}}_c t^2 + o(t^2), \tag{19}$$

where the subscript c denotes values at $t = 0$.

By using (18) and (19), the problem (17) can be recast in a sequence of linear systems of differential equations with corresponding boundary conditions. The first system results in an *eigenvalue problem* whose solution gives λ_c and $\tilde{v}_c$, while $\tilde{\lambda}_c$ is obtained by imposing a solvability condition to the second order system whose solution leads to $\tilde{\tilde{v}}_c$ and so on. As the aim of this section is just to introduce the symbols used in the next sections, the interested reader is referred to Budiansky (1974).

4 Simply Supported Beam Bent by Terminal Couples

In this section we will perform a bifurcation analysis of an initially straight beam of length l, simply supported at both ends, where the warping is free and the axial rotation is restrained, loaded by two opposite conservative bending couples.

Remembering that dead couples are not conservative, we will recover such a property if each couple is obtained by attaching two (opposite) dead forces to two (different) points of the same end section. Denoting by d the distance between the lines of action of the two forces and by f their intensity, we set

$$\mathbf{T}(l) = \lambda\mathbf{D}_1 \wedge \mathbf{d}_2, \quad \mathbf{T}(0) = -\mathbf{T}(l), \quad \text{with} \quad \lambda := fd. \tag{20}$$

In order to simplify the analysis, we assume the beam to be flexural indeformable in the plane spanned by the applied forces in the reference configuration, by adding the following internal constraint:

$$\mu_3 = 0. \tag{21}$$

The appropriate boundary conditions for the case at hand are

$$\mathbf{u} = \mathbf{0}, \quad M_1 = M_2 = \theta_1 = \varpi = 0, \quad \text{at} \quad s = 0,$$

$$v_2 = v_3 = Q_1 = M_1 = M_2 = \theta_1 = \varpi = 0, \quad \text{at} \quad s = l. \tag{22}$$

It can be easily seen that the stated boundary value problem, due to the assumed constitutive functions, has at least the following solution:

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{R} = \mathbf{I}, \quad \alpha = 0,$$

$$\mathbf{s} = \mathbf{0}, \quad \varpi = 0, \quad \mathbf{S} = \lambda\mathbf{D}_1 \wedge \mathbf{D}_2, \tag{23}$$

which means that the reference configuration is balanced for any value of λ.

In order to investigate the possible bifurcations from the given solution, we linearize the field equations together with the boundary conditions near the above solution and arrive at the following equations:

$$b_2\tilde{v}_3'''' - \lambda\tilde{\theta}_1'' = 0, \quad k^2h\tilde{\theta}_1'''' - c\tilde{\theta}_1'' - \lambda\tilde{v}_3'' = 0, \tag{24}$$

with the boundary conditions

$$\tilde{v}_3 = \tilde{v}_3'' = 0 \quad \text{at} \quad s = 0, \quad \text{and} \quad s = l,$$

$$\tilde{\theta}_1 = \tilde{\theta}_1'' = 0 \quad \text{at} \quad s = 0, \quad \text{and} \quad s = l, \tag{25}$$

where $\tilde{}$ has the meaning introduced in Section 3.

By solving the eigenvalue problem (24), (25), one obtains the buckling load

$$\lambda_c = \frac{\pi}{l}\sqrt{b_2 c}\sqrt{1 + \frac{k^2h}{c}\left(\frac{\pi}{l}\right)^2}, \tag{26}$$

with the associate buckling mode

$$\tilde{v}_3 = -\frac{l}{\pi}\sqrt{\frac{c}{b_2}}\sqrt{1 + \frac{k^2h}{c}\left(\frac{\pi}{l}\right)^2}\sin\left(\frac{\pi s}{l}\right),$$

$$\tilde{\theta}_1 = \sin\left(\frac{\pi s}{l}\right) \tag{27}$$

normalized by setting $\tilde{\theta}_1(l/2) = 1$.

The increment of the load parameter corresponding to the solution of the eigenvalue problem is given by

$$\tilde{\lambda}_c = \frac{\pi}{l^2}\left(q_2 + \frac{\pi^2}{l}\frac{b_2 k r}{\lambda_c}\right), \qquad (28)$$

showing that the beam considered can have an asymmetric bifurcation, that is, it *can be imperfection sensitive.*

Such a behavior originates from the fact that both the bending couple M_2 and the bimoment ϖ depend on the torsion μ_1, through the constants q_2 and r, respectively. It can be shown that if the beam cross section has at least one axis of symmetry, then $r = 0$ (Tatone, 1992), while q_2 vanishes only if the axis of symmetry is parallel to $\mathbb{D}_2$. Therefore one can have $\tilde{\lambda}_c \neq 0$ also if the beam section has one axis of symmetry, as shown in the following example.

It is interesting to examine Eqs. (V.1.10) obtained by Vlasov (1961) in its classical work, in order to evaluate the critical mode for "beams loaded at the ends by longitudinal forces and moments." Despite the approaches being completely different, in the balance Eq. (V.1.10)$_3$ can be recognized a constitutive function for the twisting couple M_1. By comparing that expression with (16)$_1$, and assuming a cartesian coordinate system with the origin at the centroid of the cross section and basis vectors $\mathbb{D}_1$, $\mathbb{D}_2$ collinear to the principal axes of inertia, the following identifications result:

$$a = EA,\quad q_2 = EI_{q_2},\quad b_2 = EI_2,\quad q_3 = EI_{q_3},$$

$$b_3 = EI_3,\quad p = EI_p,\quad c = GI_c,\quad k^2h = EI_\omega. \qquad (29)$$

In (29), E and G are the Young and shear moduli, A is the area of the cross section, I_2 and I_3 the principal moments of inertia, I_c and I_p the moments of inertia with respect to the centroid and the shear center, I_ω is the sectorial warping rigidity, and $I_{q_2} = \int_D x_2\rho^2 dA$, $I_{q_3} = -\int_D x_3\rho^2 dA$ where x_2, x_3 are the coordinates of a point of the cross section and ρ its distance from the shear center. Note that as a consequence of the above identifications, the expression (26) becomes identical to the classical result of Timoshenko (1910).

Now, with the purpose of showing a numerical computation, let us consider a C-section beam with the end couples acting on a plane parallel to the web. By using the expressions for the constants in (29) taken from Vlasov's theory, we can first evaluate them and then compute the numerical values of λ_c and $\tilde{\lambda}/\lambda_c$.

Denoting by b the width of the flanges, h the height of the web, t the thickness, for the values $E = 2.1\ 10^4$ kg/mm^2, $G = 0.84\ 10^4$ kg/mm^2, $t = 3$ mm, $h = 100$ mm we get the following results:

b	l	$\lambda_c \times 10^6$	$\tilde{\lambda}/\lambda_c$
50	1000	1.520	0.794
100	1000	9.020	1.360
150	1000	26.700	1.990
50	2000	0.487	0.622
100	2000	2.440	1.250
150	2000	6.930	1.920

5 Axially Compressed Cantilever Beam

We consider next a straight beam of length l clamped at one end, where the warping is constrained to be zero, and compressed by an axial force λ applied at the free end. Thus the boundary conditions are

$$\mathbf{u} = \mathbf{0},\quad \mathbb{R} = \mathbb{I},\quad \alpha = 0,\quad \text{at}\quad s = 0,$$

$$\mathbf{s} = -\lambda\mathbb{D}_1,\quad \mathbf{S} = \mathbf{0},\quad \varpi = 0,\quad \text{at}\quad s = l. \qquad (30)$$

In the case of double symmetric cross sections ($q_2 = q_3 = r = 0$ in the (15), (16)), the resulting boundary value problem can easily be seen to have the following solution:

$$v_1 = -\lambda/a,\quad v_2 = 0,\quad v_3 = 0,\quad \alpha = 0,\quad \mathbb{R} = \mathbb{I},$$

$$Q_1 = -\lambda,\quad Q_2 = 0,\quad Q_3 = 0,\quad \mathbb{S} = \mathbf{0},\quad \varpi = 0. \qquad (31)$$

It can be shown that the previous solution has bifurcating branches that usually correspond to an axial-torsional behavior for the beam or to an axial-flexural behavior. By selecting from each of the two classes the branch corresponding to the smallest critical load, the following results are obtained.

The flexural mode is characterized by

$$\lambda_c = \frac{a}{2}\left(1 - \sqrt{1 - \frac{\pi^2}{l^2}\frac{b_3}{a}}\right),\quad \tilde{\lambda}_c = 0,$$

$$\frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} = \frac{\pi^2 a^2}{16 l^2}\frac{a - 4\lambda_c}{(a - \lambda_c)^2(a - 2\lambda_c)}. \qquad (32)$$

There is nothing interesting about it but the fact that $0 \le \lambda_c \le a/2$. As a result, the ratio $\tilde{\tilde{\lambda}}_c/\lambda_c$ becomes negative whenever $\lambda_c > a/4$. The bifurcated branch, up to the second order, is given by

$$\tilde{v}_1 = 0,\quad \tilde{v}_2 = \cos\frac{\pi s}{2l} - 1,$$

$$\tilde{\tilde{v}}_2 = 0,\quad \tilde{\tilde{v}}_1 = \frac{\pi}{2l}\frac{a(a - 2\lambda_c)}{4(a - \lambda_c)^2}\left[\sin\left(\frac{\pi s}{l}\right) - \frac{\pi s}{l}\right], \qquad (33)$$

being $v_3 = \theta_1 = 0$. Here the normalization condition $\tilde{v}_2(l) = -1$ has been assumed.

The torsional bifurcation mode is characterized by

$$\lambda_c = \frac{a}{p}\left(c + \frac{\pi^2}{4}\frac{k^2h}{l^2}\right) = \frac{a(4cl^2 + \pi^2k^2h)}{4l^2p},\quad \tilde{\lambda}_c = 0,$$

$$\frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} = -\frac{1}{\lambda_c}\frac{3\pi^2 p}{16 l^2} = -\frac{3\pi^2 p^2}{4a(4cl^2 + \pi^2k^2h)}, \qquad (34)$$

and the associate bifurcated solution up to the second order is

$$\tilde{\tilde{v}}_1 = 0,\quad \tilde{\theta}_1 = \cos\left(\frac{\pi s}{2l}\right) - 1,$$

$$\tilde{\tilde{v}}_1 = \frac{\pi p}{8al}\left[\sin\left(\frac{\pi s}{l}\right) - \frac{\pi s}{l}\right],\quad \tilde{\tilde{\theta}}_1 = 0, \qquad (35)$$

being $v_1 = v_3 = 0$. The normalization has been chosen so that $\tilde{\theta}_1(l) = -1$. Note that such a bifurcation can occur only if $p > c + \pi^2k^2h/4l^2$ and the value of $\tilde{\tilde{\lambda}}_c/\lambda_c$ is negative no matter the values of the constants of the constitutive functions are. This means that, despite what can be guessed by looking only at the results of the linearized analysis, torsional buckling due to an axial force is *imperfection sensitive.*

If the constitutive parameters and the length of the beam are such that the buckling loads in (32) and (34) take the same value, the two buckling modes occur simultaneously and the first-order part of the bifurcation branches can be put in the form

$$\tilde{u} = \nu_1\tilde{u}_I + \nu_2\tilde{u}_{II}, \qquad (36)$$

where $\tilde{u}_I$ and $\tilde{u}_{II}$ stand for the torsional mode and the flexural mode, respectively.

The resulting mode interaction problem has been solved by following the procedure outlined by Pignataro and Rizzi (1982). It turns out that $\tilde{\lambda}_c = 0$ while, besides the torsional and flexural buckling modes occurring separately, two more solutions appear. Normalizing the coefficients ν_1, ν_2 by assuming that $\nu_1^2 + \nu_2^2 = 1$, the expressions for $\tilde{\tilde{\lambda}}_c$ and ν_1^2 can be easily obtained by solving the following two equations:

$$\frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} = -\frac{3p\omega_1^2\nu_1^2}{4\lambda_c} - \left[\frac{16\tan(l\omega_3)\omega_1^2(\omega_1^2-\omega_3^2)^2}{lp(a-\lambda_c)\omega_3^3(4\omega_1^2-\omega_3^2)^2} + \frac{2(a-\lambda_c)(\omega_3^2-\omega_1^2)(\omega_3^2+8\omega_1^2) - 3\omega_1^2\omega_3^2p(4\omega_1^2-\omega_3^2)}{4p(a-\lambda_c)^2(4\omega_1^2-\omega_3^2)\omega_3^2}\right]$$

$$\times a^2l^2(1-\nu_1^2)\frac{a^2l^2\omega_1^2(a-4\lambda_c)}{8p(a-\lambda_c)^3}(1-\nu_1^2)$$

$$+\left[\frac{8\tan(l\omega_3)\omega_1^2(\omega_1^2-\omega_3^2)^2}{lp\omega_3^3(4\omega_1^2-\omega_3^2)^2} + \frac{2(a-\lambda_c)(\omega_3^2-\omega_1^2)(\omega_3^2+8\omega_1^2) - 3\omega_1^2\omega_3^2p(4\omega_1^2-\omega_3^2)}{8p(a-\lambda_c)(4\omega_1^2-\omega_3^2)\omega_3^2}\right]\times\nu_1^2$$

$$-\frac{(a-2\lambda_c)}{2p(a-\lambda_c)}\frac{\tilde{\tilde{\lambda}}_c}{\lambda_c} = 0 \quad (37)$$

where $\omega_1 = \pi/(2l)$ and $\omega_3 = |\sqrt{\lambda_c(a-\lambda_c)/(ab_2)}|$.

It must be noted that, depending on the values of ω_1 and ω_3, the value of $\tilde{\tilde{\lambda}}_c/\lambda_c$ given by the expression (37) can become negative and hence the beam can be imperfection sensitive.

6 Conclusions

The aim of this work was to describe a one-dimensional theory for thin-walled beams and to apply it to two simple problems. The given results show that, within the formal setting of the mechanical theory, a simple choice of the constitutive relations makes the model able to describe qualitatively some global behavior like flexural-torsional or axial-torsional bifurcations.

As an example of application of the theory some numerical results have been shown for the case of a beam bent by terminal couples. To this end the constitutive parameters have been identified with those underlying Vlasov's linear theory.

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