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Foreword

The International Society for the Interaction of Mechanics and Mathematics (ISIMM) was founded in 1977. Its purpose is to promote cooperative research involving the fields of mechanics and pure mathematics.

Its Executive Committee decided that, from time to time, scholarly works relevant to the Society's interests should, by invitation, be published under its auspices. The present volume is one in this series which, it is hoped, will help to advance the objective of the Society.

The Editorial Board

A. TATONE AND N. RIZZI

A one-dimensional model for thin-walled beams

Abstract

A one-dimensional continuum model endowed with local structure, whose kinematical states are characterized by a four-dimensional differentiable manifold, is presented. It is shown that this model allows a suitable description of the relevant aspects of the mechanical behaviour of thin-walled beams. Further it is shown that, by introducing a suitable characterization of the material, the linearized model coincides with Vlasov's model.

1. Introduction

In the first decades of this century there was a large increase in the use of open-section beams with thinner walls, essentially due to the needs of the aeronautical industry. The growing interest in the structural behaviour of such members resulted in efforts to give a general theory to be adopted as a proper design tool.

In a pioneer work H. Wagner [1] pointed out as distinctive features of the behaviour of a thin-walled beam: (a) the considerable increase in its torsional rigidity when warping is prevented in some section; (b) the arising of flexural-torsional buckling under axial load long before the reaching of the Eulerian buckling load. Then, starting from the classical Saint-Venant results for torsion and introducing the so called *unitary warping* concept, he proposed a *linear one-dimensional theory* able to describe the aforementioned phenomena.

That theory, later refined by R. Kappus [2] in order to attain a better agreement with experimental results, was finally recast in a more systematic way by Vlasov [3] and so accepted as the standard, technical, one-dimensional *linear* theory for thin-walled beams. Furthermore, progresses in nonlinear mechanics stimulated researches also in the field of thin-walled beams.

Looking at these works it seems to the writers that they share a basic idea: one-dimensional models do have a meaning only if obtained by a two- or three-dimensional one, via more or less consistent projection methods (see e.g. [4]). A different point of view - as far as the authors know - can be found only in a work by M. Epstein [5], who resorts to the *direct* theory of one-dimensional structured continua in order to introduce a model that, conceived for beams with particular cross sections, turns out to be very complicated.

The aim of this work is to present in a direct way a one-dimensional continuum model with a local structure conceived to be the *minimal* one able to describe at least the phenomena remarked by Wagner. From the kinematical point of view the configuration space for the local structure is a four-dimensional differential manifold obtained by adding one parameter to the descriptor of the rotations of the beam section. The additional parameter is a coarse descriptor of the warping of the section.

By introducing a standard expression for the mechanical power, *contact* and *body* dynamical actions are defined as quantities algebraically dual to those describing the velocity field. An expression for the mechanical power in terms of the sole contact actions is obtained.

A characterization of the material is given by introducing an internal constraint between warping and torsion and by defining a suitable class of constitutive functions. In particular it is assumed that the torsional component of the contact couple depends on the axial and flexural components of the strain measures.

It is finally shown that Vlasov's model can be recovered by the proposed model in the context of a linearized theory.

2. One-dimensional continuum model

We regard a beam as a differentiable manifold $\mathcal{B}$, and its motion as a function

$$\chi : \mathcal{B} \times I \rightarrow \mathcal{E} \times \mathcal{F} \tag{2.1}$$

where I is an interval in the field of real numbers $\mathcal{R}$, $\mathcal{E}$ is a three-dimensional Euclidean space, whose translation space is $\mathcal{V}$, $\mathcal{F}$ is the four-dimensional product manifold $SO(\mathcal{V}) \times \mathcal{R}$.

Let the motion of $\mathcal{B}$ be such that for any placement $\chi_t := \chi(\cdot, t)$ the natural projection $C_t := p(\chi_t(\mathcal{B}))$ of $\chi_t(\mathcal{B})$ into $\mathcal{E}$ is a one-dimensional differentiable manifold.

Let us consider a reference placement

$$\kappa : \mathcal{B} \rightarrow \mathcal{E} \times \mathcal{F}, \tag{2.2}$$

such that $\kappa : \mathcal{B} \rightarrow \kappa(\mathcal{B})$ is a diffeomorphism, and a parametrization $\mathbf{X}(s)$ of $C_\kappa := p(\kappa(\mathcal{B}))$, being $s \in [0, L]$ a curve-length parameter. With respect to the reference shape $\kappa(\mathcal{B})$, any placement χ_t can be described by a function $\mathbf{x}$ which maps C_κ to C_t and, for each point of C_κ, by a proper rotation $\mathbf{R}$ and a scalar quantity α. By using the parametrization of C_κ we will regard $\mathbf{x}, \mathbf{R}$ and α as functions on $[0, L] \times I$ and will assume them to be sufficiently smooth.

In our interpretation, $\mathbf{R}(s, t)$ and $\alpha(s, t)$ describe the rotation and the *warping*, with respect to the reference shape, of the beam cross-section corresponding to the point $\mathbf{x}(s, t)$ of the beam axis C_t.

The velocity is defined by

$$\mathbf{v} := \dot{\mathbf{x}},$$

$$\mathbf{W} := \dot{\mathbf{R}}\,\mathbf{R}^{\mathrm{T}}, \tag{2.3}$$

$$a := \dot{\alpha},$$

where a dot denotes differentiation with respect to time, while a suitable definition of strain measures, suggested by a similar definition in [6], is given as follows

$$\mathbf{e} := \mathbf{R}^{\mathrm{T}}\,\mathbf{x}' - \mathbf{X}',$$

$$\mathbf{E} := \mathbf{R}^{\mathrm{T}}\mathbf{R}', \tag{2.4}$$

$$\beta := \alpha',$$

where a prime denotes differentiation with respect to s.

Corresponding to each point $\mathbf{x}(s, t)$, the contact actions are characterized by a vector $\mathbf{t}$, a skew-symmetric tensor $\mathbf{T}$ and a scalar quantity ϖ, which in this context will be given the meaning of *bimoment*, while the body actions are characterized by a vector $\mathbf{b}$, a skew-symmetric tensor $\mathbf{B}$ and a scalar quantity η. Such a characterization of the actions results from the following expression of the power[1]

$$\mathcal{W} := \int_0^L (\mathbf{b}\cdot\mathbf{v} + \mathbf{B}\cdot\mathbf{W} + \eta a)\mathrm{d}S + [\,\mathbf{t}\cdot\mathbf{v} + \mathbf{T}\cdot\mathbf{W} + \varpi\, a\,]_0^L . \tag{2.5}$$

The conditions for the power to be objective are expressed by the following balance equations

$$\mathbf{t}' + \mathbf{b} = \mathbf{0},$$

$$\mathbf{T}' + \mathbf{B} - \mathbf{x}' \wedge \mathbf{t} = \mathbf{0}. \tag{2.6}$$

Substituting these relations into the expression (2.5) we obtain the stress power formula

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[1] The inner product for skew-symmetric tensors has been defined as

$$\mathbf{B}\cdot\mathbf{W} = \frac{1}{2}\mathrm{tr}\,(\mathbf{B}\mathbf{W}^{\mathrm{T}}).$$

$$\mathcal{W} = \int_0^L [\mathbf{t} \cdot (\mathbf{v}' - \mathbf{W}\mathbf{x}') + \mathbf{T} \cdot \mathbf{W}' + \pi a + \varpi a'] \, ds$$

$$= \int_0^L [\mathbf{s} \cdot \dot{\mathbf{e}} + \mathbf{S} \cdot \dot{\mathbf{E}} + \pi a + \varpi a'] \, ds \tag{2.7}$$

where $\mathbf{s} := \mathbf{R}^T\mathbf{t}$ e $\mathbf{S} := \mathbf{R}^T\mathbf{T}\mathbf{R}$ and

$$\pi = \eta + \varpi'. \tag{2.8}$$

It is worth noting that this expression is a special case of (3.17) in [7].

Assuming that the material is elastic and homogeneous, it can be proved that the most general constitutive relations are of the form

$$\begin{aligned} \mathbf{s} &= \hat{\mathbf{s}}(\mathbf{e}, \mathbf{E}, \alpha, \beta), \\ \mathbf{S} &= \hat{\mathbf{S}}(\mathbf{e}, \mathbf{E}, \alpha, \beta), \\ \pi &= \hat{\pi}(\mathbf{e}, \mathbf{E}, \alpha, \beta), \\ \varpi &= \hat{\varpi}(\mathbf{e}, \mathbf{E}, \alpha, \beta). \end{aligned} \tag{2.9}$$

By choosing an orthonormal basis $\{\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3\}$ of $\mathcal{V}$, such that $\mathbf{D}_3 = \mathbf{X}'$, we can introduce the components of $\mathbf{s}, \mathbf{S}, \mathbf{E}$, which will be used in the next section, as follows

$$\mathbf{s} = Q_1\mathbf{D}_1 + Q_2\mathbf{D}_2 + N\mathbf{D}_3,$$

$$\mathbf{S} = M_1\,\mathbf{D}_2 \wedge \mathbf{D}_3 + M_2\,\mathbf{D}_3 \wedge \mathbf{D}_1 + T\mathbf{D}_1 \wedge \mathbf{D}_2, \tag{2.10}$$

$$\mathbf{E} = \mu_1\mathbf{D}_2 \wedge \mathbf{D}_3 + \mu_2\mathbf{D}_3 \wedge \mathbf{D}_1 + \tau\mathbf{D}_1 \wedge \mathbf{D}_2.$$

It also turns out to be useful to define the vector $\mathbf{u} := \mathbf{x} - \mathbf{X}$ and its components v_i with respect to the basis

$$\mathbf{d}_i := \mathbf{R}\,\mathbf{D}_i, \quad i = 1, 2, 3. \tag{2.11}$$

3. Constrained model and comparison with Vlasov's model

We now want to show that, by assuming a suitable characterization of the material, Vlasov's model can be obtained as a special case of the continuum model described in the previous section.

Let us first note that, if we want to cast into our model the property that, as shown by experimental observations, the warping is an effect of the other strain measures, it seems reasonable to assume an internal constraint of the form

$$\alpha = k(\mathbf{e}, \mathbf{E}), \quad k(\mathbf{o}, \mathbf{O}) = 0. \tag{3.1}$$

For any objective scalar function k, this constraint turns out to be objective because any change of frame leaves both $\mathbf{e}$ and $\mathbf{E}$ unchanged.

By the axiom of determinism for constrained materials, which states that the stress is determined by the motion only to within an arbitrary additive part which does no work in any motion compatible with the constraint, the indetermined part of the stress, which will be denoted by a subscript c, corresponding to the above constraint has to satisfy the following relations

$$\mathbf{s}_c = -\pi_c k_{\mathbf{e}},$$
$$\mathbf{S}_c = -\pi_c k_{\mathbf{E}}, \tag{3.2}$$
$$\varpi_c = 0,$$

where k_e and $k_{\mathbf{E}}$ stand for the derivatives of k with respect to its arguments.

Let us assume now a special form of the above constraint, which could be seen as reflecting more closely the observation that warping is related to torsion, by replacing (3.1) with

$$\alpha = \check{k}(\tau), \quad \check{k}(0) = 0. \tag{3.3}$$

In general we do not expect this constraint to model the observed property more accurately than the previous constraint. We only want to lay down the assumptions which lead our model to match Vlasov's model. In the same spirit we shall introduce all the assumptions below.

Besides the internal constraint (3.3) let us assume, as in Vlasov's model, the material to be shear-undeformable. This constraint is defined in our model by the relation

$$\mathbf{e} = \varepsilon \mathbf{X}'. \tag{3.4}$$

The indetermined part of the stress, corresponding to the constraints (3.4) and (3.3), must satisfy the following scalar relations

$$N_c = 0,\ M_{1c} = 0,\ M_{2c} = 0,\ T_c = -\pi_c(d\check{k}/d\tau),\ \varpi_c = 0, \tag{3.5}$$

while Q_{1c}, Q_{2c}, as well as π_c, can take any value.

In order to compare the two models let us consider an elastic beam, with a straight reference shape, subject to terminal loads which consist of an axial force and a couple at each end defined by $\mathbf{t}(L) = -\mathbf{t}(0) = \lambda f \mathbf{D}_3, \mathbf{T}(L) = -\mathbf{T}(0) = \lambda F\, \mathbf{D}_2 \wedge \mathbf{D}_3$, where f and F are constants and λ is a parameter.

Let us assume that, for $\lambda = \lambda_0 \neq 0$, a solution exists which is characterized by

$$\begin{gathered}\mu_2 = 0,\ \tau = 0,\\ Q_1 = 0,\ M_2 = 0, T = 0,\\ \pi_a = 0,\ \varpi_a = 0,\end{gathered} \tag{3.6}$$

where the subscript a denotes the determined part of the stress. Linearization of the scalar form of equations (2.6) and (2.8) at the above solution yields, denoting by a tilde differentation with respect to λ, the following equations

$$\begin{gathered}\tilde{Q}_1' - N\tilde{\mu}_2 + Q_2\tilde{\tau} = 0,\\ \tilde{Q}_2' + \mu_1\tilde{N} + N\tilde{\mu}_1 = 0,\\ \tilde{N}' - \mu_1\tilde{Q}_2 - Q_2\tilde{\mu}_1 = 0,\\ \tilde{M}_1' + (1+\varepsilon)\tilde{Q}_2 + Q_2\tilde{\varepsilon} = 0,\\ \tilde{M}_2' + \mu_1\tilde{T} - M_1\tilde{\tau} - (1+\varepsilon)\tilde{Q}_1 = 0,\\ \tilde{T}' + M_1\tilde{\mu}_2 - \mu_1\tilde{M}_2 = 0,\\ \tilde{\pi}_c = -\tilde{\pi}_a + \tilde{\varpi}_a',\end{gathered} \tag{3.7}$$

where, as a consequence of $(3.5)_3$,

$$\tilde{T} = \tilde{T}_a - \tilde{\pi}_c\,\frac{d\check{k}}{d\tau}\ . \tag{3.8}$$

By linearizing equation (2.4) and (3.4) we obtain as well

$$\tilde{\varepsilon} = \tilde{v}_3' - \mu_1 \tilde{v}_2,$$

$$\tilde{\mu}_1 = \frac{1}{1+\varepsilon}(\tilde{v}_2'' + \mu_1' \tilde{v}_3 + \mu_1 \tilde{v}_3') - \frac{\varepsilon'}{(1+\varepsilon)^2}(\tilde{v}_2' + \mu_1 \tilde{v}_3),$$

$$\tilde{\mu}_2 = \frac{\varepsilon'}{(1+\varepsilon)^2}\tilde{v}_1' - \frac{1}{1+\varepsilon}v_1'' - \mu_1 \tilde{\theta}, \tag{3.9}$$

$$\tilde{\tau} = -\tilde{\theta}' + \frac{\mu_1}{1+\varepsilon}\tilde{v}_1',$$

where $\tilde{\theta}$ denotes the component of $\tilde{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ with respect to the first tensor of the basis $\{\mathbf{d}_2 \wedge \mathbf{d}_1, \mathbf{d}_3 \wedge \mathbf{d}_2, \mathbf{d}_1 \wedge \mathbf{d}_3\}$.

Let us now assume the constitutive functions to be analytic and their series expansion near the reference shape to be

$$N = \hat{N}(\varepsilon, \tau) = A\varepsilon + o(\tau),$$

$$M_1 = \hat{M}_1(\mu_1, \tau) = B_1\mu_1 + o(\tau),$$

$$M_2 = \hat{M}_2(\mu_2, \tau) = B_2\mu_2 + o(\tau),$$

$$T_a = \hat{T}(\varepsilon, \mu_1, \mu_2, \tau) = C\tau + (C_\varepsilon\varepsilon + C_{\mu_1}\mu_1 + C_{\mu_2}\mu_2)\tau + o(\tau), \tag{3.10}$$

$$\varpi_a = \hat{\varpi}(\beta) = D\beta + o(\beta),$$

$$Q_{1a} = 0, Q_{2a} = 0, \ \pi_a = 0,$$

where $A, B_1, B_2, C, C_\varepsilon, C_{\mu_1}, C_{\mu_2}, D$ are constants. Note that such specifications are consistent with the solution (3.6). Due to (3.10) and (3.8) equations (3.7) become

$$\tilde{Q}_1' - N\tilde{\mu}_2 + Q_2\tilde{\tau} = 0,$$

$$\tilde{Q}_2' + \mu_1\tilde{N} + N\tilde{\mu}_1 = 0,$$

$$A\tilde{\varepsilon}' - \mu_1 \tilde{Q}_2 - Q_2 \tilde{\mu}_1 = 0,$$

$$B_1 \tilde{\mu}_1' + (1+\varepsilon)\tilde{Q}_2 + Q_2 \tilde{\varepsilon} = 0, \tag{3.11}$$

$$B_2\tilde{\mu}_2' + \mu_1 (C + C_\varepsilon \varepsilon + C_{\mu_1}\mu_1)\tilde{\tau} - \mu_1 D \left(\frac{d\check{k}}{d\tau}\right)^2 \tilde{\tau}'' - M_1\tilde{\tau} - (1+\varepsilon)\tilde{Q}_1 = 0,$$

$$(C + C_\varepsilon \varepsilon + C_{\mu_1}\mu_1)\tilde{\tau}' - D\left(\frac{d\check{k}}{d\tau}\right)^2 \tilde{\tau}''' + (M_1 - B_2\mu_1)\tilde{\mu}_2 = 0.$$

At first glance equations (3.11) appear to be quite different from Vlasov's classical buckling equations (see [3], Ch. V (1.10)). But we must consider that, while in deriving equations (3.11) the exact solution (3.6) has been adopted, in the other case such a solution has been replaced by a fictitious one in which stresses are linearized near the reference shape and strains are assumed to be zero. Although clearly inconsistent, these assumptions are very usual in 'technical' theories. Only by accepting these assumptions the last three of equations (3.11), after substitution of (3.9), become

$$B_1 \tilde{v}_2'''' - N\tilde{v}_1'' = 0,$$

$$B_2 \tilde{v}_1'''' - N\tilde{v}_1'' - M_1\tilde{\theta}'' = 0, \qquad (3.12)$$

$$C_0 \tilde{\theta}'''' - (C + \frac{C_\varepsilon}{A}N + \frac{C_{\mu_1}}{B_1}M_1)\tilde{\theta}'' - M_1\tilde{v}_1'' = 0,$$

where $C_0 := D(d\check{k}/d\tau)^2$.

These equations are equal to Vlasov's equations in [3], even though the constants have different meanings. In particular note that the last equation contains the term $C_0\tilde{\theta}''''$ corresponding to the term in Vlasov's equations due to the bimoment. What is illuminating here is that that term originates from nothing but a constraint reaction. Note also that the expression for the coefficient of $\tilde{\theta}''$ in the last equation arises from the assumption that the torsional component T of the contact couple depends on ε, μ_1 and μ_2 (hence for an hyperelastic material N, M_1 and M_2 must depend on τ as well).

Finally it is interesting to note that the constitutive relations in [8], which have been derived from a two-dimensional continuum through a projection method, satisfy the restrictions (3.10).

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