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BIFURCATION ANALYSIS OF A CIRCULAR ARCH UNDER HYDROSTATIC PRESSURE

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SOMMARIO. Nel presente lavoro si studia la biforcazione di un arco circolare soggetto a pressione idrostatica nel campo elastico facendo uso di un modello di trave cinematicamente esatto. Le corrispondenti equazioni di campo nonlineari vengono risolte utilizzando una tecnica perturbativa. Vengono riportati in diagramma una serie di risultati numerici riguardanti la dipendenza del carico critico e del parametro di carico del secondo ordine dai parametri geometrici e meccanici.

SUMMARY. In this paper the bifurcation analysis of a circular arch under hydrostatic pressure in the elastic postbuckling range is performed by means of a geometrically exact beam model. The relevant nonlinear field equations are solved by utilizing a perturbation technique. A number of numerical results regarding the dependence of the critical load and the second order load parameter on the geometric and mechanical parameters are plotted in diagrams.

1. INTRODUCTION

The field of nonlinear analysis of one-dimensional continua has been investigated to a very large extent in the last decades, following essentially two lines [1].

The first – and most common – approach consists in developing nonlinear «technical» theories in which geometrical approximations are introduced, in order to «simplify» the analysis of some particular problems. Even if such theories lead, in many cases, to good results for specific problems, their range of applicability is often tortuously assessed by declaring such restrictions as «sufficiently inextensible» or «sufficiently shallow», whose exact meaning is left obscure.

The second line relies on the approach of modern continuum mechanics, wich has led to the development of *geometrically exact* models of rods [2]. These can be considered as refinements of the Bernoulli-Euler model in the sense that, by selecting appropriate kinematical descriptors, effects as extensibility, shearability and «section deformation», can be taken into account in addition to flexure.

Furthermore, the possibility of assuming very general constitutive relations allows these models to give a sufficiently good description of a very large class of mechanical phenomena.

Compared with the enormous quantity of results available in the current literature and obtained by applying technical theories, analyses performed in the framework of the exact ones appear to be very few also in fields like asymptotic postbuckling analysis where, in the authors opinion, their use is particularly suitable.

In this work the authors perform a local bifurcation analysis of a planar circular arch under hydrostatic pressure by using a geometrically exact model with linear hyperelastic constitutive relationships.

Following the steps outlined in [3], the asymptotic expansion of the field equations near the bifurcation point is furnished up to the third order. Each perturbation step, owing to the model used, results in a set of equations in which all the problems connected with the reliability of the results as an effect of the approximations embedded in the geometrical description [4-7] are overcome.

In view of obtaining parametric results in terms of the constitutive and shape parameters, and in order to encompass problems connected with heavy (and so hardly reliable) hand computations, the perturbation problems have been solved making use of an automatic system for symbolic computation [8].

A number of diagrams has been plotted which illustrate the buckling and postbuckling behaviour in terms of the mechanical and geometrical parameters.

2. KINEMATICAL MODEL AND EQUILIBRIUM EQUATIONS

Let us consider [2] a body $\mathscr{B}$ as a set of material points whose motions are described by the function

$$\chi : \mathscr{B} \times \mathbb{R} \to E^2 \times V^2$$
$$\chi : (P, t) \mapsto (\mathbf{x}, \mathbf{d}) \tag{1}$$

that maps each body point $P \in \mathscr{B}$ and time $t \in \mathbb{R}$ into a position $\mathbf{x} \in E^2$ and a director $\mathbf{d} \in V^2$, with $\| \mathbf{d} \| = 1$, where E^2 is a 2-*D* Euclidean point space and V^2 is its translation space. If at any t the projection of χ on E^2 is a smooth curve and that on V^2 is a smooth vector field we call $\mathscr{B}$ a plane one-dimensional directed continuum. Further an orthonormal right-handed basis associated with each space point belonging to the beam axis, namely $(\mathbf{b}_1 = \mathbf{d}, \mathbf{b}_2 | \mathbf{b}_1 \cdot \mathbf{b}_2 = 0)$ is introduced. We shall denote by

$$k : \mathscr{B} \to E^2 \times V^2 \tag{2}$$

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$$k : P \mapsto (\mathbf{X}, \mathbf{D}) \tag{2}$$

the reference shape and by $(\mathbf{B}_1 \equiv \mathbf{D}, \mathbf{B}_2 | \mathbf{B}_1 \cdot \mathbf{B}_2 = 0)$ the reference basis. The collection of the points will be called the reference beam axis.

Any motion χ is completely described, in referential form, through the functions

$$\begin{aligned} &\mathbf{u}(S) := \mathbf{x}(S) - \mathbf{X}(S) \\ &\mathbf{R}(S) : \mathbf{B}_i \mapsto \mathbf{b}_i \qquad i = 1, 2 \end{aligned} \tag{3}$$

where S denotes the curvilinear abscissa induced on the reference beam axis by the (usual) metric of E^2 once an origin has been chosen and $\mathbf{B}_1 = \mathbf{X}'(S)$, a prime denoting differentiation with respect to S.

Let the strain field be defined, in referential form, as follows

$$\begin{aligned} &\mathbf{e}(S) := \mathbf{x}'(S) - \mathbf{R}(S)\mathbf{X}'(S) \\ &\mathbf{C}(S) := \mathbf{R}'(S)\, \mathbf{R}(S)^{-1} \end{aligned} \tag{4}$$

Since $\mathbf{R}$ is orthogonal, $\mathbf{C}$ turns out to be skew. The rate of change of the frame fields $(\mathbf{B}_1, \mathbf{B}_2)$ and $(\mathbf{b}_1, \mathbf{b}_2)$ with respect to S can be written in the form

$$\begin{aligned} &\mathbf{B}_1' = \rho \mathbf{B}_2 \qquad \mathbf{b}_1' = \rho_t \mathbf{b}_2 \\ &\mathbf{B}_2' = -\rho \mathbf{B}_1 \qquad \mathbf{b}_2' = -\rho_t \mathbf{b}_1 \end{aligned} \tag{5}$$

where ρ coincides with the *curvature* of the reference beam axis. The scalar form for eqns. (4) is then

$$\begin{aligned} &\epsilon = u' - \rho_t \nu + \cos\theta - 1 \\ &\gamma = \nu' + \rho_t u - \sin\theta \\ &\mu = \theta' \end{aligned} \tag{6}$$

where we have posed $\mathbf{e} = \epsilon \mathbf{b}_1 + \gamma \mathbf{b}_2$, $\mathbf{u} = u \mathbf{b}_1 + \nu \mathbf{b}_2$ and θ is the amplitude of the rotation from $(\mathbf{B}_1, \mathbf{B}_2)$ to $(\mathbf{b}_1, \mathbf{b}_2)$ positive if counterclokwise.

From $(4)_2$ and (5) the relation $\mu = \rho_t - \rho$ is obtained which clarifies the kinematical meaning of ρ_t.

Let us consider now a part $\mathscr{P}$ of $\mathscr{B}$ and a time t. We will denote by $\mathbf{f}(S, t)$ and $\mathbf{g}(S, t)$ the referential form of the body force and couple fields, respectively, acting upon $\mathscr{P}$ at time t. Further we shall refer to $\mathbf{n}(S, t)$ and $\mathbf{m}(S, t)$ as the contact force and couple fields. Then the differential equilibrium equations for our continuum read

$$\begin{aligned} &\mathbf{n}'(S, t) + \mathbf{f}(S, t) = 0 \\ &\mathbf{m}'(S, t) + \mathbf{g}(S, t) + \mathbf{x}'(S, t) \wedge \mathbf{n}(S, t) = 0 \end{aligned} \tag{7}$$

whence the scalar equations are obtained by projecting eqns. (7) onto $(\mathbf{b}_1, \mathbf{b}_2)$

$$\begin{aligned} &N' - T(\mu + \rho) + f_1 = 0 \\ &T' + N(\mu + \rho) + f_2 = 0 \\ &M' - N\gamma + T(1 + \epsilon) + g = 0 \end{aligned} \tag{8}$$

3. BIFURCATION ANALYSIS

Let us assume that for a given constitutive relation and suitable boundary conditions there exists a regular solution branch to eqns. (6) and (8) depending on a parameter p $u_0(p)$, $\nu_0(p)$, $\theta_0(p)$, $N_0(p)$, $T_0(p)$, $M_0(p)$ which will be referred to as «fundamental». We are interested in looking for another solution branching off from the first one.

Let the generic solution be parametrized through a parameter t and expressed in the form

$$\begin{aligned} &u(t) = u_0(p(t)) + \tilde{u}(t), \qquad N(t) = N_0(p(t)) + \tilde{N}(t) \\ &\nu(t) = \nu_0(p(t)) + \tilde{\nu}(t), \qquad T(t) = T_0(p(t)) + \tilde{T}(t) \\ &\theta(t) = \theta_0(p(t)) + \tilde{\theta}(t), \qquad M(t) = M_0(p(t)) + \tilde{M}(t) \end{aligned} \tag{9}$$

The aim of the analysis is to obtain a series expansion, in terms of t, of the sliding variables $\tilde{u}, \tilde{\nu}, \tilde{\theta}, \tilde{N}, \tilde{T}, \tilde{M}$ and of the parameter p up to second order terms. To this end we replace eqns. (9) into (8). By remembering that the set $u_0, \nu_0, \ldots$ is a solution we obtain

$$\begin{aligned} &-\tilde{T}(\rho + \mu_0 + \tilde{\mu}) + \tilde{N}' - T_0 \tilde{\mu} + \tilde{f}_1 = 0 \\ &\tilde{T}' + \tilde{N}(\rho + \mu_0 + \tilde{\mu}) + N_0 \tilde{\mu} + \tilde{f}_2 = 0 \\ &\tilde{T}(1 + \epsilon_0 + \tilde{\epsilon}) - \tilde{N}(\gamma_0 + \tilde{\gamma}) + \tilde{M}' + T_0 \tilde{\epsilon} - N_0 \tilde{\gamma} + \tilde{g} = 0 \end{aligned} \tag{10}$$

after f_1, f_2, g have been splitted according to (9). Following the same way, the strain-displacement relations (6) lead to

$$\begin{aligned} &\tilde{\epsilon} = \tilde{u}' - \nu_0 \tilde{\theta}' - \tilde{\nu}\theta_0' - \tilde{\nu}\tilde{\theta}' - \rho\tilde{\nu} - \cos\theta_0 + \cos(\theta_0 + \tilde{\theta}) \\ &\tilde{\gamma} = \tilde{\nu}' + u_0 \tilde{\theta}' + \tilde{u}\theta_0' + \tilde{u}\tilde{\theta}' + \rho\tilde{u} - \sin(\theta_0 + \tilde{\theta}) + \sin\theta_0 \\ &\tilde{\mu} = \tilde{\theta}'. \end{aligned} \tag{11}$$

Finally the constitutive law which so far needs not to be specified, has to be added to eqns. (10) and (11).

A perturbation technique can now be applied to obtain an asymptotic solution [3]. Replacing each sliding variable by its series expansion in terms of the parameter t, starting from a generic point of the fundamental solution, and grouping terms of the same order, three sets of equations (equilibrium, strain-displacement, constitutive equations) are obtained at each order, as well as corresponding kinematic boundary conditions. The solution can be constructed by solving the first order equations, then the subsequent equations of second and third order.

4. CIRCULAR ARCH UNDER HYDROSTATIC PRESSURE

a) *Perturbation equations*

Let us consider a circular arch under hydrostatic pressure p [9] with constraints defined by (Fig. 1)

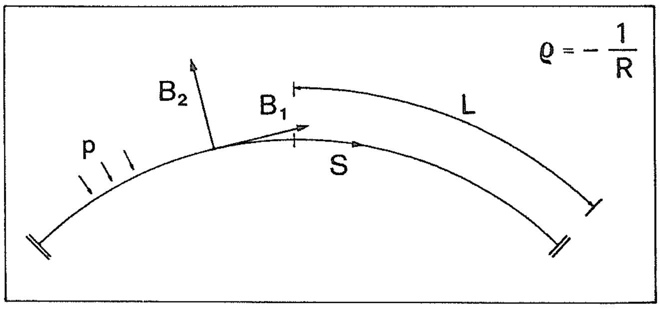

Fig. 1. Circular arch.

$$\theta(\pm L) = 0, \quad u(\pm L) = 0, \quad T(\pm L) = 0 \tag{12}$$

and characterized by linear elastic constitutive functions

$$N = A\epsilon$$
$$T = G\gamma \tag{13}$$
$$M = B\mu$$

Equilibrium eqns. (8) specialize in this case into [9, 10]

$$N' - T(\mu + \rho) + p\gamma = 0$$
$$T' + N(\mu + \rho) - p(1 + \epsilon) = 0 \tag{14}$$
$$M' - N\gamma + T(1 + \epsilon) = 0$$

where p is the inward hydrostatic pressure per unit length of the reference shape. Eqns. (14), (6) together with the boundary conditions (12) admit the fundamental solution

$$\begin{array}{lll} \epsilon_0 = -\rho\nu_0, & N_0 = p(1 + \epsilon_0)/\rho, & u_0 = 0 \\ \gamma_0 = 0, & T_0 = 0, & \nu_0 = -p/\rho(\rho A - p) \\ \mu_0 = 0, & M_0 = 0, & \theta_0 = 0 \end{array} \tag{15}$$

The perturbed equilibrium and compatibility equations of the first, second and third order, in terms of sliding variables, are listed in the Appendix.

By writing eqns. (13) in terms of sliding variables and taking successive derivatives with respect to t we complete the set of equations at each order.

In view of eqns. $(13)_2$ and $(6)_2$, the boundary conditions $(12)_3$ become $\nu'(\pm L) = 0$. Perturbation of this boundary condition along with eqns. (12) furnishes the boundary conditions pertaining to the asymptotic problem at each step.

The mixed formulation of eqns. A.1 to A.6 and (13) is suitable for solving problems under initial internal constraints by allowing the axial and shear rigidities grow to infinity. The relevant expressions for this problem are furnished in [11] which is broadly inspired by the treatment given in [3]. Since we are interested to general results for unconstrained beam, we shall solve the system of eqns. A.1 to A.6 and (13) in terms of displacements.

Substituting eqns. A.4, A.5, A.6 into A.1, A.2, A.3 and into the perturbed equations obtained from (13) by differentiation with respect to t and hence by eliminating the generalized forces we obtain a set of equilibrium equations in terms of displacements. These equations, as well as all the perturbation equations A.1 to A.6 derived from eqns. (6), (14), have been generated by using a symbolic computation system (REDUCE-2) [8]. They turn out to be very cumbersome and therefore their expression has been omitted. The solution to the equilibrium equations is given in subsection b.

b) *Solution to perturbation equations*

The critical values of the pressure p_c – that is the values corresponding to bifurcation points along the fundamental path – are furnished by the solution of the characteristic equation which leads to either sin $(hL) = 0$ or cos $(hL) = 0$ where h is defined by

$$\{BR^4p^4 + (G + 3A)BR^3p^3 + [3(G + A)B + R^2A^2]\ AR^2p^2$$
$$+ [(3G + A)B + R^2GA]\ A^2Rp \tag{16}$$
$$+ GBA^3\}/[(pR + A)^2GBA] = h^2R^2 = h^2L^2/\alpha^2$$

and $\alpha = L/R$. From eqn. (16) the following expressions are derived

$$\frac{pR^3}{B} = \frac{1}{2B}\left\{-(B + R^2G) \pm\right.$$
$$\left.\pm\left[(B + R^2G)^2 - 4R^2GB\,\frac{1 - h^2L^2}{\alpha^2}\right]^{0.5}\right\} \tag{17}$$

$$\frac{pR^3}{B} = \left(\frac{h^2L^2}{\alpha^2}\right) - 1 \bigg/ \left\{\frac{B}{AR^2} + [1 + \epsilon_0(p)]^2\right\} \tag{18}$$

$$\frac{pR^3}{B} = \frac{h^2L^2}{\alpha^2} - 1 \tag{19}$$

valid for $A \to \infty$, $G \to \infty$ $G, = A \to \infty$, respectively, where $\epsilon_0(p)$ is furnished by eqns. (15).

For $\alpha = hL/2$ eqn. (19) furnishes the critical load for the inextensional ring that matches the well known result due to M. Lévy (1884) reported in [12], while eqn. (16) coincides with eqn. (4.6) derived by Antman and Dunn in [9]. If we consider the case $hL = \pi/2$, the corresponding eigensolution to the first order equations is

$$\dot{\tilde{u}} = a_1 \cos(hS)$$
$$\dot{\tilde{\nu}} = -a_2 \sin(hS) \tag{20}$$
$$\dot{\tilde{\theta}} = a_3 \cos(hS)$$

where a_1, a_2, a_3 stand for the following expressions

$$a_1 = \frac{R[-p^3R^3 - p^2R^2(G + 2A) + pRA(h^2R^2G - 2G - A) - GA^2]}{hR^2A^2(pR + G)}$$

$$a_2 = 1 \tag{21}$$

$$a_3 = \frac{p^3R^3 + p^2R^2(G + 2A) - pRA(h^2R^2G - 2G - A) + GA^2(1 - h^2R^2)}{hR^2A^2(pR + G)}$$

and a dot denotes differentiation with respect to the parameter t. Note that it has been normalized according to $\dot{\tilde{\nu}}(-L) = 1$. The value of $\dot{p}_c$ turns out to be zero while the resulting expressions for the solution to the second order equations are

$$\ddot{\tilde{u}} = da_1 \cos(hS) + d_1 \cos(hS)\sin(hS)$$
$$\ddot{\tilde{\nu}} = -d\sin(hS) + d_2\cos^2(hS) + d_3\sin^2(hS) \tag{22}$$
$$\ddot{\tilde{\theta}} = da_3\cos(hS) + d_4\cos(hS)\sin(hS)$$

where $d = -d_3$ is a normalizing factor such that $\ddot{\tilde{\nu}}(-L) = 0$ and d_1, d_2, d_3, d_4 are constants pertaining to the particular solution of the second order perturbation equations. The expression for $\ddot{p}_c$ is too cumbersome and has not been reported here.

In the case $hL = \pi$ the critical mode reads

$$\dot{\tilde{u}} = a_1 \sin(hS)$$
$$\dot{\tilde{v}} = a_2 \cos(hS) \qquad (23)$$
$$\dot{\tilde{\theta}} = a_3 \sin(hS)$$

a_1, a_2, a_3 being the same as (21). The second order solution can be recast in the same form as (22) but with different expressions for the constants d_1, d_2, d_3, d_4. The modes have been normalized in the same way as for the case $hL = \pi/2$.

By letting $p_2 = \ddot{p}_c/2p_c$ and remembering that $\dot{p}_c = 0$, the asymptotic expansion of the function $p(t)$ near the bifurcation point ($t = 0$) up to the second order can be written in the form

$$p(t) = 1 + p_2 t^2 \qquad (24)$$

where, due to the chosen normalization, $t = \nu(-L)$.

5. NUMERICAL RESULTS

From the previous equations obtained through symbolic computation, some results of thecnical interest can be obtained by appropriate choice of the geometric and mechanical parameters.

A first interesting result obtained for $A = G \to \infty$ is shown in Fig. 2 where the dimensionless critical load $\bar{p}_c = p_c R^3/B$ is plotted against the shape parameter $\alpha = L/R$ according to (19) with hL replaced by $\pi/2$ and π, respectively.

It is seen that the curve corresponding to $hL = \pi/2$ furnishes always lower values for the critical load and therefore it dominates the problem for $\alpha \leqslant 90°$. On the other hand, for $90° < \alpha \leqslant 180°$ also the second curve comes to importance due to the fact that negative values of the critical load come into play. Fig. 3 shows the dependence of $\bar{p}_c$ on A by correspondence with different values of α for the case $G \to \infty$ and $hL = \pi/2$ (see eqn. 18). It is apparent that for low values of α each curve exhibits a maximum and then approaches asymptotically the limiting values shown in Fig. 2 as $A \to \infty$. For $\alpha \to 90°$ the maximum tends to disappear, and for $\alpha = 90°$ the curve coincides with the horizontal axis. For $\alpha > 90°$ negative values of $\bar{p}_c$ are obtained, the corresponding curves resulting monotonically, slowly decreasing. Note that $A = 0 \Rightarrow \bar{p}_c = 0$ due to the fact that the system becomes kinematically indeterminate.

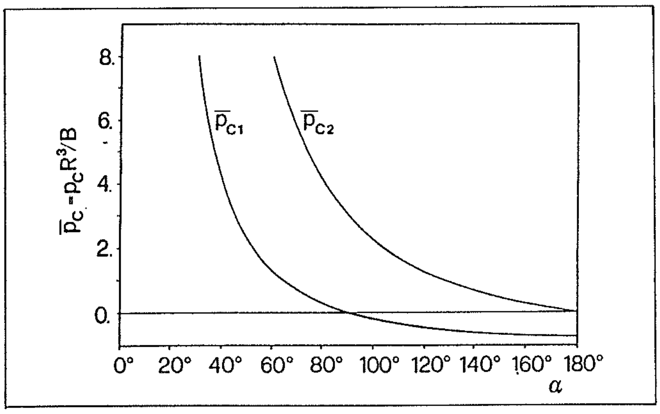

Fig. 2

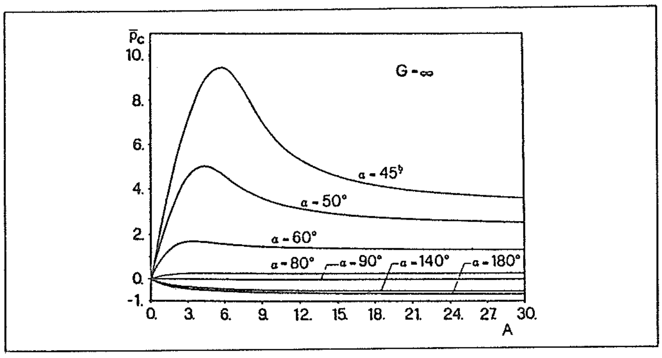

Fig. 3

The dependence of p_2 defined in (24) on A for different values of α and $G \to \infty$ is illustrated in Fig. 4. It is seen that for low values of α and high values of A all curves exhibit positive values for p_2 and decrease with decreasing A until they reach a negative minimum by correspondence with the maximum value of the critical stress. For $\alpha = 90°$ the curve degenerates into the horizontal axis while for $\alpha > 90°$ p_2 is always positive and slowly increasing with A. As for $\bar{p}_c$, p_2 is always zero for $A = 0$. It is interesting to note that whenever

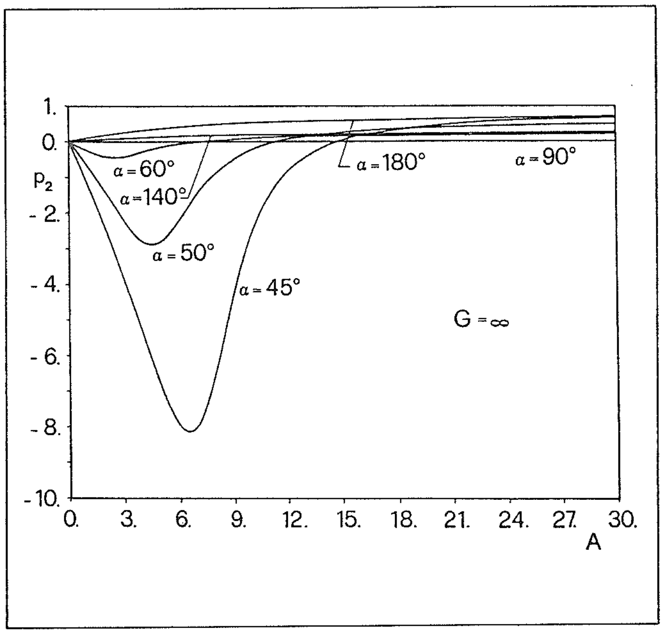

Fig. 4

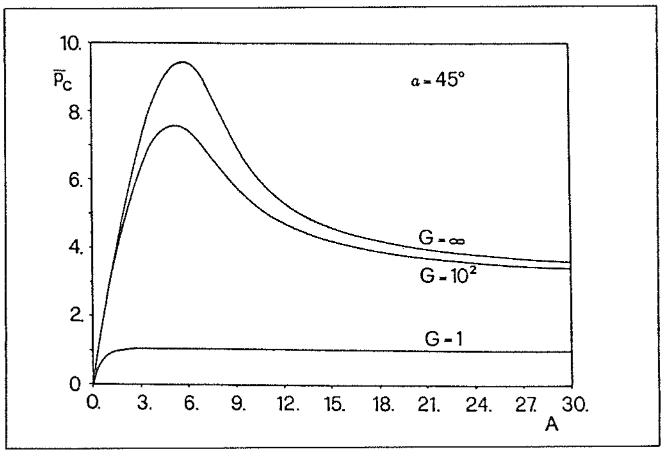

Fig. 5

the postbuckling behaviour is governed by the axial deformability of the arch, sensitivity to initial imperfections arises. The corresponding increasing erosion of the maximum theoretical critical stress is however counterbalanced by increasing values of $\bar{p}_c$. The limiting values of $\bar{p}_c$ and p_2 furnished in [4, 5] are here recovered for $\alpha = 45°$.

Figs. 5, 6 show the dependence of $\bar{p}_c$ and p_2 on A for different values of G and $hL = \pi/2$. α has been taken equal to 45° in order to describe the ring behaviour. It is apparent that the mechanical parameter G plays an analogous role as the geometrical parameter α of Figs. 3, 4.

Finally the dependence of $\bar{p}_c$ on G for $A \to \infty$, $hL = \pi/2$ and different values of the shape parameter α has also been investigated. Fig. 7 describes the analytical findings that for each G and α two critical loads of opposite sign are obtained the negative load being, in absolute value, much larger than the positive one. For sufficiently large values of G the positive critical load approaches an asymptote, whereas the negative one goes to infinity. Note that for G approaching zero the positive curves tend to zero while the negative ones approach -1. Fig. 8 shows that also for p_2 two family of curves are obtained. Curves associated with positive values of the critical load, have always a positive p_2 and curves corresponding to negative values of $\bar{p}_c$ are characterized by a value of p_2 which is negative for small values of G becoming positive for large values of G.

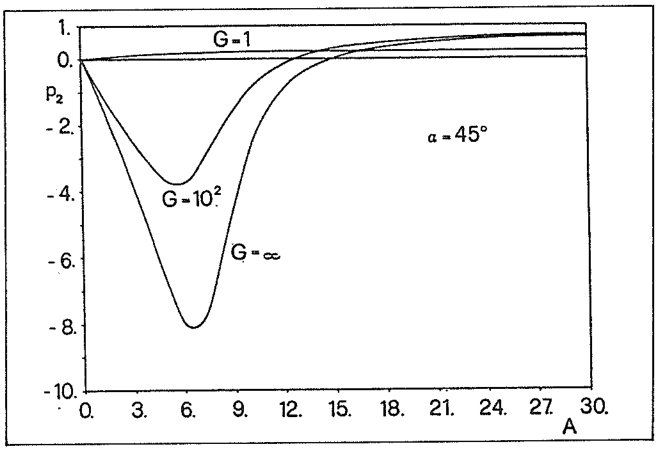

Fig. 6

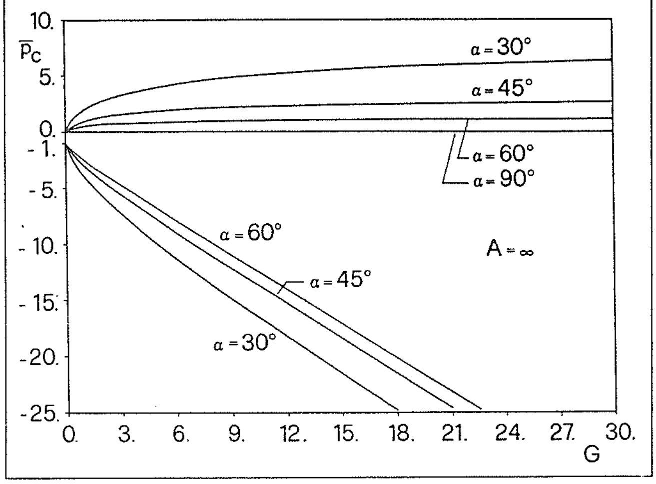

Fig. 7

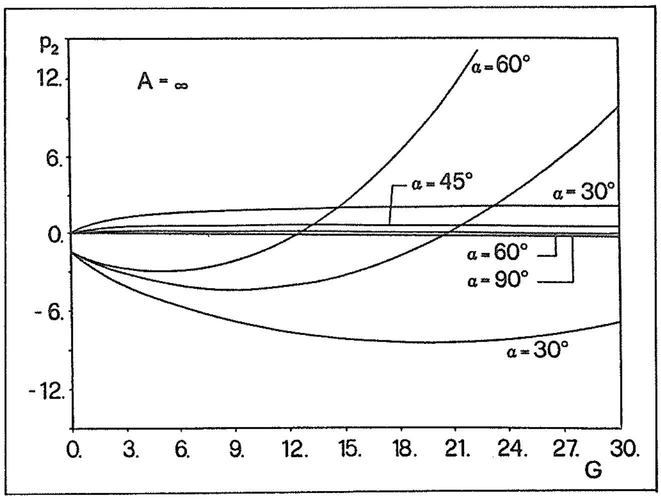

Fig. 8.

6. CONCLUSIONS

The analysis of a circular arch under hydrostatic pressure in the elastic postbuckling range has been performed by utilizing a geometrically exact beam model. The perturbation equations suitable for local bifurcation analysis have been derived directly from the nonlinear field equations. Numerical results show that for $hL = \pi/2$ the critical load $\bar{p}_c$ is always negative for any value of A, G whenever $\alpha > 90°$ and is positive for $\alpha < 90°$. On the other hand p_2 is always found to be positive except for very small values of A and for sufficiently large values of G. It should be observed that positive peaks of the $\bar{p}_c$ curves correspond to negative peaks of the p_2 curves.

7. APPENDIX

The perturbed equilibrium equations of the first, second and third order are:

$$-\dot{T}\rho + \dot{N}' + p\dot{\gamma} = 0$$
$$\dot{T}' + \dot{N}\rho - \dot{\epsilon}p + N_0\dot{\mu} = 0 \qquad (A.1)$$
$$\dot{T}(1+\epsilon_0) + \dot{M}' - N_0\dot{\gamma} = 0$$

$$-\ddot{T}\rho + \ddot{N}' + p\ddot{\gamma} - 2(\dot{T}\dot{\mu} - \dot{p}\dot{\gamma}) = 0$$
$$\ddot{T}' + \ddot{N}\rho - \ddot{\epsilon}p + N_0\ddot{\mu} + 2(\dot{N}\dot{\mu} - \dot{p}\dot{\epsilon} + \dot{N}_0\dot{\mu}) = 0 \qquad (A.2)$$
$$\ddot{T}(1+\epsilon_0) + \ddot{M}' - N_0\ddot{\gamma} + 2[\dot{T}(\dot{\epsilon}_0 + \dot{\epsilon}) - \dot{N}\dot{\gamma} - \dot{N}_0\dot{\gamma}] = 0$$

$$-\dddot{T}\rho + \dddot{N}' + p\dddot{\gamma} - 3(\ddot{T}\dot{\mu} + \dot{T}\ddot{\mu} + \ddot{\gamma}\dot{p} - \dot{\gamma}\ddot{p}) = 0 \qquad (A.3)$$
$$\dddot{T}' + \dddot{N}\rho + N_0\dddot{\mu} + 3(\ddot{N}\dot{\mu} + \dot{N}\ddot{\mu} + \dot{N}_0\ddot{\mu} - \ddot{p}\dot{\epsilon} + \ddot{N}_0\dot{\mu} - \ddot{\epsilon}\dot{p}) = 0$$
$$\dddot{T}(1+\epsilon_0) + \dddot{M}' - N_0\dddot{\gamma} + 3(\ddot{T}(\dot{\epsilon}_0 + \dot{\epsilon}) + \dot{T}(\ddot{\epsilon} + \ddot{\epsilon}_0) - \ddot{N}\dot{\gamma} -$$
$$- \dot{N}\ddot{\gamma} - \dot{N}_0\ddot{\gamma} - \ddot{N}_0\dot{\gamma}) = 0$$

The corresponding perturbed compatibility equations are:

$$\dot{\epsilon} = \dot{u}' - \dot{v}\rho - \dot{\theta}'v_0$$
$$\dot{\gamma} = \dot{u}\rho + \dot{v}' - \dot{\theta} \qquad (A.4)$$
$$\dot{\mu} = \dot{\theta}'$$

$$\ddot{\tilde{\epsilon}} = \ddot{\tilde{u}}' - \ddot{\tilde{v}}\rho - 2\dot{\tilde{v}}\dot{\tilde{\theta}}' - \ddot{\tilde{\theta}}'\nu_0 - 2\dot{\tilde{\theta}}'\dot{\nu}_0 - \dot{\tilde{\theta}}^2$$

$$\ddot{\tilde{\gamma}} = \ddot{\tilde{u}}\rho + 2\dot{\tilde{u}}\dot{\tilde{\theta}}' + \ddot{\tilde{v}}' - \ddot{\tilde{\theta}} \tag{A.5}$$

$$\ddot{\tilde{\mu}} = \ddot{\tilde{\theta}}'$$

$$\dddot{\tilde{\epsilon}} = \dddot{\tilde{u}}' - \dddot{\tilde{v}}\rho - 3\ddot{\tilde{v}}\dot{\tilde{\theta}}' - 3\dot{\tilde{v}}\ddot{\tilde{\theta}}' - \dddot{\tilde{\theta}}'\nu_0 - 3\ddot{\tilde{\theta}}'\dot{\nu}_0 - 3\dot{\tilde{\theta}}'\ddot{\nu}_0 - 3\ddot{\tilde{\theta}}\dot{\tilde{\theta}}$$

$$\dddot{\tilde{\gamma}} = \dddot{\tilde{u}}\rho + 3\ddot{\tilde{u}}\dot{\tilde{\theta}}' + 3\dot{\tilde{u}}\ddot{\tilde{\theta}}' + \dddot{\tilde{v}}' - \dddot{\tilde{\theta}} - \dot{\tilde{\theta}}^3 \tag{A.6}$$

$$\dddot{\tilde{\mu}} = \dddot{\tilde{\theta}}' .$$

where a dot denotes time differentiation.

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