MECHANICS OF SOLIDS AND MATERIALS

Description of motion of a body

By body point we mean a small particle, something that we can describe with a spot

BODY : collection of points

We are interested in the positions of the same point, and different positions of different points

$\cdot A,t_1$ $\quad\cdot A,t_2$ $\quad\cdot A,t_3$ $\quad\cdot A,t_4$ different positions of point A

$\cdot B,t_1$ $\quad\cdot B,t_2$ $\quad\cdot B,t_3$ $\quad\cdot B,t_4$ different positions of point B

we can use the notation $p(A,t_i)$ or $P(B,t_i)$ thus use the function p to describe the positions of the body i.e.

$$p_A(t_i) \equiv p(A,t_i)$$

<u>DEFINITION</u> the motion of a point A

$$p_A : I \subset \mathbb{R} \longrightarrow \mathbb{E} \quad \text{EUCLIDEAN SPACE}$$

The image of this function is the trajectory

If we consider a collection of body points: for two such points

$$p : \{A,B\} \times I \longrightarrow \mathbb{E}$$

in general the motion of a body $\mathcal{B}$

$$p : \mathcal{B} \times I \longrightarrow \mathbb{E}$$

where $\mathcal{B}$ is not necessarily countable

1.

E.g.

$p : \{A\} \times I \longrightarrow \mathbb{E}$ the elts of this set are called POSITIONS

$p : \mathcal{B} \times \{t_1\} \longrightarrow \mathbb{E}$ PLACEMENT

(describes the position of $\underbrace{\text{the body}}_{\text{each body point}}$ at time t_1

$P_{t_1}(\mathcal{B})$ is the collection of positions of the body points := SHAPE

What we can say about the dimension of the body?

If we want to compare time we can say

$$\underset{t_1}{+}\!\!-\!\!-\!\!\underset{t_2}{+} \Rightarrow t_2 - t_1$$

what about distance?

$\cdot\, x_2$

$\cdot\, x_1$

we have to fix our frame

↳ sistema di riferimento

$x_1 \longrightarrow x_2$

it is a vector i.e.

$(x_1, x_2) \longmapsto u$ vector

$(x_2, x_1) \longmapsto -u$

we mean $x_1 \longrightarrow x_2$ $u := (x_2 - x_1) \in V$

$(x_1 - x_2) \in V$

<u>DEF</u>: $+ : \mathbb{E} \times V \longrightarrow \mathbb{E}$ is the TRANSLATION

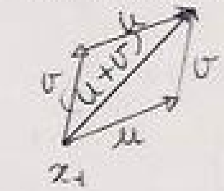
$u = (x_2 - x_1)$

$x_1 + u = x_2$

For each eucledian elt there is a vector such that $\exists$ a fuction which transforms the position and a vector in a position.

PROPERTIES

- $x_1 + 0 = x_1$
- $x_1 + (u + v) = (x_1 + u) + v$



If we repeace u+v with v+u (look at the drawn)

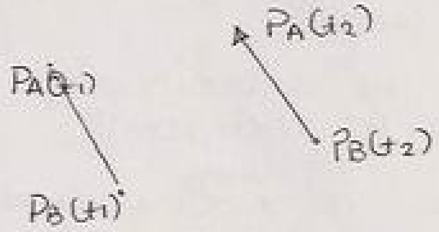
Thus the position plus a vector is equal to the position $x_1 + (x_2 - x_1) = x_2$

the distance between these two position is tavially the norm of the vector.

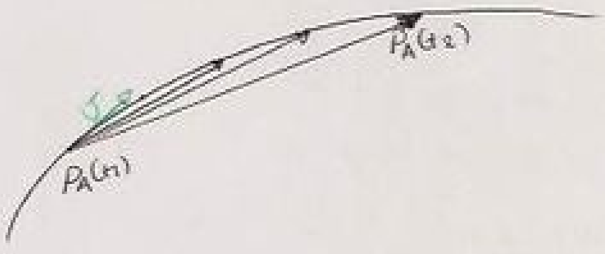
$$d(x_1, x_2) = \| x_2 - x_1 \| \geq 0$$

with all the propecties for a norm.

Now



(*)



$P_A(t_2) - P_A(t_1)$

$P_A(t_1 + \Delta t) - P_A(t_1)$

$\lim\limits_{\Delta t \to 0} \dfrac{(P_A(t_1 + \Delta t) - P_A(t_1))}{\Delta t}$ is the velocity vector of A at time t_1 $= v_A(t_1)$

2.

$\dot{P}_A(t_1) = v_A(t_1)$.

Going back to the question (*), what can be those distances? Are not trajectory anymore.

We define a function

$$C_{t_1} : [0, l] \longrightarrow \mathcal{E}$$

PARAMETRIZATION

$$C_{t_1}(0) = P_B(t_1)$$

$$C_{t_1}(l) = P_A(t_1)$$

We want to compare the positions of the body at the same time

$C_{t_1}(h) - C_{t_1}(0)$

sequence of h

$$\lim_{h \to 0} \left(C_{t_1}(h) - C_{t_1}(0) \right) = 0$$

$$\lim_{h \to 0} \frac{C_{t_1}(h) - C_{t_1}(0)}{h} = C'_{t_1}(0)$$

- the derivative of the function is not a velocity anymore
- a tangent vector

a straight line $\to$ when the derivative is equal at each point

$$C_{t_1}(h) = C_{t_1}(0) + h u$$

vector

str. line if it described by this eq

we can translate the origin by that vector along the direction

$$\Rightarrow h u = C_{t_1}(h) - C_{t_1}(0)$$

$$u = \frac{C_{t_1}(h) - C_{t_1}(0)}{h} \xrightarrow[h \to 0]{} C'_{t_1}(0) = u$$

email: amabile.tatone@univaq.it
ing.univaq.it/tatone

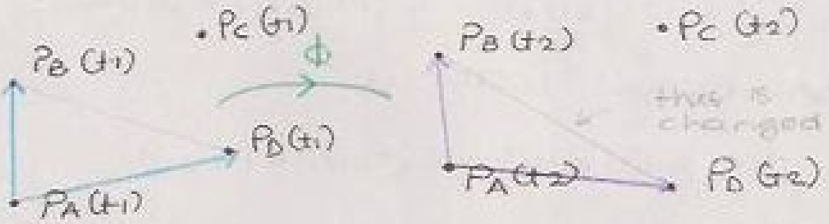
BOOKS
- CHATWICK
CONTINUUM MECHANICS

- OGDEN
SOLID MECHANICS

1.03.2011

$p : \mathcal{B} \times I \longrightarrow \mathbb{E}$
MOTION

Now we want to find a transformation for the positions



$\phi : \mathbb{E} \longrightarrow \mathbb{E}$

what we can do is compare the vectors, and check if they are the same or not.
Consider the particular case when the shapes are equal (i.e. the lenght of the vectors are equal)
What can we do to get the same shape?

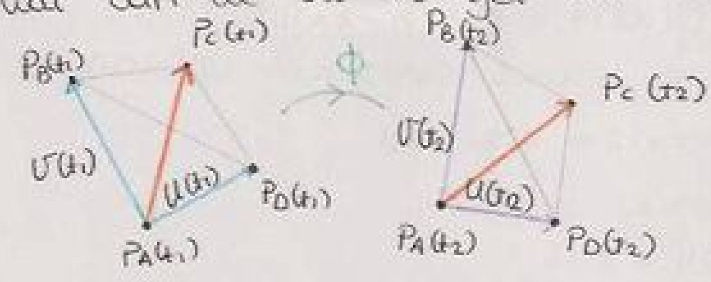


FIG. 1

The deformation is s.t. the norms of the vectors are the same respectively

3

$$u(t_1) \xmapsto{R} u(t_2) \qquad \|u(t_1)\| = \|u(t_2)\|$$

$$v(t_1) \xmapsto{R} v(t_2) \qquad \|v(t_1)\| = \|v(t_2)\|$$

Then we consider the vector joining body A & C

$$\alpha u(t_1) + \beta v(t_1) \xmapsto{R} \alpha u(t_2) + \beta v(t_2)$$

linear combination is transformed in L.C.

$$= \alpha R(u(t_1)) + \beta R(v(t_1))$$

$\Rightarrow$ R IS A LINEAR FUNCTION

By the definition of the deformation

$$P_B(t_2) = \phi(P_B(t_1)) = \underbrace{\phi(P_A(t_1))}_{P_A(t_2)} + R(v(t_1))$$

and $P_D(t_2) = \phi(P_D(t_1)) = \phi(P_A(t_1)) + R(u(t_1))$

we can replace vector u

$$\phi(P_D(t_1)) = \phi(P_A(t_1)) + R(P_D(t_1) - P_A(t_1))$$

In general

$$\circledast \quad \boxed{\phi(x) = \phi(x_A) + R(x - x_A)} \quad \text{R-ROTATION TENSOR}$$

MECHANICS LINEAR TRANSFORMATION

DEFORMATION leaves distances unchanged by transformation $\circledast$

By definition we can also write

$$\phi(x) - \phi(x_A) = R(x - x_A).$$

Has the property

$$\|\phi(x) - \phi(x_A)\| = \|x - x_A\|$$

if the deformation leaves distances unchanged

$$\Rightarrow \|R(x - x_A)\| = \|x - x_A\|$$

setting $u = x - x_A$

$$\|R(u)\| = \|u\|$$

Using the scalar product

$\|u\| = \sqrt{u \cdot u}$

$\|R(u)\|^2 = \|u\|^2$

$Ru \cdot Ru = u \cdot u$ scalar product

Given a linear operator

$A : V \to V$

$u \cdot v$

$Au \cdot v = u \cdot A^* v \quad \forall u, \forall v$ ADJOINT OPERATOR

$A^* = A^T$ the TRANSPOSE OPERATOR

Using the transpose operator

$Ru \cdot Ru = u \cdot u$

$u \cdot R^T Ru = u \cdot u \quad \forall u$

Then $\boxed{R^T R = I}$ i.e. are ORTHOGONAL TENSOR

If the deformation leaves distances unchanged can be given a representation like this ⊙ where R is the orthogonal tensor.

We call this deformation $\underbrace{\text{RIGID DEFORMATION}}$

deform. leaving distance unchanged or equivalently a transf. like this

$\phi(x) = \phi(x_A) + R(x - x_A)$.

We have seen that the transformation leaves

$\|u(t_1)\| = \|u(t_2)\|$

simplifing

$\|\bar{u}\|^2 = \|u\|^2$

■ $\bar{u} \cdot \bar{u} = u \cdot u \quad \forall u$

4.

$$(\bar u+\bar v)\cdot(\bar u+\bar v) = (u+v)\cdot(u+v)$$

$$\Rightarrow \cancel{\bar u\cdot\bar u} + \bar u\cdot\bar v + \bar v\cdot\bar u + \cancel{\bar v\cdot\bar v} = \cancel{u\cdot u}+u\cdot v+v\cdot u+\cancel{v\cdot v}$$

by ■ $\Rightarrow 2\,\bar u\cdot\bar v = 2\,u\cdot v$

$$\boxed{u\cdot v = \bar u\cdot\bar v}$$

By this information

$$R\bar u\cdot R\bar v = \bar u\cdot\bar v$$

$$\bar u\cdot R^T R\,\bar u = \bar u\cdot\bar v \quad \text{then} \quad R^T R = I.$$

Consider the angle between two vectors

$$\cos\alpha = \frac{u\cdot v}{\|u\|\,\|v\|}$$

in rigid transformation

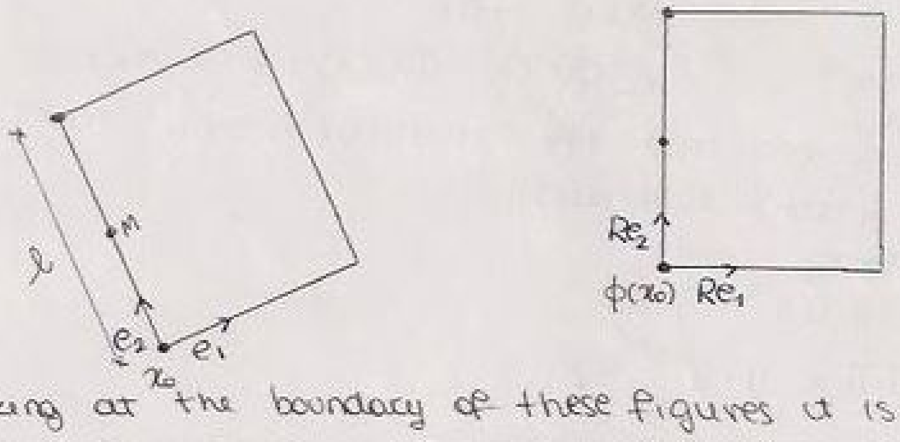
$$\cos\alpha = \frac{\bar u\cdot\bar v}{\|\bar u\|\,\|\bar v\|}$$

lenght of the segments are the same.

As in FIG.1 the angles are the same, using the fact that the scalar product does not change.

For any rigid trasformation the positions are changed according to this transf.

$$\phi(x) = \phi(x_0) + R(x-x_0) \quad \text{where } R^T R = I$$



Looking at the boundary of these figures it is ok. prec.
But we have to be sure that each point goes in the right

Consider the position given by

$$x_0 + he_2$$

$$\phi(x_0 + he_2) = \phi(x_0) + R(x_0 + he_2 - x_0)$$
$$= \phi(x_0) + h\,R(e_2)$$

For different value of h we get the descrip-tion of the second shape.

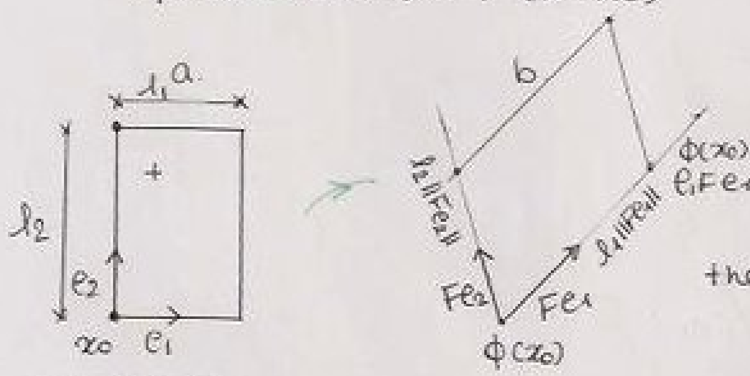
If we consider the middle point $x_0 + \frac{l}{2} e_2$

$$\phi(x_0 + \frac{l}{2} e_2) = \phi(x_0) + \frac{l}{2} R e_2$$

We've used the orthogonality property of the tensor. Now a general tensor called $\underline{F}$

$$\phi(x) = \phi(x_0) + F(x - x_0)$$

- that could or could not be an orthog. tensor.
- is a linear transformation



the STRETCH

$$\frac{\|Fe_1\|}{\|e_1\|}$$

$0 \le h \le l_2$

$$\phi(x_0 + he_2) = \phi(x_0) + h\,Fe_2$$

the lenght of hFe_2 is not necessary the same.

a. $\|x_0 + l_2 e_2 - x_0\| = \|l_2 e_2\| = l_2$

b. $\|\phi(x_0) + l_2 Fe_2 - \phi(x_0)\| = \|l_2 Fe_2\| = l_2 \|Fe_2\|$

In the previous deformation F was s.t. $F^T F = I$, in this case the norms are different $\Rightarrow$ the lenghts are changed. What else can be changed by F? The angles between vectors.

The only property that we can get by F is the linearity and that transforms straight lines into s.l.

5.

The STRETCH corresponding Fe_1 is $\dfrac{\|Fe_1\|}{\|e_1\|}$ which is a positive quantity.

we want to describe this side

$(x_0 + l_2 e_2) + h e_1$ by the properties of the translation

$0 \le h \le l_1$

$x_0 + (l_2 e_2 + h e_1).$

$$\phi(x_0 + (l_2 e_2 + h e_1)) = \phi(x_0) + F(l_2 e_2 + h e_1) =$$

$$= \underbrace{\phi(x_0) + l_2 F(e_2)}_{\text{what we already have}} + h F(e_1)$$

b.

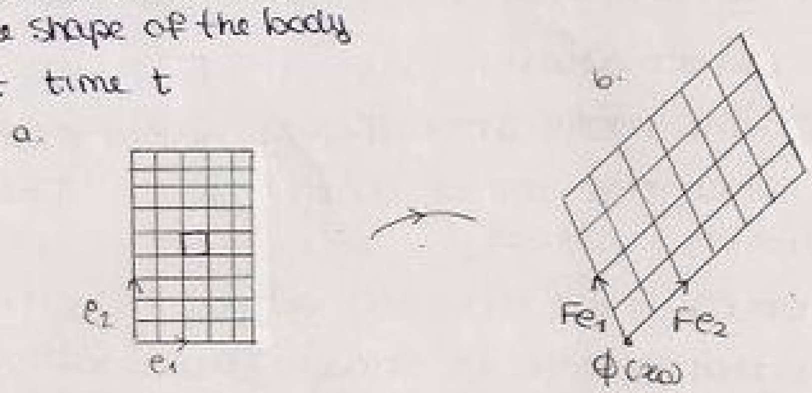
If we want to describe a point inside a.

using $0 \le h_1 \le l_1$, $0 \le h_2 \le l_2$

$h_1 = 0$ describe left side
$h_1 = l_1$ " right "
$h_2 = 0$ bottom
$h_2 = l_2$ top

$$\phi(x_0 + h_1 e_1 + h_2 e_2) = \phi(x_0) + h_1 Fe_1 + h_2 Fe_2$$

PARAMETRIZATION of the shape of the body at time t



The grid is stretched in different ways. The description is given by ● even if they have not the same dimension.

In 3D

a. PARALLELEPIPED
   6 faces
   12 edges
   8 corners

F: is not necessary an orthogonal tensor. Instead it is an AFFINE TRANSFORMATION and it is called the DEFORMATION GRADIENT

Shapes are subdomains in $\mathbb{R}^3$. Which is the dim $\mathbb{E}$? It will be the vector space that we will use for the transf.

$$\mathbb{E} \xrightarrow{\text{COORDINATE FUNCTION}} \mathbb{R}^3$$

A tensor can be written in components

$$Fe_1 = f_{11} e_1 + f_{21} e_2$$
$$Fe_2 = f_{12} e_1 + f_{22} e_2$$

$$\begin{aligned} F(u_1 e_1 + u_2 e_2) &= u_1 Fe_1 + u_2 Fe_2 \\ &= u_1 (f_{11} e_1 + f_{21} e_2) + \\ &\quad + u_2 (f_{12} e_1 + f_{22} e_2) \\ &= (u_1 f_{11} + u_2 f_{12}) e_1 + \\ &\quad + (u_1 f_{21} + u_2 f_{22}) e_2 \end{aligned}$$

$$\overset{[F]}{\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Fe_1 Fe_2

6.

The matrix of the rotation

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$R^T R = I$$

02.03.2010

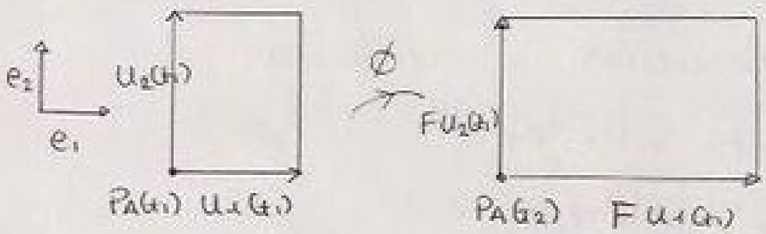
Examples $\phi(x) = \phi(x_0) + F(x - x_0)$

i. $F = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$\phi(x) = \phi(x_0) + (x - x_0)$$

UNIFORM TRANSLATION

ii. $F = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$ $\quad F(x - x_0) = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1(t_1)\\ u_2(t_1)\end{bmatrix}$

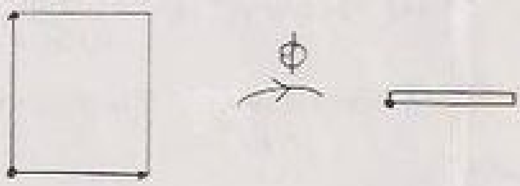


$$Fe_1 = 2e_1$$
$$Fe_2 = e_2$$
$$Fu_1(t_1) = F(l_1 e_1 + 0 \cdot e_2) = l_1 F e_1 = \overbrace{2 l_1 e_1}^{u_1(t_2)}$$
$$Fu_2(t_1) = F(0 \cdot e_1 + l_2 e_2) = l_2 F e_2 = \underbrace{l_2 e_2}_{u_2(t_2)}$$

Then by linearity we can conclude the figure.

The deformation ϕ stretches the figure

iiia. $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 10^{-12} \end{bmatrix}$



just a line

in mechanics this object is forbidden we will consider like a stick or thin as you like

iiib.

$F = \begin{bmatrix} 1 & 0 \\ 0 & -0.1 \end{bmatrix}$

iv.

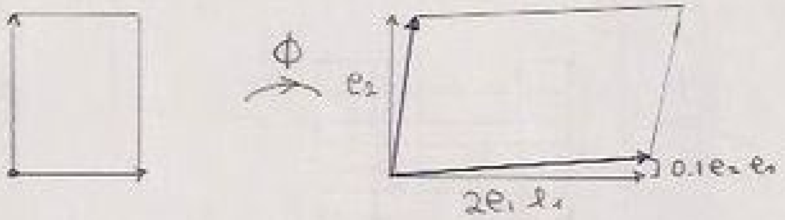
$F = \begin{bmatrix} 2 & 0.1 \\ 0.1 & 1 \end{bmatrix}$

$$\begin{cases} Fe_1 = 2e_1 + 0.1e_2 \\ Fe_2 = 0.1e_1 + e_2 \end{cases}$$

$$F u_1(t_1) = F(l_1 e_1 + 0 \cdot e_2) = l_1 F e_1 = 2 l_1 e_1 + 0.1\, l_1 e_2$$

$$F u_2(t_1) = F(0 \cdot e_1 + l_2 e_2) = l_2 F e_2 = 0.1 l_2 e_1 + l_2 e_2$$

7.



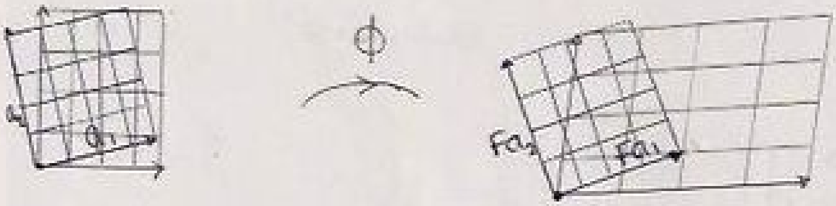
We will use the diagonalization to stretch the basis

$$\begin{pmatrix} 2 & 0.1 \\ 0.1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \sqcup & 0 \\ 0 & \sqcup \end{pmatrix} \text{ STRETCHES}$$

we already know that the stretch is $\frac{\|U(\epsilon)\|}{\|U(\epsilon)\|}$

So the eigenvalues are called PRINCIPAL STRETCHES λ_1, λ_2

If a_1 and a_2 are eigenvectors



$$Fa_1 = \lambda_1 a_1$$
$$Fa_2 = \lambda_2 a_2$$

using eigenvectors the grid will be the same no stretching the lines will be orthogonal.

DIRECTIONS a_1, a_2

If we have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

eigenvalues?

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\cos\theta\,\lambda + 1 = 0$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta$$

Since the eigenvalues are complex we cannot say anything about the stretch.

In general if the matrix is not symmetric we will not get real eigenvalues.

How much this deformation is far from rigid deformation?

We are interested in symmetric matrix with det $F>0$, so

$F = RU$ where $R^T R = I$, $U^T = U$

- a tensor which is orthogonal
- a tensor which is symmetric

We require that U is positively defined i.e. $Uv \cdot v > 0$

e.g. $Ua_1 \cdot a_1 = \lambda_1 \underbrace{a_1 \cdot a_1}_{>0} \Rightarrow \lambda_1 > 0$

We can prove that for any F we can find the composition RU

we are composing our deformation in two deformations.

PROOF: $F = RU$ det $F > 0$

Hp: $R^T R = I$ — R rotation

$U^T = U$ — U stretch (is a tensor)

$\downarrow$

then det $R = \pm 1$ — the principal stretches will

but we assume det $R = 1$. — be the eigenvalues of U

$C = F^T F = (RU)^T RU = U^T \underbrace{R^T R}_{\text{rot.}} U = U^T U \overset{\text{symm}}{=} U^2$

$\boxed{\text{CAUCHU-GREEN TENSOR}}$

C is def. pos. since $F^T F v \cdot v$

E.g.

$$F = \begin{bmatrix} 2 & 0.1 \\ 0.2 & 1 \end{bmatrix}$$

$$F^T F = \begin{bmatrix} 2 & 0.2 \\ 0.1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0.1 \\ 0.2 & 1 \end{bmatrix} = \begin{bmatrix} 4.04 & 0.4 \\ 0.4 & 1.01 \end{bmatrix} = U^2$$

Let's call η_1, η_2 eigenvalues of U^2 then

$\lambda_1 = \sqrt{\eta_1}$ $\quad \lambda_2 = \sqrt{\eta_2}$

Using the PROJECTORS P_1 and P_2 where

$\operatorname{Im} P_1 = \ker(C - \eta_1 I)$

$\operatorname{Im} P_2 = \ker(C - \eta_2 I)$

$V = P_1 \oplus P_2$ since $P_1 \cap P_2 = \{0\}$

$P_1 P_2 = 0 \quad P_1 P_1 = P_1 \quad P_1 + P_2 = I$

$\boxed{C = \eta_1 P_1 + \eta_2 P_2}$ SPECTRAL DECOMPOSITION

How to compute P:

$$\begin{aligned} C - \eta_1 I &= C - \eta_1 (P_1 + P_2) \\ &= \cancel{\eta_1 P_1} + \eta_2 P_2 - \cancel{\eta_1 P_1} - \eta_1 P_2 \\ &= (\eta_2 - \eta_1) P_2 \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} P_2 &= \frac{C - \eta_1 I}{\eta_2 - \eta_1} \\ P_1 &= I - P_2 = \frac{C - \eta_2 I}{\eta_2 - \eta_1} \end{aligned}}$$

$\boxed{U := \sqrt{\eta_1}\, P_1 + \sqrt{\eta_2}\, P_2}$ is pos. def

Now we check if $C = U^2$

$$\begin{aligned} UU &= \eta_1 P_1^2 + 2\sqrt{\eta_1 \eta_2}\, \overset{0}{\overset{\shortparallel}{P_1 P_2}} + \eta_2 P_2^2 \\ &= \eta_1 P_1 + \eta_2 P_2 = C \end{aligned}$$

Finally $F = RU \Rightarrow \boxed{R := F U^{-1}}$

We have to check that R is the rotation matrix:

$$U = \sqrt{\eta_1}\, P_1 + \sqrt{\eta_2}\, P_2$$

U is a diagonal matrix

$$U^{-1} = \frac{1}{\sqrt{\eta_1}} P_1 + \frac{1}{\sqrt{\eta_2}} P_2 \qquad \left(U U^{-1} = P_1 + P_2 = I \right)$$

Thus

$$\begin{aligned} R^T R &= (F U^{-1})^T F U^{-1} \\ &= (U^{-1})^T F^T F U^{-1} = (U^T)^{-1} C\, U^{-1} \\ &= (U^T)^{-1} U\, U U^{-1} \\ &= U^{-1} U = I \end{aligned}$$

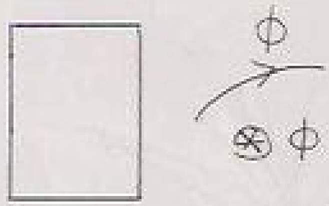
Then

$$F = RU \quad \forall F \ \det F > 0. \qquad \text{QED}$$

$F = R\underset{\text{ROTATION}}{}\overset{\text{STRETCH}}{U}$ is called the POLAR DECOMPOSITION

F: DEFORMATION GRADIENT

07.03.2011



$$\circledast \quad \phi(P_B(t_1)) = \phi(P_A(t_1)) + F(P_B(t_1) - P_A(t_1))$$

$$P_B(t_2) = P_A(t_2) + F(P_B(t_1) - P_A(t_1))$$

$F = RU$ s.t. $U^T = U$, $R^T R = I$, $\det R = 1$

$C = F^T F = U^2$ RIGHT CAUCHY TENSOR

We will show that we can write $F = VR$ where R is orthogonal and V is symmetric and pos def

$$B = F F^T = V R R^T V^T = V^2 \quad \text{LEFT CAUCHY TENSOR}$$

Note: V is different from U but R is the same

9.

The eigenvalues of C and B are the same.

$F = VR$ LEFT DECOMPOSITION

Then consider
$C = \eta_1 P_1 + \eta_2 P_2$ $R := F U^{-1}$

$U := \sqrt{\eta_1}\, P_1 + \sqrt{\eta_2}\, P_2$

$B = FF^T \rightarrow F^{-1} B F = F^{-1} F F^T F = F^T F = C$

$B = F C F^{-1}$

If a_1 is an eigenvector of C : $C a_1 = \eta_1 a_1$

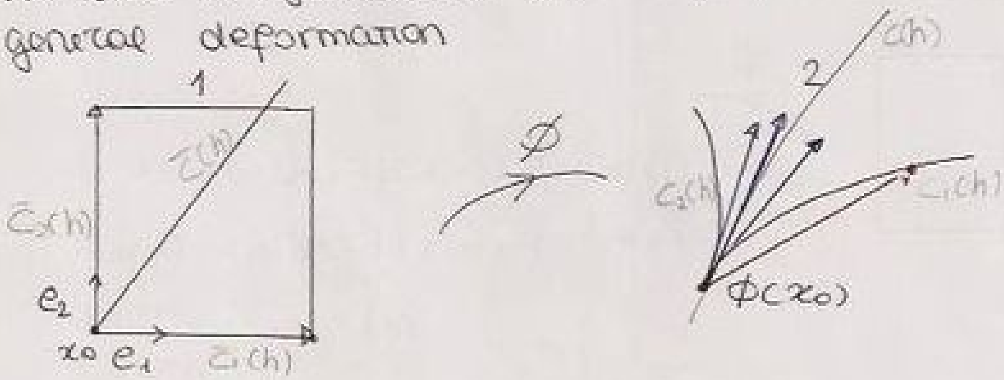
$F^{-1} B F a_1 = \eta_1 a_1$

$B(F a_1) = \eta_1 (F a_1)$ and $F a_1$ is an eigenvector for B with the same eigenvalue η_1

We can relate U and V saying

$VR = RU \rightarrow V = R U R^{-1}$

We want to generalize the expression ⊗ considering general deformation



$\{e_1, e_2\}$ orthonormal basis

w.r.t. dominium

$\bar{c}_1(h) = x_0 + h e_1$

$c_1(h) = \phi(\bar{c}(h))$

$\bar{c}(h) = x_0 + h a$

$\stackrel{!}{=} x_0 + h(\alpha_1 e_1 + \alpha_2 e_2)$

$\bar{c}_2 = x_0 + h e_2$

then $\lim\limits_{h \to 0} \frac{1}{h} (c(h) - c(0)) = c'(0)$

The tg vector of (1) is e_1 $\lim\limits_{h \to 0} \frac{1}{h} (\bar{c}_1(h) - x_0) = e_1$

i.e

$$\bar{c}_1'(0) = \lim_{h \to 0} \frac{1}{h} \left(\bar{c}_1(h) - \bar{c}_1(0) \right) = e_1$$

If $\bar{c}(h) = x_0 + ha$ then a is the tg vector

$$c'(0) = \lim_{h \to 0} \frac{1}{h} \left(c(h) - c(0) \right) = a$$

How these tan vectors are related to each other? Consider F s.t. $e_1 = \bar{c}_1' \longmapsto c_1'$

$$e_2 = \bar{c}_2' \longmapsto c_2'$$

$$\alpha_1 e_1 + \alpha_2 e_2 = \bar{c}' \longmapsto c'$$

A linear transformation is made up

$$x = \underbrace{\sqcup}_{\text{origin}} + \sqcup\, e_1 + \sqcup\, e_2$$

PARAMETRIZATION

we are interested in this two numbers

$$x = o + s_1 e_1 + s_2 e_2$$

$$\psi(x) = (s_1, s_2)$$

$$k(s_1, s_2) = x = o + s_1 e_1 + s_2 e_2$$

$$\phi(x) = \phi(k(s_1, s_2)) = o + \phi_1(s_1, s_2) e_1 + \phi_2(s_1, s_2) e_2$$

$$c_1(h) = \phi(\bar{c}_1(h))$$

$$\begin{aligned}
\bar{c}_1(h) &= x_0 + h e_1 \\
&= k(s_1, s_2) + h e_1 \\
&= o + s_1 e_1 + s_2 e_2 + h e_1 = o + (s_1 + h) e_1 + s_2 e_2 \\
&= k(s_1 + h, s_2)
\end{aligned}$$

$$c_1(h) = \phi(\bar{c}_1(h)) = \phi(k(s_1 + h, s_2))$$

$$c_2(h) = \phi(\bar{c}_2(h)) = \phi(k(s_1, s_2 + h))$$

where

$$\phi(k(s_1 + h, s_2)) = o + \phi_1(s_1 + h, s_2) e_1 + \phi_2(s_1 + h, s_2) e_2$$

$$\phi(k(s_1, s_2 + h)) = o + \phi_1(s_1, s_2 + h) e_1 + \phi_2(s_1, s_2 + h) e_2.$$

10.

Then

$$\frac{1}{h}(c_1(h) - c_1(0)) = \frac{\phi_1(s_1+h, s_2) - \phi_1(s_1,s_2)}{h} e_1 + \frac{\phi_2(s_1+h) - \phi_2(s_1,s_2)}{h} e_2$$

taking $h \to 0$

i.e.

$$c_1' = \partial_1\phi_1 e_1 + \partial_1\phi_2 e_2 = Fe_1$$

$$c_2' = \partial_2\phi_1 e_1 + \partial_2\phi_2 e_2 = Fe_2$$

Thus

$$\begin{aligned}\bar{c}(h) = x_0 + ha &= x_0 + h\alpha_1 e_1 + h\alpha_2 e_2 \\ &= o + s_1 e_1 + s_1 e_2 + h\alpha_1 e_1 + h\alpha_2 e_2 \\ &= k(s_1 + h\alpha_1, s_2 + h\alpha_2)\end{aligned}$$

and

$$\frac{1}{h}(c(h) - c(0)) = \quad ?$$

$$c(h) = \phi(k(s_1 + h\alpha_1, s_2 + h\alpha_2)) = o + \phi_1(s_1 + h\alpha_1, s_2 + h\alpha_2)e_1 + \phi_2(\quad)e_2$$

$$\frac{1}{h}(c(h) - c(0)) = \frac{1}{h}(\phi_1(s_1 + h\alpha_1, s_2 + h\alpha_2) - \phi_1(s_1,s_2))e_1 + \frac{1}{h}(\phi_2(s_1 + h\alpha_1, s_2 + h\alpha_2) - \phi_2(s_1,s_2))e_2$$

taking the limit

$$\begin{aligned}c' &= (\alpha_1\partial_1\phi_1 + \alpha_2\partial_2\phi_1)e_1 + (\alpha_1\partial_1\phi_2 + \alpha_2\partial_2\phi_2)e_2 \\ &= \alpha_1(\partial_1\phi_1 e_1 + \partial_1\phi_2 e_2) + \alpha_2(\partial_2\phi_1 e_1 + \partial_2\phi_2 e_2) \\ &= \alpha_1 Fe_1 + \alpha_2 Fe_2\end{aligned}$$

the linearity is preserved $\Rightarrow$ F is linear

$$[F] = [Fe_1 \;\; Fe_2]$$

$$F = \begin{pmatrix} \partial_1\phi_1 & \partial_1\phi_2 \\ \partial_2\phi_1 & \partial_2\phi_2 \end{pmatrix}$$

GRADIENT is totally different from the gradient of a vector field

In general straight lines are transformed into curves, but tan vectors are transformed into tan vectors (also linearly)

What do we describe?

$$\phi(\bar{c}(h)) = \phi(x_0) + \underbrace{F(\bar{c}(h) - \bar{c}(0))}_{\text{this will be a special curve}}$$

$$= \phi(x_0) + hFa$$

a straight line with tan vector Fa

$$c' = F(\alpha_1 e_1 + \alpha_2 e_2) = Fa$$

Sometimes we can say that

$$c(h) = \phi(\bar{c}(h)) = \phi(x_0) + hFa + \boxed{}$$

is an approximation plus a rest, let's say $o(h)$

$$o(h) = c(h) - (\phi(x_0) + hFa)$$
$$= (c(h) - \phi(x_0)) - hFa$$
$$= (c(h) - c(0)) - h\,c'(0)$$

Then $\lim\limits_{h\to 0} \dfrac{o(h)}{h} = 0$

Thus

$$c(h) = \phi(x_0) + hFa + \overbrace{o(h)}^{\text{vector}}$$

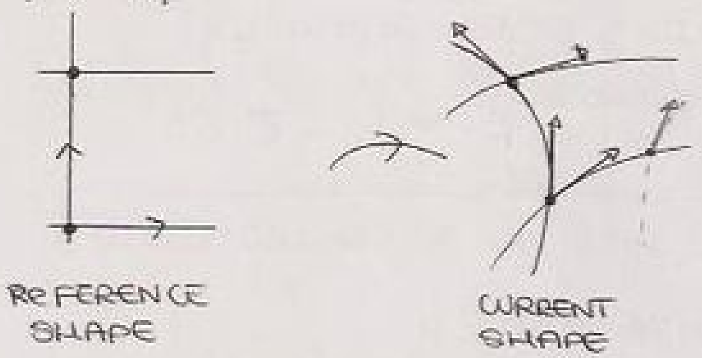
is an approximately description of the curve

1° order approx since $\dfrac{o(h)}{h} \underset{h\to 0}{\longrightarrow} 0$

11.

DEFORMATION GRADIENT 08.03.2011

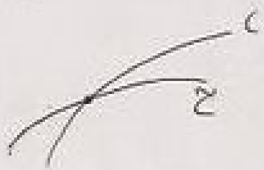
$F = \nabla \phi$



If we consider another point the deformation gradient will be different since it depends on the position in the reference shape.
The gradient is given by a field of Tensors
$$F(x) = F(x_0)$$
It can be shown that if F is uniform then can describe the deformation

If we consider a vector field on the current shape

$$\lim_{h \to 0} \frac{1}{h} (v(c(h)) - v(c(0)) = \underbrace{\nabla v}_{\text{gradient but it is completely different from the gradient is a TENSOR}} c'(0)$$



They will have the same tangent line

$\bar{v}(\bar{c}(h)) = v(c(h))$

How the two gradients are related?

$$\lim_{h \to 0} \frac{1}{h} (\bar{v}(\bar{c}(h)) - \bar{v}(\bar{c}(0))) = \nabla \bar{v} \, \bar{c}'(0)$$

Maybe the limit will be the same
$$\nabla \bar{v} \, \bar{c}'(0) = \nabla v \, c'(0)$$
How $\bar{c}'(0)$ and $c'(0)$ are related?

Since F transforms tangent vectors into tan vectors

$$c'(0) = F\,\bar{c}'(0)$$

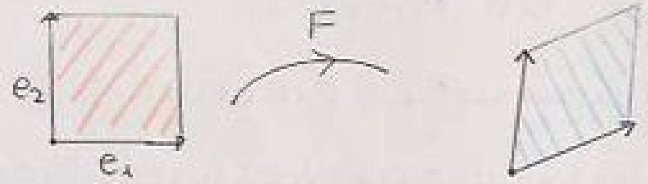
this is true for every one.

Then

$$\boxed{\nabla \bar{v} = \nabla v\, F}$$

PROPERTIES of F

- $\det F$. $F : V \to V$



sometimes it is better to observe the area of the figures.

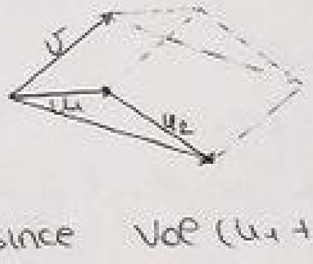
$$\boxed{\det F = \frac{\mathrm{Vol}(Fe_1, Fe_2)}{\mathrm{Vol}(e_1, e_2)}}$$

So we have to understand what is $\mathrm{Vol}(Fe_1, Fe_2)$

$\mathrm{Vol}(u, v)$

- $\mathrm{Vol}(u_1 + u_2, v) = \mathrm{Vol}(u_1, v) + \mathrm{Vol}(u_2, v)$

linear in the first argument



since $\mathrm{Vol}(u_1 + 2v, v) = \mathrm{Vol}(u_1, v)$

$$\mathrm{Vol}(u_1, v) + \mathrm{Vol}(2v, v) = \mathrm{Vol}(u_1, v)$$

$$\Rightarrow \mathrm{Vol}(2v, v) = 0$$

i.e. $\mathrm{Vol}(v, v) = 0$

- $\mathrm{Vol}(u, v_1 + v_2) = \mathrm{Vol}(u, v_1) + \mathrm{Vol}(u, v_2)$
- $\mathrm{Vol}(u, v) = \mathrm{Vol}(u, v-u) = \mathrm{Vol}(u + (v-u), v-u) =$
$= \mathrm{Vol}(v, v-u) = \mathrm{Vol}(v, -u) = -\mathrm{Vol}(v, u)$ 12.

$$Vol(u,v) = -Vol(v,u)$$

- $Vol(2u,v) = 2\,Vol(u,v)$

The volume is a bilinear func in both arguments

$$Vol : V \times V \longrightarrow \mathbb{R}$$

- $Vol(u_1+u_2, v) = Vol(u_1, v) + Vol(u_2, v)$
- $Vol(u,v) = -Vol(v,u)$ SKEW FUNCTION

antisimmetrica

(If a tensor is skew symmetric $A = -A^T$)

Note: $\nexists$ a basis s.t. $Vol(e_1,e_2)=0$

$$Vol(u,v) = Vol(u_1e_1+u_2e_2\,,\,v_1e_1+v_2e_2)$$

$$= Vol(u_1e_1, v_1e_1) + Vol(u_1e_1, v_2e_2) + Vol(u_2e_2, v_1e_1) + Vol(u_2e_2, v_2e_2)$$

$$= u_1v_1\,\cancel{Vol(e_1,e_1)}^{\,0} + u_1v_2\,Vol(e_1,e_2) + u_2v_1\,Vol(e_2,e_1) + u_2v_2\,\cancel{Vol(e_2,e_2)}^{\,0}$$

$$Vol(u,v) = \underbrace{(u_1v_2 - u_2v_1)}_{\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}} Vol(e_1,e_2)$$

If we consider another vector

$$Vol(\tilde{u},\tilde{v}) = (\tilde{u}_1\tilde{v}_2 - \tilde{u}_2\tilde{v}_1)\,Vol(e_1,e_2)$$

then $\dfrac{Vol(u,v)}{Vol(\tilde{u},\tilde{v})}$ does not depend on (e_1,e_2)

$$Vol(Fu,Fv) = Vol(u_1Fe_1+u_2Fe_2\,,\,v_1Fe_1+v_2Fe_2)$$

$$= u_1v_2\,Vol(Fe_1,Fe_2) - u_2v_1\,Vol(Fe_1,Fe_2)$$

$$= (u_1v_2 - u_2v_1)\,Vol(Fe_1,Fe_2)$$

Then $\dfrac{Vol(Fu,Fv)}{Vol(u,v)} = \dfrac{Vol(Fe_1,Fe_2)}{Vol(e_1,e_2)} = \det F$ it depends only on F

$\det F \neq 0$ since if $\det F = 0$ –

$\Rightarrow Vol(Fe_1, Fe_2) = 0$ just a line

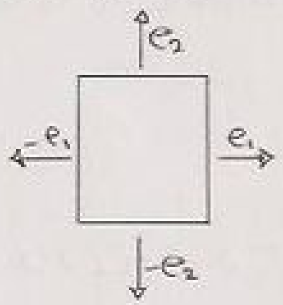
In order to ensure that the volume does not change sign we require

$$\det F > 0$$

In general we consider $\{e_1, e_2, e_3\}$

$$\det F := \frac{Vol(Fe_1, Fe_2, Fe_3)}{Vol(e_1, e_2, e_3)}$$

If we consider in general one of the six faces



NOTATION: we'll use $\bar{y}_1$ face wrt e_1

$\bar{y}_{-1}$ " $-e_1$

and so on...

Then we are interested in

$$\bar{y}_1 \mapsto y_1$$

$$\bar{y}_{-1} \mapsto y_1$$

$Vol_{\bar{R}}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ $\qquad$ $Vol_R(u_1, u_2, u_3)$

We want a definition for $A_{\bar{y}_1}$

$A_{\bar{y}_1} = Vol(\bar{n}_1, \bar{u}_2, \bar{u}_3)$

13

$\bar{h}_1 = \bar{u}_1 \cdot \bar{n}_1$ $\qquad\qquad$ $\bar{w}_1 := \bar{u}_1 - \bar{h}_1 \bar{n}_1$ orthogonal projection

something related to the two faces

$$\bar{w}_1 \bar{n}_1 = \underbrace{\bar{u}_1 \bar{n}_1}_{\bar{h}_1} - \bar{h}_1 \underbrace{\bar{n}_1 \bar{n}_1}_{1} = 0 \rightsquigarrow w_1 \perp n_1$$

$$\bar{u}_1 = \bar{w}_1 + \bar{h}_1 \bar{n}_1$$

$$V_{\bar{R}} = \mathrm{Vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = \mathrm{Vol}(\bar{w}_1 + \bar{h}_1 \bar{n}_1, \bar{u}_2, \bar{u}_3)$$

$$= \cancel{\mathrm{Vol}(\bar{w}_1, \bar{u}_2, \bar{u}_3)}^{\,0} + \mathrm{Vol}(\bar{h}_1 \bar{n}_1, \bar{u}_2, \bar{u}_3)$$

since $w_1 \perp n_1 \Rightarrow w_1$ is a linear combination of u_2 and u_3

$$V_{\bar{R}} = \bar{h}_1 \mathrm{Vol}(\bar{n}_1, \bar{u}_2, \bar{u}_3)$$

$$= \bar{h}_1 A_{\bar{y}_1}.$$

Assume $V_{\bar{R}} > 0$ then $A_{\bar{y}_1} > 0 \iff \bar{h}_1 > 0$

Finally $V_R = h_1 A_{y_1}$.

Now we compare $V_{\bar{R}} = \bar{h}_1 A_{\bar{y}_1}$

$V_R = h_1 A_{y_1}$

in this way $\dfrac{V_R}{V_{\bar{R}}} = \dfrac{h_1 A_{y_1}}{\bar{h}_1 A_{\bar{y}_1}}$

$$\det F = \frac{h_1}{\bar{h}_1} \frac{A_{y_1}}{A_{\bar{y}_1}}$$

$$\frac{\bar{h}_1}{h_1} \det F = \frac{A_{y_1}}{A_{\bar{y}_1}}$$

it is hard to compute A_{y_1} as before so we'll use this relation to know that ratio.

In the new face

$$h_1 := u_1 \cdot n_1$$

$u_2 \cdot n_1 = 0 \qquad u_3 \cdot n_1 = 0$ since $n_1 \perp$ to the face generated by u_2, u_3

$F\bar{u}_1 \cdot n_1 = h_1 \quad F\bar{u}_2 \cdot n_1 = 0 \quad F\bar{u}_3 \cdot n_1 = 0$

$$\bar{u}_2 \cdot \underbrace{F^T n_1}_{\perp \bar{u}_2} = 0 \qquad \bar{u}_3 \cdot \underbrace{F^T n_1}_{\perp \bar{u}_3} = 0$$

$$F^T n_1 = \alpha \bar{n}_1 \Rightarrow n_1 = \alpha \underbrace{F^{-T} \bar{n}_1}$$

F^{-T} trasforms unit normal vectors

$$\boxed{F^{-T} \bar{n}_1 = k_1 n_1}$$

$$F^{-T} \bar{n}_1 \cdot u_1 = k_1 n_1 \cdot u_1 = k_1 h_1$$

$$\bar{n}_1 \cdot F^{-1} u_1 = \bar{n}_1 \cdot \bar{u}_1 = k_1 h_1$$

$$\Rightarrow \bar{h}_1 = k_1 h_1 \quad \text{i.e.} \quad k_1 = \frac{\bar{h}_1}{h_1}$$

Then $F^{-T} \bar{n}_1 = \dfrac{\bar{h}_1}{h_1} n_1 = \dfrac{1}{\det F} \dfrac{A_{y_1}}{A_{\bar{y}_1}} n_1$

$$\boxed{(\det F) F^{-T} \bar{n}_1 = \frac{A_{y_1}}{A_{\bar{y}_1}} n_1}$$

If we know F we can compute this equality

$(\det F) F^{-T}$ COFACTOR OF F

14.

09.03.2011

$$\det F = \frac{Vol(Fe_1, Fe_2)}{Vol(e_1, e_2)} = (f_{11} f_{22} - f_{21} f_{12}) \frac{Vol(e_1, e_2)}{Vol(e_1, e_2)}$$

$$Fe_1 = f_{11} e_1 + f_{21} e_2$$

$$Fe_2 = f_{12} e_1 + f_{22} e_2$$

is independent from the basis.
By comparing the areas of each faces of the solid, the cofactor of F

$$\left((\det F) F^{-T}\right) \bar{n}_1 = \frac{A y_1}{A \bar{y}_1} n_1$$

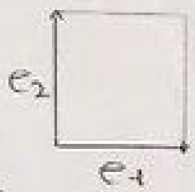
then

$$\boxed{\| \operatorname{cof} F \, \bar{n}_1 \| = \frac{A y_1}{A \bar{y}_1}} .$$

Now we want to see what is

$$\left((\det F) F^{-T}\right) e_1 \cdot e_2 \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad - cb$



$\|e_i\| = 1$

$$\operatorname{cof} F e_1 = A y_1 n_1 = Vol(n_1, Fe_2, Fe_3) n_1$$

$$\operatorname{cof} F e_1 \cdot e_2 = (n_1 \cdot e_2) \, Vol(n_1, Fe_2, Fe_3)$$

$$= Vol((n_1 \cdot e_2) n_1, Fe_2, Fe_3)$$

$$w = e_2 - (n_1 \cdot e_2) n_1$$

$$w \cdot n_1 = (e_2 \cdot n_1) - (n_1 \cdot e_2) \underbrace{n_1 \cdot n_1}_{1} = 0 \quad \Rightarrow w \perp n_1$$

$$\Rightarrow Vol(w, Fe_2, Fe_3)$$

$w \perp n_1 \Rightarrow n_1$ can be expressed as lin comb of $Fe_2, Fe_3 \Rightarrow Vol(w, Fe_2, Fe_3) = 0$

$\rightarrow \mathrm{Vol}(e_2 - w,\ Fe_2,\ Fe_3) = \mathrm{Vol}(e_2, Fe_2, Fe_3)$

We get the matrix

$$\begin{bmatrix} 0 & f_{12} & f_{13} \\ 1 & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \end{bmatrix} \Rightarrow \det = f_{13}f_{32} - f_{33}f_{12}$$

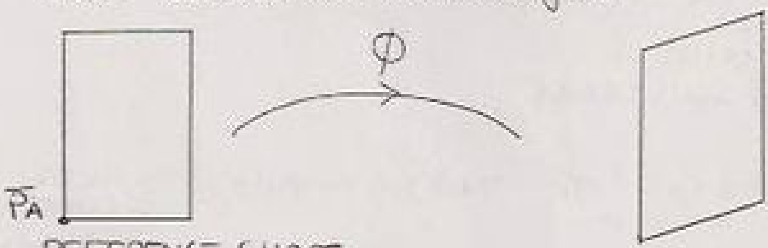
$$\mathrm{Vol}(e_2, e_1, e_3) + \mathrm{Vol}(e_2, e_3, e_1)$$
$$= \mathrm{Vol}(e_2,\ f_{12}e_1 + f_{22}e_2 + f_{32}e_3,\ f_{13}e_1 + f_{23}e_2 + f_{33}e_3)$$
$$= (-f_{12}f_{33} + f_{32}f_{13})\,\mathrm{Vol}(e_1, e_2, e_3)$$

In elementary calculus is useful to compute the inverse of a matrix.

$$(\det F)\,F^{-T} = \mathrm{cof}\,F$$
$$\Rightarrow F^{-1} = \frac{1}{(\det F)}(\mathrm{cof}\,F)^T$$

In this case the COFACTOR of F transforms unit normal vectors in unit normal vectors.

By a RIGID MOTION we mean, by a reference shape, the deformation at any time t that leaves the distances unchanged.



REFERENCE SHAPE

$$P_A(t) = \phi(\bar P_A, t) = P_0(t) + R(t)(\bar P_A - \bar P_0)$$

If we want to define the velocity vector at time t

$$\dot P_A(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(P_A(t+\Delta t) - P_A(t)\right)$$

TIME DERIVATIVE OF THE ROTATION

$$\dot P_A(t) = \dot P_0(t) + \dot R(t)(\bar P_A - \bar P_0)$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(R(t+\Delta t)(\bar P_A - \bar P_0) - R(t)(\bar P_A - \bar P_0)\right)$$

15.

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \underbrace{(R(t+\Delta t) - R(t))}\,(\bar{p}_A - \bar{p}_0)$$

note that ~~th~~ is not a rotation since

$$(R(t+\Delta t)^T - R(t)^T)(R(t+\Delta t) - R(t))$$

$$= I - R(t+\Delta t)^T R(t) - R(t)^T R(t+\Delta t) + I$$

we don't get the Identity.

$$\bar{p}_A - \bar{p}_0 = R(t)^{-1}(p_A(t) - p_0(t))$$

$$\boxed{\dot{p}_A(t) = \dot{p}_0(t) + \dot{R}(t) R(t)^{-1}(p_A(t) - p_0(t))}$$

This eq. is useful to describe the velocity field of the deformed shape

$$\dot{p}_A(t) = \dot{p}_0(t) + \underbrace{\boxed{\dot{R}(t) R^T(t)}}_{W(t)}(p_A(t) - p_0(t))$$

$W(t)$ SPIN of the velocity field
is a tensor

PROPERTY of $W(t)$:

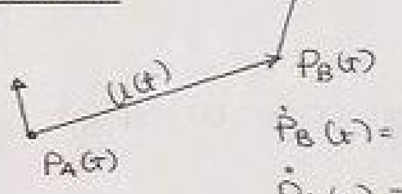
- $R(t) R^T(t) = I$

$$\underbrace{\dot{R}(t) R^T(t)}_{W(t)} + \underbrace{R(t)\dot{R}^T(t)}_{\text{adjoint of the spin tensor}} = 0$$

$$W(t) + W(t)^T = 0 \implies W(t) = -W(t)^T$$

SKEW SYMMETRIC TENSOR

MEANING



$$\dot{p}_B(t) = \dot{p}_0(t) + W(t)(p_B(t) - p_0(t))$$

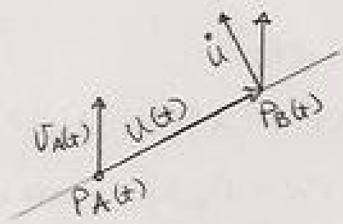
$$\dot{p}_A(t) = \dot{p}_0(t) + W(t)(p_A(t) - p_0(t))$$

$$\dot{p}_B(t) - \dot{p}_A(t) = W(t)(p_B(t) - p_A(t))$$

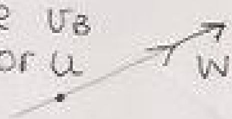
$$\dot{u}(t) = W(t)\, u(t)$$

$\dot{u} = W u$

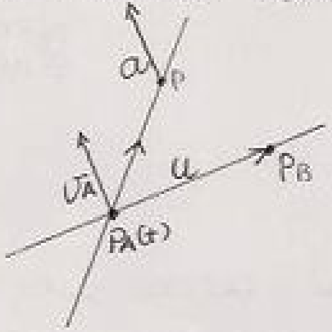
$\dot{u} \cdot u = W u \cdot u = 0 \Rightarrow \dot{u} \perp u$



If $v_A = 0$ we cannot have v_B along the direction o/w the vector u will not have $\dot{u} \perp u$.



$\exists$ any scalar s.t. $Wa = \lambda a$ holds?
We found $Wa \cdot a = 0$ then $0 = \lambda a \cdot a$ then $\exists$ any vector st the equality holds but we should take $\lambda = 0$.



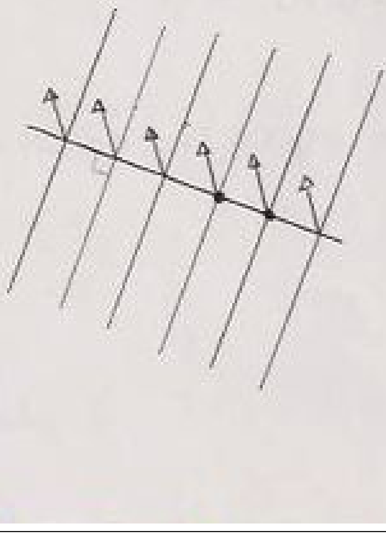
$\dot{P}_B - \dot{P}_A = W(P_B - P_A)$

we can do the same thing on a?

$\dot{P} - \dot{P}_A = W\, \alpha a = 0$

$\dot{P} = \dot{P}_A$ they have the same velocity

Consider many lines // to vector a. Cut them with a plane, orthogonally and move it to evaluate the velocity.



Consider a couple of points then the difference of the velocities will be $\perp$ to the plane.

16.

14.03.2011

$v_A = v_o + W(P_A - P_o)$

$\dot{P}_A(t) = \dot{P}_o(t) + W(t)(P_A(t) - P_o(t))$

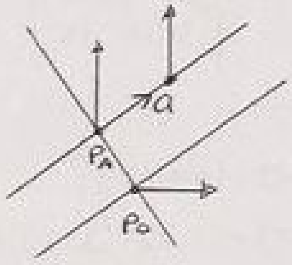
Description of the vector field

$v(P_A(t)) = v(P_o(t)) + W(t)(P_A(t) - P_o(t))$

$v(P_A) - v(P_o) = W(P_A - P_o)$ where $W = -W^T$

and $Wa = \lambda a \Rightarrow \lambda = 0$

and $(P_A - P_o) \perp W$



$(v(P_A) - v(P_o)) \cdot a = W(P_A - P_o) \cdot a = (P_A - P_o) W a = 0$

since a is an eigenvector

then $(v(P_A) - v(P_o)) \perp a$

If

$v_A = \alpha_A a + \tilde{v}_A$

$v_o = \alpha_o a + \tilde{v}_o$

$v_A - v_o = (\alpha_A - \alpha_o) a + (\tilde{v}_A - \tilde{v}_o)$ before we proved that $(v_A - v_o) \perp a$ then the projection $\tilde{v}_A - \tilde{v}_o$ should be zero

$a \cdot (v_A - v_o) = (\alpha_A - \alpha_o) + \overset{0}{\cancel{(\tilde{v}_A - \tilde{v}_o) a}} = 0$

$\Rightarrow \alpha_A = \alpha_o$

The orthogonal projections are equal to each other

Given P_0 and the spin W we can find a position x_0 s.t.

$$v(x_0) = v(P_0) + W(x_0 - P_0)$$

in particular $v(x_0) \to 0$

$$0 = v(P_0) + W(x_0 - P_0).$$

Consider W, how it is made? Let's calculate

$We_1 \cdot e_1 = 0$
$We_1 \cdot e_2 = -e_1 \cdot W^T e_2$
$We_3 \cdot e_1 = -e_3 \cdot We_1$
$We_3 \cdot e_2 = -e_3 \cdot We_2$

$$W = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad \text{in an orthonormal basis}$$

In the basis $\{e_1, e_2, a\}$

$$W = \begin{array}{c} \begin{array}{ccc} We_1 & We_2 & Wa \end{array} \\ \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Let's see it:

$$\begin{aligned} v(P_1) &= v(P_0) + W(\alpha_1 e_1 + \alpha_2 e_2) \\ &= v(P_0) + \alpha_1 We_1 + \alpha_2 We_2 \\ &= v(P_0) + \alpha_1 \omega e_2 + \alpha_2(-\omega) e_1 \\ &= v(P_0) + \omega(-\alpha_2 e_1 + \alpha_1 e_2) \end{aligned}$$

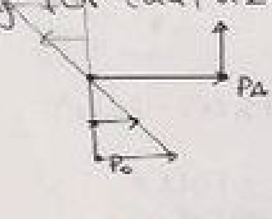
Which is the position x_0 s.t. $v(x_0) = 0$?

$$0 = v(P_0) + W(x_0 - P_0)$$

What I know is $v(P_0)$ and W, also P_0. In 2-dim

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = -\begin{pmatrix} \sqcup \\ \sqcup \end{pmatrix} \text{ coordinates of } v(P_0)$$

solving for (α_1, α_2).



A.

PROPERTY:

$v_A = v_0 + \omega \times (P_A - P_0)$ $\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$

$e_1 \times e_2 = e_3$
$e_2 \times e_3 = e_1$
$e_3 \times e_1 = e_2$ ← – – $\underline{W u} = \underbrace{\omega \times u}$ $\forall u$

then given W $\exists \omega$ s.t the equality holds is a property of the skew-simmetry tensor in 3-dim space

the cross product is s.t.

$(u_1 \times u_2)\cdot u = \mathrm{vol}(u_1, u_2, u)$

PROOF:

In any basis vector

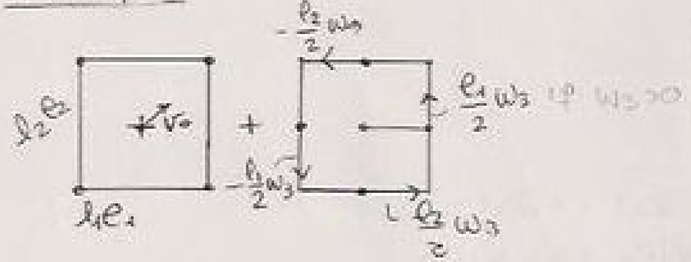
$$W e_1 = \omega \times e_1$$

$$\omega_3 e_2 - \omega_2 e_3 = +\omega_3 e_2 - \omega_2 e_3$$

and so on...

The SPIN TENSOR W is also called the AXIAL VECTOR.

Example



$$v(P_A(t)) = v(P_0(t)) + W\underbrace{(P_A(t) - P_0(t))}$$

$$W \frac{l_1}{2} e_1 = \frac{l_1}{2} W e_1 = \frac{l_1}{2}\omega_3 e_2 - \frac{l_1}{2}\omega_2 e_3$$

$$W \frac{l_2}{2} e_2 = \frac{l_2}{2} W e_2 = \frac{l_2}{2}\left(-\omega_3 e_1 + \omega_1 e_3\right)$$

Computation that is consistent with the property showed before.

If we consider the deformation

$$P_A(t) - P_0(t) = F(t)(\bar{P}_A - \bar{P}_0)$$

$$\dot{P}_A(t) = \dot{P}_0(t) + \dot{F}(t)(\bar{P}_A - \bar{P}_0) \qquad L = \dot{F}$$

$$= \dot{P}_0(t) + \dot{F}(t)F^{-1}(t)(P_A(t) - P_0(t))$$

$$v(P_A(t)) = v(P_0(t)) + L(t)(P_A(t) - P_0(t))$$

$L(t)$ IS THE VELOCITY GRADIENT

where $L(t) = \dot{F}(t)\,F^{-1}(t)$. We know

$$F(t) = \text{grad}\,\phi,$$

if F is rotation

$$L = \dot{F} F^{-1}$$
$$= \dot{R} R^{-1} = W \quad \text{then } L \text{ is skew-symm. tensor}$$

But in general L is not skew-symm tensor

Note that in the case of deformations, I can only consider the composition. In the case of velocities I can consider the sums. For this reason is right to consider

$$L = \underbrace{W}_{\text{SKEW SYMMETRIC TENSOR}} + \underbrace{D}_{\text{SYMMETRIC TENSOR}}$$

$$L^T = W^T + D^T$$

$$\Rightarrow L + L^T = 2D \Rightarrow \boxed{\begin{aligned} D &= \tfrac{1}{2}(L + L^T) \\ W &= \tfrac{1}{2}(L - L^T) \end{aligned}}$$

STRETCHING (VELOCITY)

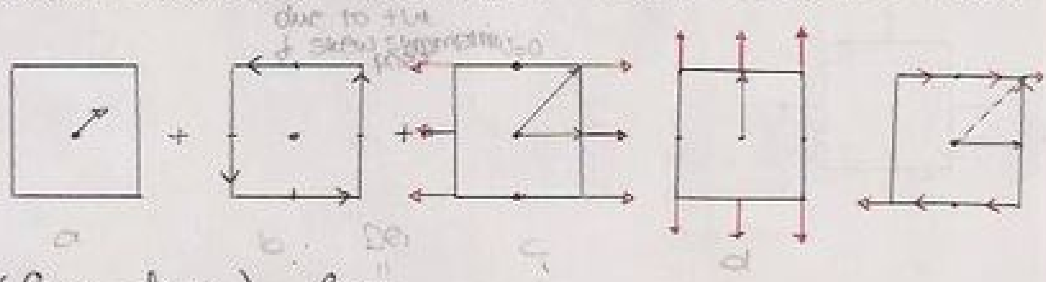
SPIN

15.03.2011

Since $L = W + D$ then

$$v(P_A(t)) = v(P_0(t)) + W(t)(P_A(t) - P_0(t)) + D(t)(P_A(t) - P_0(t))$$

due to the skew symmetric = 0



$$D\left(\frac{l_1}{2} e_1 + \frac{l_2}{2} e_2\right) = \frac{l_1}{2} e_1$$

The matrix basis that we consider

$$\alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

18.

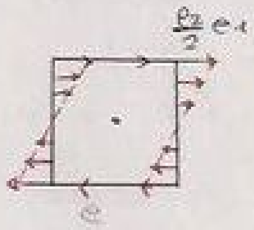
$D\left(\frac{\ell_2}{2} e_2\right) = \frac{\ell_2}{2} e_1$

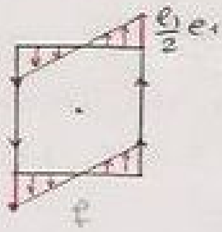
$D\left(\frac{\ell_1}{2} e_1 + \frac{\ell_2}{2} e_2\right) = \frac{\ell_1}{2} e_1$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$D\left(\frac{\ell}{2} e_1 + h e_2\right) = h e_1$

$D\left(-\frac{\ell}{2} e_1 + h e_2\right) h e_1$





(c) is stretching in the direction of the arrows

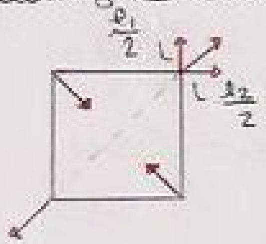
(d) "

(e) is sliding "

(f) "

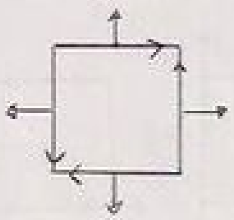
(b) is characterized by the fact that $\boxed{\cdot}$ $v=0$

Adding (e) and (f)



$D\left(\frac{\ell_1}{2} e_1 + \frac{\ell_2}{2} e_2\right) = \frac{\ell_2}{2} e_1$

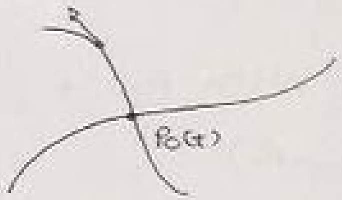
considering a square $\frac{\ell_1}{2} = \frac{\ell_2}{2}$



$p_A(t) = p_0(t) + F(t)(\bar{p}_A - \bar{p}_0) + \Theta$

we saw $\nabla v = \dot{F}F^{-1}$. What we already know $c'(t) = F(t)\bar{c}'$

$$\dot{c}'(t) = \dot{F}(t)\bar{c}' = \dot{F}(t)F^{-1}(t)c'(t) \quad \triangle$$



$$c'(0,t) = \lim_{h \to 0} \frac{1}{h}(c(h,t) - c(0,t))$$

$$\dot{c}'(0,t) = \lim_{h \to 0} \frac{1}{h}(\dot{c}(h,t) - \dot{c}(0,t))$$

by def of the gradient of the velocity field

$$\dot{c}'(0,t) = \nabla v\, c'(0,t)$$

comparing it with $\triangle$ we get $\nabla v = \dot{F}F^{-1}$

What is a FORCE? The most abstract way to define a force is:

$$p_A(t) = p_0(t) + F(t)(\bar{p}_A - \bar{p}_0)$$

$$v(p_A(t)) = v(p_0(t)) + L(t)(p_A(t) - p_0(t))$$

We assume that for each velocity field there is a function such that transforms these velocities into scalars $f(v) \in \mathbb{R}$.

In physics the power is $f_A \cdot v_A = f(v_A)$

$$f : V \to \mathbb{R}$$

For this reason someone used $P(v) \in \mathbb{R}$.

If we consider work $= f_A u_A \Rightarrow \underset{\text{WORKING}}{\mathcal{W}(u)} \in \mathbb{R}$

We'll use $\underset{\text{POWER}}{\mathcal{W}(v)} \in \mathbb{R}$

Consider a body made $\{0, A, B, C\}$

$$\mathcal{W}(v) = f_A v_A + f_B v_B + f_C v_C$$

19

$$v(P_A(t)) = \underbrace{v(P_o(t)) + W(t)(P_A(t) - P_o(t))}_{\text{rigid part}} + \underbrace{D(t)(P_A(t) - P_o(t))}_{\text{stretching part}}$$

$$\mathcal{W}(v) = f_A\,(v_o + W(P_A - P_o)) + f_B\,(v_o + W(P_B - P_o)) + f_C\,(v_o + W(P_C - P_o))$$

$$= (f_A + f_B + f_C)\cdot v_o + f_A \cdot W(P_A - P_o) + f_B \cdot W(P_B - P_o) + f_C \cdot W(P_C - P_o)$$

Using the spin vector ω : $W(P_A - P_o) = \omega \times (P_A - P_o)$

so $f_A \cdot \omega \times (P_A - P_o)$ by the property of the cross product

$\omega \cdot (P_A - P_o) \times f_A$

$$\mathcal{W}(v) = \underbrace{(f_A + f_B + f_C)}_{\text{TOTAL FORCE}}\cdot v_o + \omega \cdot \underbrace{((P_A - P_o)\times f_A + (P_B - P_o)\times f_B + (P_C - P_o)\times f_C)}_{\text{MOMENT VECTOR WRT } P_o}$$

MEANING: If we consider some forces applied to the body points the power is made by two parts

Now we are considering the velocity vector field with the rigid part and the stretching part

$$f_A \cdot v_A = f_A \cdot v_o + f_A \cdot W(P_A - P_o) + f_A \cdot D(P_A - P_o)$$

the pb: $D(P_A - P_o)$ is possible to replace it with $d \times (\text{something})$? No

$$f_A \cdot D(P_A - P_o) = (\underbrace{(P_A - P_o)\otimes f_A}_{\text{TENSOR PRODUCT}})\cdot D$$

$$f_A \cdot W(P_A - P_o) = ((P_A - P_o)\otimes f_A)\cdot W$$

so $\quad f_A \cdot W(P_A - P_o) + f_A \cdot D(P_A - P_o) = ((P_A - P_o)\otimes f_A)\cdot L$.

We used $f \cdot Lu = \underbrace{(u \otimes f)}\cdot L$

is a tensor transforming vectors into vectors

For fixed u and fixed f we can define $(u \otimes f)e = (u \cdot e)f$. Let's have a look to the matrix of this tensor.

HOMEWORK: compute the matrix using the def

$(u \otimes f)e_1 = (u \cdot e_1)f$ $\quad u = u_1 e_1 + u_2 e_2 + u_3 e_3$

$= u_1 f_1 e_1 + u_1 f_2 e_2 + u_1 f_3 e_3 \quad f = f_1 e_1 + f_2 e_2 + f_3 e_3$

$(u \otimes f)e_2 = (u \cdot e_2)f$

$= u_2 f_1 e_1 + u_2 f_2 e_2 + u_2 f_3 e_3$

$(u \otimes f)e_3 = (u \cdot e_3)f$

$= u_3 f_1 e_1 + u_3 f_2 e_2 + u_3 f_3 e_3$

$$\begin{bmatrix} u_1 f_1 & u_2 f_1 & u_3 f_1 \\ u_1 f_2 & u_2 f_2 & u_3 f_2 \\ u_1 f_3 & u_2 f_3 & u_3 f_3 \end{bmatrix} \qquad Rg = 1 \, ?$$

$A \cdot B = \operatorname{tr}(A^T B)$

$$\operatorname{tr} C = \frac{vol(Ce_1, e_2, e_3) + vol(e_1, Ce_2, e_3) + vol(e_1, e_2, Ce_3)}{vol(e_1, e_2, e_3)}$$

$$vol(Ce_1, e_2, e_3) = vol(C_{11} e_1 + C_{21} e_2 + C_{31} e_3, e_2, e_3) \underset{LIN}{\overset{\uparrow}{=}} C_{11}\, vol(e_1, e_2, e_3)$$

$vol(e_1, Ce_2, e_3) = C_{22}\, vol(e_1, e_2, e_3)$

$$\Rightarrow \operatorname{tr} C = \frac{(C_{11} + C_{22} + C_{33})\, vol(e_1, e_2, e_3)}{vol(e_1, e_2, e_3)} = C_{11} + C_{22} + C_{33}$$

is indep from the basis

$A \cdot B = a_{11} b_{11} + a_{21} b_{21} + a_{31} b_{31} + a_{12} b_{12} + a_{22} b_{22}$

$A \cdot B = \operatorname{tr}(A^T B) = \operatorname{tr}(A B^T)$ $\qquad$ 16.03.2011

$$f_A \cdot v_A = f_A \cdot (v_0 + L(P_A - P_0)) = f_A \cdot v_0 + \underbrace{f_A \cdot L(P_A - P_0)}_{(P_A - P_0) \otimes f_A \cdot L}$$

consider another point

$f_B \cdot v_B = f_B \cdot (v_0 + L(P_B - P_0)) = f_B \cdot v_0 + (P_B - P_0) \otimes f_A \cdot L$

20

$$\mathcal{W}(v) = \underbrace{(f_A + f_B)}_{\text{TOTAL FORCE}} \cdot v_o + \underbrace{\left((p_A - p_o)\otimes f_A + (p_B - p_o)\otimes f_B\right)}_{\substack{M_{p_o} \\ \text{TOTAL MOMENT TENSOR}}} \cdot L$$

$$\boxed{\mathcal{W}(v) = f \cdot v_o + M_{p_o} \cdot L}$$

Only on rigid velocity fields

$$f_A \cdot v_A = f_A \cdot (v_o + L(p_A - p_o)) = f_A \cdot v_o + f_A \underbrace{W(p_A - p_o)}_{f_A \cdot \omega \times (p_A - p_o)}$$

ω SPIN

For two bodies

$$\mathcal{W}(v) = (f_A + f_B) \cdot v_o + ((p_A - p_o)\times f_A + (p_B - p_o)\times f_B) \cdot \omega.$$

$$L = W + D \qquad W = \operatorname{skew} L = \frac{1}{2}(L - L^T)$$

$$M \cdot L = M \cdot (W + D)$$

$$(\operatorname{skew} M + \operatorname{sym} M)(W + D) = \operatorname{skew} M \cdot W + \operatorname{skew} M \cdot D + \operatorname{sym} M \cdot W + \operatorname{sym} M \cdot D$$

E.g. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = -a_{12}\omega + a_{12}\omega = 0$

$$0 = A \cdot W = \operatorname{Tr}(A^T W) = \operatorname{Tr}(AW)$$
$$= \operatorname{Tr}(A W^T) = -\operatorname{Tr}(AW)$$

are orthogonal

Then $(\operatorname{skew} M + \operatorname{sym} M) \cdot (W + D) = \operatorname{sym} M \cdot D + \operatorname{skew} M \cdot W$

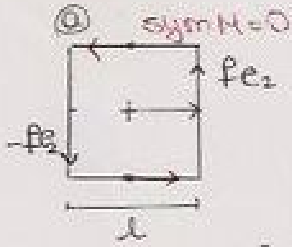
$$\mathcal{W}(v) = f \cdot v_o + M_{p_o} \cdot W$$

if forces are balanced $\mathcal{W}(v) = 0 \quad \forall v$ TEST VELOCITY FIELD

$$v(x) = v(x_o) + W(x - x_o)$$

$$\mathcal{W}(v) = f \cdot v_o + M_{p_o} \cdot W = 0 \Rightarrow f = 0$$
$$\operatorname{skew} M = 0$$

Example



$$\frac{l}{2} e_1 \otimes f e_2 = \frac{l}{2} f (e_1 \otimes e_2) \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[u \otimes f] = \begin{bmatrix} u_1 f_1 & u_2 f_1 \\ u_1 f_2 & u_2 f_2 \end{bmatrix}$$

$$-\frac{l}{2} e_1 \otimes (-f e_2) = \frac{l}{2} f (e_1 \otimes e_2)$$

$$M_{P_0} = l f\, e_1 \otimes e_2 \qquad [M_{P_0}] = l f \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$ this matrix is neither skewsymm neither symm

$$\frac{l}{2} e_2 \otimes (-f e_1) = -\frac{l}{2} f (e_2 \otimes e_1)$$

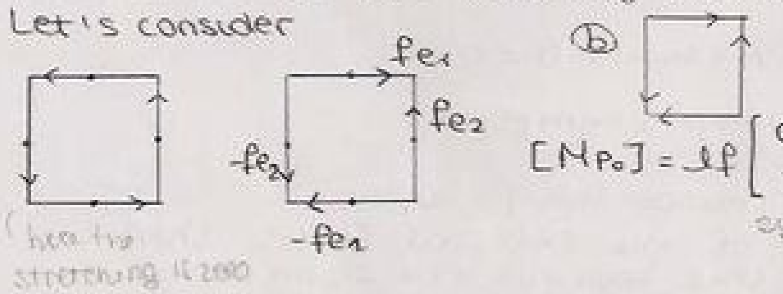
$$-\frac{l}{2} e_2 \otimes (f e_1) = -\frac{l}{2} f (e_2 \otimes e_1)$$

$$M_{P_0} = l f (e_1 \otimes e_2 - e_2 \otimes e_1) \qquad [M_{P_0}] = l f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ the symm part is zero

$$M_{P_0} \cdot W = l f \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \omega = 2 l f \omega$$

Descriptions of forces must be related with the descriptions of the velocity fields

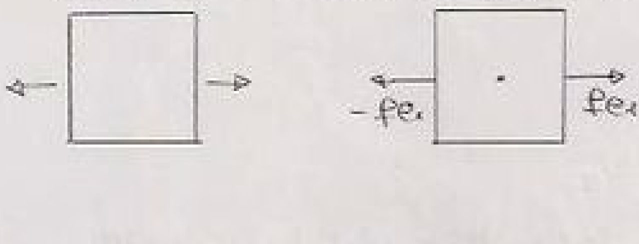
Let's consider



$$[M_{P_0}] = l f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

symmetric

What is the moment tensor related to this picture?

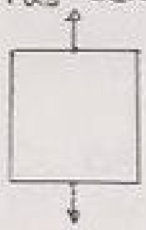


$$l e_1 \otimes (f e_1)$$

$$l f \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

21

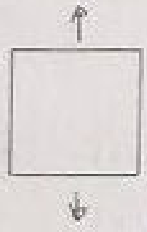
In this case



$$l e_2 \otimes f e_2 = l f (e_2 \otimes e_2)$$

$$l f \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The velocity component will be



Considering the power of ⓐ forces corresponding to ⓑ velocity fields $\Rightarrow \mathcal{W} = 0$.

If the body is a $\boxed{\text{rigid body}}$, we use the <u>balance principle</u> to say

$$\mathcal{W}(v) = 0 \quad \forall v.$$

This implies that $f = 0$

$$\text{skw} M = 0$$

Only forces satisfying that principle are allowed.

If I consider bodies described by an AFFINE deformation we will call them AFFINE BODIES.

If we assume that

$$f \cdot v_0 + \text{skw} M \cdot W + \text{sym} M \cdot D = 0$$

then $f = 0$, $\text{skw} M = 0$, $\text{sym} M = 0$

There is a way to change this principle:
consider the power of the external forces then
there is a power that balance the system

$$\mathcal{W}^{ext}(v) + \mathcal{W}^{int}(v) = 0 \qquad \forall v$$

$$\mathcal{W}^{ext}(v) = f \cdot v_0 + M_R \cdot L$$

to balance

$$\mathcal{W}(v) = (\underset{\text{0-stress}}{z} \cdot v_0 + \underset{\text{stress}}{T} \cdot L) V_R$$

and assume z and T densities w.r.t. to the volume

21.03.2011

The Power of a test velocity

$$\mathcal{W}(v) = f \cdot v_0 + M_{P_0} \cdot L$$

considering affine motion $v(x) = v_0 + L(x - x_0)$

BALANCE PRINCIPLE FOR RIGID BODY

- $\{\mathcal{W}(v) = 0 \quad \forall v \Leftrightarrow \forall v_0, \forall W\} \Leftrightarrow \begin{cases} f = 0 \\ \text{skw} M = 0 \end{cases}$

$\begin{cases} f = 0 \\ m = 0 \end{cases}$ f - total force, m - total moment

then

$$\mathcal{W}(v) = f \cdot v_0 + M_{P_0} \cdot L = f \cdot v_0 + M \cdot W = f \cdot v_0 + m \cdot \omega$$

We found that

$\begin{cases} f = 0 \\ \text{skw} M = 0 \end{cases}$ $\begin{cases} f = 0 \\ m = 0 \end{cases}$

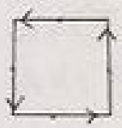
are strongly related $Wu = \underset{\text{axial vector}}{\omega} \times u \quad \forall u$

When we are considering affine motion, is not satisfactory that principle, saying

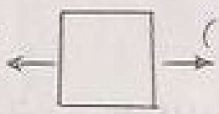
$$\{\mathcal{W}(v) = 0 \quad \forall v \Leftrightarrow \forall v_0, \forall L\} \Leftrightarrow \begin{cases} f = 0 \\ \text{skw} M = 0 \\ \text{sym} M = 0 \end{cases}$$

is not good since is too much selective.

Applied to



the moment is not zero, instead in rigid bodies



these forces are allowed

In the case of affine motion we want apply forces like $\leftarrow\square\rightarrow$ but we have to work on the power.

$$\mathcal{W}(v) = (f \cdot v_0 + M_{P_0} \cdot L) - \underbrace{(z \cdot v_0 + T \cdot L)}_{\text{we assume that this part comes from the same vector field}}$$

22.

i.e. comes from the materiae of the body for this reason we muetipey it by V_R

depends on the materiae of the body

$$\mathcal{W}(v) = \underbrace{(f \cdot v_0 + M_{P_0} \cdot L)}_{\text{EXTERNAL POWER}} - \underbrace{(\not{z} \cdot v_0 + T \cdot L) V_R}_{\text{INTERNAL POWER}} \quad (1)$$

defined by quantities described by the force that we apply

$$\mathcal{W}(v) = \mathcal{W}^e + \mathcal{W}^i$$

or OUTER POWER INNER POWER

$\not{z}$: O. STRESS

T : CAUCHY STRESS

$\downarrow$

STRESS POWER

$$\mathcal{W}(v) = (f - \not{z} V_R) \cdot v_0 + (M_{P_0} - T V_R) \cdot L$$

thus

$$\{ \mathcal{W}(v) = 0 \quad \forall v \Leftrightarrow \forall v_0, \forall L \} \Leftrightarrow \begin{cases} f = \not{z} V_R \\ M = T V_R \end{cases} \quad \#$$

i.e. the tot force is balanced by a stress which is internal and the total moment " by the cauchy stress

How ● and # are related? We can evaluate (1) on the subset $v(x) = v_0 + W(x - x_0)$ getting

$$\mathcal{W}(v) = (f \cdot v_0 + M_{P_0} \cdot W) - (\not{z} \cdot v_0 + T \cdot W) V_R$$

then

$$\{ W(v) = 0 \quad \forall v \Leftrightarrow \forall v_0, \forall W \} \Leftrightarrow \begin{cases} f = \not{z} V_R \\ \operatorname{skw} M = (\operatorname{skw} T) V_R \end{cases}$$

since $L = W + D$

$$\{ W(v) = 0 \ \forall v \Leftrightarrow \forall v_0, \forall L \} \Leftrightarrow \begin{cases} f = \not{z} V_R \\ \operatorname{skw} M = (\operatorname{skw} T) V_R \\ \operatorname{sym} M = (\operatorname{sym} T) V_R \end{cases}$$

Modify the power adding $(z_0 \cdot v_0 + T \cdot L) V_R$ with the restriction

$$\boxed{\not{z} \cdot v_0 + T \cdot W = 0}$$

MATERIAL OBJECTIVITY PRINCIPLE

is an invariance property of (1). The power will depend only on the symmetric part of the gradient velocity

The consequence of this principle

$$\ell \cdot v_0 + T \cdot W = 0 \iff \begin{cases} \ell = 0 \\ \text{skw}\,T = 0 \end{cases}$$

it comes from

$$(\text{sym}\,T + \text{skw}\,T) \cdot W = \text{skw}\,T \cdot W = 0$$

With this additional principle states finally

$$\{ \mathcal{W}(v) = 0 \quad \forall v \iff \forall v_0, \forall L \} \iff \begin{cases} f = \ell V_R = 0 \\ \text{skw}\,M = (\text{skw}\,T) V_R = 0 \\ \text{sym}\,M = (\text{sym}\,T) V_R \end{cases}$$

$$\text{i.e.} \iff \begin{cases} f = 0 \\ \text{skw}\,M = 0 \\ \text{sym}\,M = T\, V_R \end{cases}$$

$\left.\begin{matrix} f = 0 \\ \text{skw}\,M = 0 \end{matrix}\right\}$ contribute of rigid motion

symmetric part is balanced by the stress

The physical dim of these quantities:

$$\mathcal{W}^{ext} = f \cdot v_0 + M_{P_0} \cdot L$$

$$\frac{N \cdot m}{s} \qquad m N \frac{1}{s}$$

since $v(x) - v_0 = L(x - x_0)$

$$\frac{m}{s} = \alpha \cdot m \Rightarrow \alpha = \frac{1}{s}$$

$$M = (p_A - p_0) \otimes f_A$$

$$m \cdot N$$

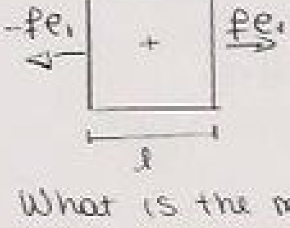
dimension of stress:

$$\frac{1}{V_R} \text{sym}\,M = T$$

$$\frac{1}{m^{\cancel{3}2}}\, m N = \frac{N}{m^2}$$ → force density per unit area

Example

Consider a face of a cube. What is the moment?



$$M = \frac{\ell}{2} e_1 \otimes (f e_1) + \left(-\frac{\ell}{2} e_1\right) \otimes (-f e_1)$$

$$= \ell f\, e_1 \otimes e_1$$

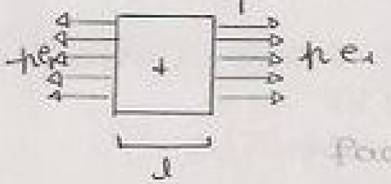
What is the meaning of $\frac{M}{V}$? $\frac{M}{V} = \frac{\ell f}{\ell^{\cancel{3}2}} e_1 \otimes e_1$

23.

$$\frac{M}{V} = \frac{f}{\underbrace{l^2}} e_1 \otimes e_1$$

area of the face where is applied f

Consider a density to [illegible] that is uniform



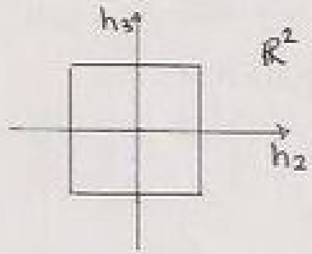
$$M = \int_{\mathcal{Y}_1} (x - p_0) \otimes (p\, e_1)\, dA$$

face no. 1 $\to \mathcal{Y}_1$

set $x = p_0 + \frac{l}{2} e_1 + h_2 e_2 + h_3 e_3$ then

$$M = \int_{\mathcal{Y}_1} \Big(\frac{l}{2} e_1 + \underbrace{h_2 e_2 + h_3 e_3} \Big) \otimes (p e_1)\, dA$$

they parametrize the face $\mathcal{Y}_1$



$$M = \int_{\mathcal{Y}_1} \frac{l}{2} e_1 \otimes (p e_1)\, dA + \underbrace{\int_{-\frac{l}{2}}^{\frac{l}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}} h_2 \underbrace{e_2 \otimes (p e_1)}_{\text{tensor}} + h_3 \underbrace{e_3 \otimes (p e_1)}_{\text{tensor}}\, dh_2\, dh_3}_{=0}$$

$$= \frac{l}{2} p\, e_1 \otimes e_1 \int_{\mathcal{Y}_1} dA$$

$$= \frac{l}{2} p\, A_{\mathcal{Y}_1} e_1 \otimes e_1$$

since we are integrating on a symm. interval its like

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} h_2\, dh_2\, dh_3 = l \frac{1}{2} \left(\frac{l^2}{4} - \frac{l^2}{4} \right)$$

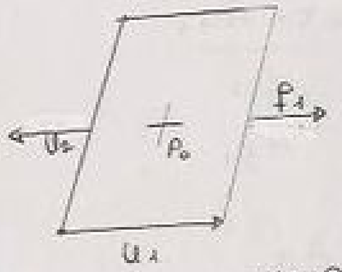
On the other face we get

$$M = l\, p\, A_{\mathcal{Y}_1} e_1 \otimes e_1 = p \underbrace{l A_{\mathcal{Y}_1}}_{\text{volume } V_R} e_1 \otimes e_1$$

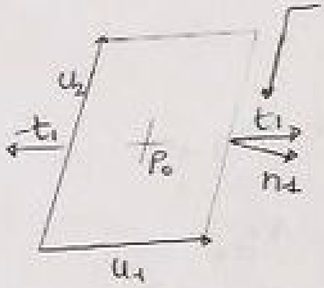
$$M = p V_R\, e_1 \otimes e_1 \implies \frac{M}{V_R} = p\, e_1 \otimes e_1$$

22.03.2011

$\{u_1, u_2, u_3\}$



$$\frac{1}{2} u_1 \otimes f_1 - \frac{1}{2} u_1 \otimes (-f_1) = u_1 \otimes f_1$$



uniform distribution of face No 1

$$\int_{y_1} (x - p_0) \otimes t_1 \, dA$$

where $\int_{y_1} dA = A_{y_1}$

then we can define a position g

$$g_{y_1} := p_0 + \frac{1}{A_{y_1}} \int_{y_1} (x - p_0) \, dA$$

that is called the CENTER OF FACE No 1. Is a property for this face.
BARICENTER

$$g_{y_1} - p_0 = \frac{1}{A_{y_1}} \int_{y_1} (x - p_0) \, dA = \frac{1}{A_{y_1}} \int_{y_1} \frac{1}{2} u_1 \, dA \; +$$

$$x = p_0 + \frac{1}{2} u_1 + h_2 u_2 + h_3 u_3$$ where h_2 and h_3 belong to a rectangular $\subset \mathbb{R}^2$

$$-\frac{1}{2} \leq h_2 \leq \frac{1}{2}$$

$$\circledast \quad \int_{y_1} h_2 \, dA = \int_{-\frac{1}{2}}^{1/2} h_2 \|u_2\| \, dA = 0$$

$$g_{y_1} - p_0 = \frac{1}{A_{y_1}} \int_{y_1} \frac{1}{2} u_1 \, dA + \overbrace{\frac{1}{A_{y_1}} \int_{y_1} (h_2 u_2 + h_3 u_3) \, dA}^{=0}$$

by $\circledast$ and extending the result to $\int h_3 \, dA = 0$ the baricenter of the body is exactly the center of the face:

$$g_{y_1} = p_0 + \frac{1}{2} u_1 \cancel{\frac{1}{A_{y_1}}} \int_{y_1} \cancel{dA} = p_0 + \frac{1}{2} u_1$$

24.

$$\int_{\mathcal{Y}_1} (x - p_0) \otimes t_1 \, dA = \int_{\mathcal{Y}_1} (x - p_0)\, dA \otimes t_1$$

$$= (g_{\mathcal{Y}_1} - p_0) \otimes \underbrace{t_1 \, A_{\mathcal{Y}_1}}_{f_1}$$

$$= \frac{1}{2} u_1 \otimes t_1 \; A_{\mathcal{Y}_1}$$

$$\int_{\mathcal{Y}_2} (x - p_0) \otimes t_2 \, dA = \frac{1}{2} u_2 \otimes t_2 \; A_{\mathcal{Y}_2}$$

Now consider

$$\int_{\mathcal{Y}_{-1}} (x - p_0) \otimes (-t_1) \, dA = -\frac{1}{2} u_1 \otimes (-t_1) A_{\mathcal{Y}_{-1}}$$

$$\overset{A_{\mathcal{Y}_{-1}} = A_{\mathcal{Y}_1}}{=} \frac{1}{2} u_1 \otimes t_1 \; A_{\mathcal{Y}_1}$$

$$M = A_{\mathcal{Y}_1} u_1 \otimes t_1 + A_{\mathcal{Y}_2} u_2 \otimes t_2 + A_{\mathcal{Y}_3} u_3 \otimes t_3.$$

$$T = \frac{M}{V_R}$$ SURFACE FORCE $\frac{M}{V_R}$

$\frac{N}{m^2}$

$n_1 :=$ unit normal to face $\mathcal{Y}_1$

$$M n_1 = A_{\mathcal{Y}_1} (u_1 \otimes t_1) n_1 + A_{\mathcal{Y}_2} (u_2 \otimes t_2) n_1 + A_{\mathcal{Y}_3} (u_3 \otimes t_3) n_1$$

since $\mathcal{Y}_1$ is generated by u_2 and u_3 $\rightarrow$ $u_2 \cdot n_1 = 0$, $u_3 \cdot n_1 = 0$

$$M n_1 = \underbrace{A_{\mathcal{Y}_1} \underbrace{(u_1 \cdot n_1)}_{h_1}}_{\substack{\text{area of } \mathcal{Y}_1 \\ \text{times the height } t_1 \\ \text{i.e. is a volume}}} t_1 + A_{\mathcal{Y}_2} \underbrace{(u_2 \cdot n_1)}_{0} t_2 + A_{\mathcal{Y}_3} \underbrace{(u_3 \cdot n_1)}_{0} t_3$$

$$M n_1 = V_R \, t_1.$$

In the end we find that $\boxed{\frac{M}{V_R} n_1 = t_1}$ Force per unit area

force distribution applied to face 1

Since $\frac{M}{V_R} = T$ we can also consider

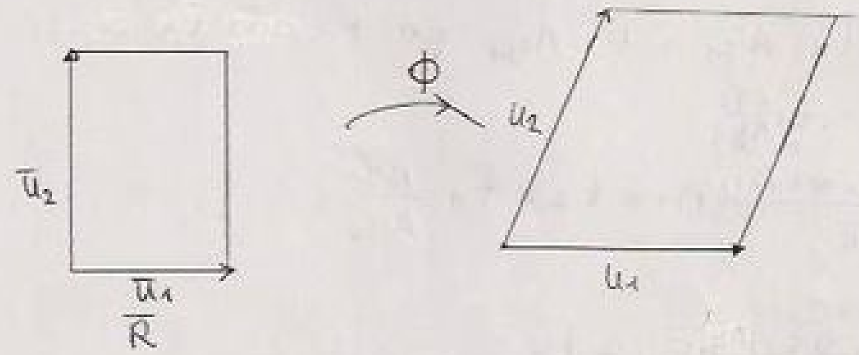
$$\boxed{T n_1 = t_1}$$

We can give a description of M giving a force distribution.
If I consider a traction on the body, a piece smaller and smaller with the same force distribution we have

$$\frac{\tilde{M}}{V_{\tilde{R}}} n_1 = \underbrace{t_1}_{L_{\Pi}}$$ will be the same

The stress T is a property related to the material of the body and depends only on $\frac{M}{V_R}$. Is independent on the shape of the body since if I consider a small parallelepiped inside the big parallelepiped the ratio M over V_R is a kind of constant.

Consider the deformation



How M change? $\phi(\bar{P}_A) = P_A + F(\bar{P}_A - \bar{P}_0)$

so $u_1 = F\bar{u}_1$

$A_{y_1} n_1 = (\mathrm{cof}\, F)\bar{n}_1 A_{\bar{y}_1}$

$$M n_1 = A_{y_1}(u_1 \otimes t_1) n_1 = (F\bar{u}_1 \otimes t_1) A_{\bar{y}_1} \mathrm{cof}\, F\, \bar{n}_1 = t_1 V_R$$

25.

$$M n_1 = t_1 V_{\bar{R}} \det F$$

$$t_1 = \left(\frac{A_{\bar{y}_1}(F\bar{u}_1 \otimes t_1)\operatorname{cof} F}{V_{\bar{R}} \det F} \right) \bar{n}_1$$

tensor

If we want to transform $T n_1 = t_1$:

$$(Au \otimes v)e = (Au \cdot e)v = (u \cdot A^T e)v = (u \otimes v)A^T e$$

we can move A outside just considering the transpose

$$(F\bar{u}_1 \otimes t_1)\operatorname{cof} F = (\bar{u}_1 \otimes t_1) F^T F^{-T} \underbrace{\det F}_{V_R}$$

then $$t_1 = \frac{A_{\bar{y}_1}(\bar{u}_1 \otimes t_1)}{V_{\bar{R}}} \bar{n}_1 = \underbrace{\left(\frac{A_{\bar{y}_1}(\bar{u}_1 \cdot \bar{n}_1)}{V_{\bar{R}}} \right)}_{\text{computed on the reference shape}} t_1 = t_1$$

if I know t_1 and the reference shape then I can compute ( ).

For $T n_1 = t_1$ we replace $n_1 = \operatorname{cof} F \, \bar{n}_1 \dfrac{A_{\bar{y}_1}}{A_{y_1}}$

$$T \operatorname{cof} F \, \bar{n}_1 \frac{A_{\bar{y}_1}}{A_{y_1}} = t_1$$

$$\underbrace{(T \operatorname{cof} F)}_{\substack{\text{PIOLA STRESS} \\ \text{PIOLA KIRCHOFF}}} \bar{n}_1 = t_1 \frac{A_{y_1}}{A_{\bar{y}_1}} = \bar{t}_1$$

$\bar{t}_1$ is s.t. $\bar{t}_1 A_{\bar{y}_1} = t_1 A_{y_1}$ i.e. $t_1 \neq \bar{t}_1$.

Then

$$\frac{\cancel{A_{\bar{y}_1}}(\bar{u}_1 \otimes t_1)}{V_{\bar{R}}} \bar{n}_1 = \underbrace{t_1}_{\bar{t}_1 \frac{A_{\bar{y}_1}}{A_{y_1}}} = \bar{t}_1 \frac{\cancel{A_{\bar{y}_1}}}{\cancel{A_{y_1}}}$$

$$\frac{A_{\bar{y}_1}(\bar{u}_1 \otimes \bar{t}_1)}{V_{\bar{R}}} \bar{n}_1 = \bar{t}_1$$

original expression of the moment evaluated on the reference shape.

$$\bar{M} = A_{\bar{y}_1} \, \bar{u}_1 \otimes \bar{t}_1$$

$$M = A_{y_1}\, u_1 \otimes t_1 = A_{y_1} F \bar{u}_1 \otimes \bar{t}_1 \frac{A_{\bar{y}_1}}{A_{y_1}}$$

$$= A_{\bar{y}_1} (\bar{u}_1 \otimes \bar{t}_1) F^T = \bar{M} F^T$$

i.e. $M = \bar{M} F^T$ or $\bar{M} = M (F^T)^{-1}$

$$\frac{\bar{M}}{V_{\bar{R}}} = \frac{M F^{-T}}{V_{\bar{R}}} = \frac{M F^{-T}}{V_R} \det F = \frac{M}{V_R} \operatorname{cof} F$$

$T n_1 = t_1$
True stress

$(T \operatorname{cof} F) \bar{n}_1 = \bar{t}_1$.
Nominal stress

$\mathcal{W}^{ext}(v) = f \cdot v_0 + M \cdot L$ velocity gradient

$\mathcal{W}^{int}(v) = -(z \cdot v_0 + T \cdot L) V_R$

In general we have to consider

$$\boxed{\begin{aligned} \mathcal{W}^{ext}(v) &= \int_R b \cdot v \, dV + \int_{\partial R} t \cdot v \, dA \\ \mathcal{W}^{int}(v) &= -\int_R (z \cdot v + T \cdot \nabla v) \, dV \end{aligned}}$$

By the balance principle $\mathcal{W}^{ext}(v) + \mathcal{W}^{int}(v) = 0 \quad \forall v$

we know that $\operatorname{div} v = \operatorname{tr}(\nabla v)$, what is the divergence of T? The $\operatorname{div} T$ is s.t.

$$\operatorname{div} T \cdot e = \operatorname{div}(T^T e)$$

$$\boxed{\operatorname{div} T \cdot v = \operatorname{div}(T^T v) - T \cdot \nabla v}$$

$\Rightarrow T \cdot \nabla v = \operatorname{div}(T^T v) - \operatorname{div} T \cdot v$

$$\mathcal{W}^{int}(v) = -\int_R z v \, dV + \int_R \operatorname{div} T \cdot v \, dV - \int_R \operatorname{div}(T^T v) \, dV$$

26.

By the Divergence theorem

$$\int_{\partial R} \underset{T^T n}{v \cdot n}\, dA = \int_R \underset{T^T n}{\operatorname{div} v}\, dV$$

thus

$$\mathcal{W}^{int}_{(v)} = -\int_R z \cdot v\, dV + \int_R \operatorname{div} T \cdot v\, dV - \int_{\partial R} Tn \cdot v\, dA$$

a necessary consequence

$$\begin{cases} \int_R (b - z + \operatorname{div} T) \cdot v\, dV = 0 \\ \int_{\partial R} (t - Tn) \cdot v\, dA = 0 \end{cases} \qquad \forall v$$

i.e.

$$\begin{cases} b - z + \operatorname{div} T = 0 \\ t - Tn = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} b - z + \operatorname{div} T = 0 \\ t = Tn \end{array}}$$

CAUCHY
CONTINUUM OF
EQUATIONS

23.03.2011

$$W^{ext}(v) = \int_R \underset{\text{BULK FORCE (or DENSITY)}}{b} \cdot v \, dV + \int_{\partial R} \underset{\text{TRACTION}}{t} \cdot v \, dA$$

$$W^{int}(v) = -\int_R (z \cdot v + T \cdot L) dV \quad \otimes$$

$W(v) = 0 \quad \forall v$ <u>balance principle</u>

$$\begin{cases} b - z + \operatorname{div} T = 0 & R \\ t = Tn & \partial R \end{cases}$$

If I neglect $T \cdot \nabla v$ in $\otimes$ then I'll have the trivial case $\begin{cases} b = z & R \\ t = 0 & \partial R \end{cases}$

i.e. $z = 0 \Rightarrow \begin{cases} b = 0 \\ t = 0 \end{cases}$ so the forces will be 0

since z cannot be $\neq 0$

$z \cdot v + T \cdot \nabla v = 0$ (if the velocity field is rigid $\Rightarrow$ sym $\nabla v = 0$ since is skew symmetric

For rigid deformation

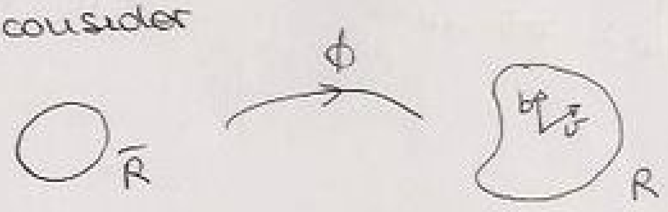
$z \cdot v + T \cdot W = 0 \quad \forall v, \forall W$ MATERIAL OBJECTIVITY

$\Rightarrow$ $z = 0$

skw $= 0$

$\Rightarrow W^{int}(v) = -\int_R T \cdot \nabla v \, dV$ reduced form of the internal power.

Let's consider



then $\int_R \underbrace{(b \cdot v)}_{\text{scalar field}} dV = \int_{\bar{R}} \tilde{b} \cdot \bar{v} \det F \, dV = \int_{\bar{R}} \underbrace{(\tilde{b} \det F)}_{\text{DENSITY WRT THE VOLUME OF THE REFERENCE SHAPE}} \cdot \bar{v} \, dV$

27

$$\boxed{\int_R b\cdot v\, dV = \int_{\bar R} \bar b\cdot \bar v\, dV} \qquad \bar b = \tilde b \det F$$

$$\boxed{\int_R T\cdot \nabla v\, dV - \int_{\bar R} S\cdot \nabla \bar v\, dV}$$

PROOF

$$T\cdot \nabla v = T\cdot \nabla \bar v F^{-1} = T F^{-T}\cdot \nabla \bar v$$

$$\int_R T\cdot \nabla v\, dV = \int_{\bar R} T F^{-T}\cdot \nabla \bar v \det F\, dV$$

but $S = T F^{-T} \det F$

$$= \int_{\bar R} S\cdot \nabla \bar v\, dV$$

$$\boxed{\int_{\partial R} t\cdot v\, dA = \int_{\partial \bar R} \bar t\cdot \bar v\, dA} \qquad \boxed{\bar t = \| \operatorname{cof} F\, \bar n \| \tilde t}$$

PROOF

Remind that

$$\| \operatorname{cof} F\, \bar n_1 \| = \left| \frac{A y_1}{A \bar y_1} \right|$$

$$\Rightarrow W^{ext}(v) = \int_{\bar R} \bar b\cdot \bar v\, dV + \int_{\partial \bar R} t\cdot \bar v\, dA$$

$$W^{int}(v) = -\int_R S\cdot \nabla \bar v\, dV \qquad \forall v \text{ velocity test}$$

HOMEWORKS amabile.tatone@univaq.it 28.03.2011

#1 1. Draw parallelograms only by drawing describe deformation

· describe deformation by deform. gradient

$[F] = \begin{pmatrix} \ \end{pmatrix}$

2. $F = RU$ ← Compute polar decomposition

3. Describe how R and U define two different deform. Illustration (drawing) of the RU decomposition

Extend the def in $\mathbb{R}^3$ adding $[F] = \begin{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Fe_1
Fe_2
$Fe_3 = e_3$

PLANE DEFORMATION

More general case $[F] = \begin{pmatrix} & & 0 \\ & & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$ $Fe_3 = \lambda_3 e_3$

$f_{33} = \lambda_3$

· Using the cofactor compute the unit normal vector to y_1 and y_{-1}

· $\dfrac{A y_1}{A \bar{y}_1}$ = ___ · $\dfrac{V_R}{V_{\bar{R}}}$

#2

1. Describe ⌐ same of hw 1

2. compute the tensor BULK moment and the

$$\int_R b \cdot v \, dV = \int_R b \, dV \cdot v_0 + \int (x - p_0) \otimes b \, dV \cdot L$$

28.

total force wrt the density and the volume

3. Check that force in pt 2 does not satisfy balance eq.

$$\begin{cases} f = 0 \\ \mathrm{skw}\, M = 0 \\ \mathrm{sym}\, M = T V_R \end{cases}$$

find a distribution on the boundary s.t.

$$\begin{cases} f = 0 \\ \mathrm{skw}\, M = 0 \end{cases}$$ are satisfied

since $\mathrm{sym}\, M = T V_R$ is satisfied by T.

Which forces could in principle balanced by $\mathrm{sym}\, M = T V_R$

4. Compute

$\frac{1}{V_R} M$ on the original shape i.e. find $\frac{\overline{M}}{\overline{V_R}}$

5. Compare the values of

$\frac{\overline{M}}{\overline{V_R}} \overline{n_1}$ and $\frac{M}{V_R} n_1$

We've given a formula for $\mathcal{W}^{ext}(v)$ and $\mathcal{W}^{int}(v)$

$$\mathcal{W}^{ext}(v) = \int_R \underset{\text{BULK}}{b} \cdot v \, dV + \int_{\partial R} \underset{\text{TRACTION}}{t} \cdot v \, dA$$

$$\mathcal{W}^{int}(v) = -\int_R (\not{f} \cdot v + T \cdot \nabla v) \, dV$$

Then the Balance principle says $\mathcal{W}(v) = 0 \quad \forall v$.

We have another principle

$$\not{f} \cdot v + T \cdot \nabla v = 0 \qquad \forall x \in R$$

$$\not{f} \cdot v_0 + T \cdot W = 0 \qquad \text{AFFINE BODIES}$$

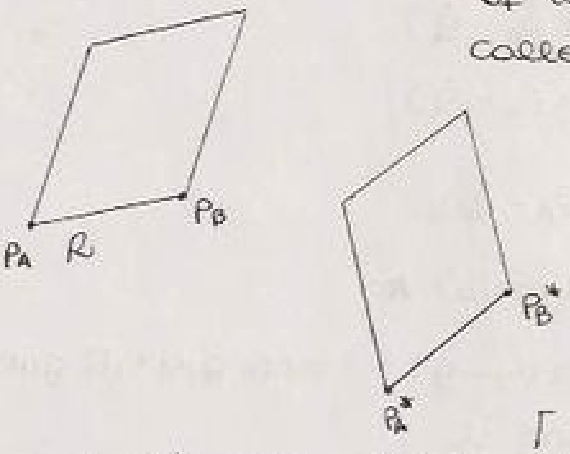
$$\mathbf{\not{z}} \cdot v_0 + T \cdot W = 0 \quad \Rightarrow \begin{cases} \not{z} = 0 \\ \text{Skw}\, T = 0 \end{cases}$$

If we consider affine bodies and the power

$$\not{z} \cdot v_0 + T \cdot L$$

If we consider what is called SUPERPOSED RIGID MOTION

Everything is moving



q, q^*, Q given

where $P_A^* = q^* + Q(P_A - q)$ and is the definition of the CHANGE OF OBSERVER (OR CHANGE OF FRAMING)

$$Q^T Q = I$$

C of Obs is another way to express a superposed rigid motion

C of Obs is used to state that: the stress is defined by two different quantities

$$\boxed{\not{z}^* \cdot v_0^* + T^* \cdot L^* = \not{z} \cdot v_0 + T \cdot L}$$

the power does not change if I superposed the motion

Invariance conditions on the group of rigid motions

in algebraic sense

should hold for any v_0, L and any change of obs q, q^*, Q

The conclusion is that

$$\boxed{\begin{cases} \not{z} = 0 \\ \text{Skw}\, T = 0 \end{cases}}$$

i.e. by 2

$$\begin{cases} \not{z}^* = Q \not{z} \\ T^* = Q\, T Q^T \end{cases}$$

we get our conclusion

29

$T^* = Q^T T Q$ relates the stress calculated by the 1° observer and the stress evaluated by the 2° observer

We try to prove ■:
we need the velocity so

$\dot{p}_A^* = \dot{q}^* + \dot{Q}(p_A - q) + Q(\dot{p}_A - \dot{q})$

$\dot{p}_o^* = \dot{q}^* + \dot{Q}(p_o - q) + Q(\dot{p}_o - \dot{q})$

$\dot{p}_A^* - \dot{p}_o^* = \dot{Q}(p_A - p_o) + Q(\dot{p}_A - \dot{p}_o)$

$v_A^* - v_o^* = \dot{Q}(p_A - p_o) + Q(v_A - v_o)$ ☆

$v_o^* = \dot{q}^* + \dot{Q}(p_o - q) + Q(v_o - \dot{q})$ since q, q^*, Q given

└ a way to relate v_o^* and v_o

by ☆
$L^*(p_A^* - p_o^*) = \dot{Q}(p_A - p_o) + QL(p_A - p_o)$
where we define $L^* Q(p_A - p_o) = (\dot{Q} + QL)(p_A - p_o)$

$\Rightarrow L^* Q = \dot{Q} + QL$

$$\boxed{L^* = \underbrace{\dot{Q}Q^T}_{} + QLQ^T}$$ velocity gradient

skew symm Tensor since it looks like $\dot{R}R$ the spin tensor.

$$\boxed{v_o^* = \dot{q}^* + \dot{Q}(p_o - q) - Q\dot{q} + Qv_o}$$ Since it is true $(\forall q, \forall q^*, \forall Q)$

We consider the special case when $Q = I$
$\dot{q} = 0$, $\dot{q}^* = 0$:

$v_o^* = Qv_o \Rightarrow \underset{\underset{v_o}{\parallel}}{\not{z}^* \cdot Qv_o} + T^* \cdot \underset{\underset{L}{\parallel}}{L^*} = \not{z} \cdot v_o + T \cdot L$

For this trivial choice $\not{z}^* = \not{z}$
$T^* = T$

We should choose another combination

Let us try if $Q = Q_0$ does not depend on time

$\dot{q} = 0, \ \dot{q}^* = 0$

$v_0^* = Q_0 v_0$

$$b^* \cdot Q_0 v_0 + T^* \cdot (Q_0 L Q_0^T) = b \cdot v_0 + T \cdot L$$

$$(Q_0^T b^* - b) \cdot v_0 + (Q_0^T T^* Q_0 - T) \cdot L = 0$$

$$\Rightarrow \quad b^* = Q_0 b$$

$$T^* = Q_0 T Q_0^T$$

29.03.2011

$$b^* \cdot (\dot{q}^* - Q\dot{q} + Q v_0 + \dot{Q}(p_0 - q)) - b \cdot v_0$$

$$+ T^* (\dot{Q}Q^T + QLQ^T) - T \cdot L = 0$$

multiplying for $QQ^T = I$

$$Q^T b^* \cdot (Q^T \dot{q}^* - \dot{q} + v_0 + Q^T \dot{Q}(p_0 - q)) - b v_0$$

reminding $A \cdot QB = tr(A^T QB) = Q^T A \cdot B$

$A \cdot QB = tr(A(QB)^T) = tr(AB^T Q^T) = AB^T \cdot Q$

$$+ Q^T T^* Q \cdot (Q^T \dot{Q} + L) - T \cdot L = 0$$

$$(Q^T b^* - b) \cdot v_0 + Q^T b^* \cdot ((Q^T q^* - \dot{q}) + Q^T \dot{Q}(p_0 - q))$$

$$+ Q^T T^* Q \cdot Q^T \dot{Q} + (Q^T T^* Q - T) \cdot L = 0$$

Since shoued be true for any v_0, $\forall L$ then

$$\begin{cases} Q^T b^* - b = 0 \\ Q^T T^* Q - T = 0 \end{cases}$$

$$Q^T b^* \cdot ((Q^T q^* - \dot{q}) + Q^T \dot{Q}(p_0 - q)) + Q^T T^* Q \cdot Q^T \dot{Q} = 0$$

$$\Rightarrow \quad \begin{matrix} Q^T b^* = 0 \\ skw\, Q^T T^* Q = 0 \end{matrix} \quad \Rightarrow \quad \begin{cases} b^* = 0 \\ skw\, Q^T T^* Q = 0 \end{cases}$$

31.

Appeying the def of skew symm

$$\frac{1}{2}(Q^T T^* Q - Q^T (T^*)^T Q) = 0$$

$$\frac{1}{2} Q^T (T^* - (T^*)^T) Q = 0$$

$$Q^T \, \mathrm{skw} T^* \, Q = 0 \quad \Rightarrow \quad \mathrm{skw} T^* = 0$$

i.e. is indep on Q

$$\begin{cases} \mathbf{t}^* = 0 \\ \mathrm{skw} T^* = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{t} = 0 \\ \mathrm{skw} T = 0 \end{cases}$$

This principle selects the stress in this way

$$\boxed{\begin{matrix} \mathbf{t} = 0 \\ \mathrm{skw} T = 0 \end{matrix}}$$

in this course we'll not consider any material with $\mathbf{t} \neq 0$. It is correct since

$$\begin{cases} f - \mathbf{t} V_R = 0 \\ M - \underset{\substack{\downarrow \\ \text{TENSIONE} \\ \text{(STRESS)}}}{T} V_R = 0 \end{cases} \Rightarrow \begin{cases} f = 0 \\ \mathrm{skw} M = 0 \\ \mathrm{sym} M = T V_R \end{cases}$$

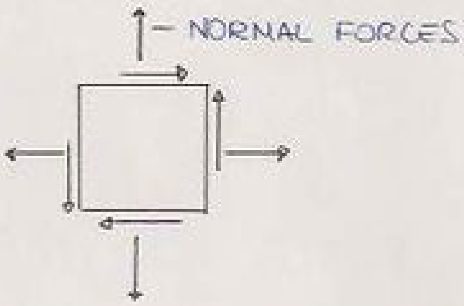
conditions about the applied forces

for any force applied by outside the body is not able to balance that force. The same for the moment.

Stress T: something describing couples applied by the outside balancing couples in the inside

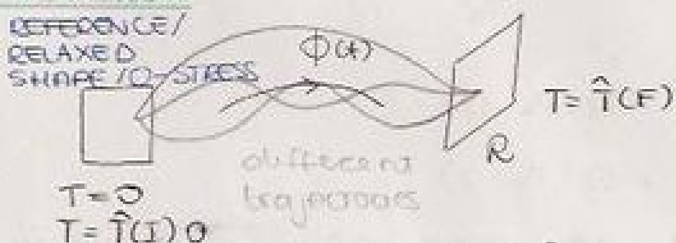
$\text{sym}\, M = T\, V_R$

means that the symmetric moment could be balanced by a stress



THE RESPONSE FUNCTION

$T = \underbrace{\hat{T}(F)}$
ELASTIC MATERIAL



F^t collection of different trajectories of the deform gradient

$T = \underbrace{\hat{T}(F^t)}$ but it depends only on the current shape
MATERIAL
for this reason we drop t and it will be elastic.

We consider only elastic material $T = \hat{T}(F)$.

Since $M = \overline{M} F^T$ then $\overline{M} F^T = \hat{T}(F)$

$\hat{T}$ depends on this function and is the true characterization of the material

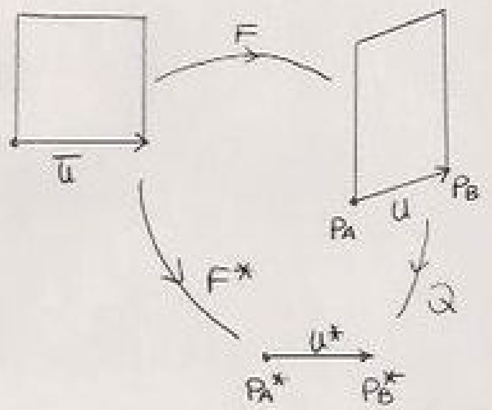
$\hat{T} : F \longrightarrow T$

$\text{Lin}(V, V) \longrightarrow \text{Sym}$

space of tensors

space of symmetric tensors

31



$$P_A^* = q^* + Q(P_A - q)$$

$$P_B^* = q^* + Q(P_B - q)$$

$$u^* = P_B^* - P_A^* = Q(P_B - P_A) = Qu = \underbrace{QF}_{F^*}\,\bar{u}$$

$$\boxed{F^* = QF}$$

$$T^* = QTQ^T$$

$T = \hat{T}(F)$; if the second observer use the same function

$$T^* = \hat{T}(F^*)$$

$$\boxed{\hat{T}(F^*) = Q\hat{T}(F)Q^T}$$

from different

$$\hat{T}(QF) = Q\hat{T}(F)Q^T \quad \forall Q, \forall F$$

rotation of polar decomp.

then it is true for a special value of $Q = R^T$

$$\hat{T}(R^T F) = R^T\hat{T}(F)R$$

$$\hat{T}(\underbrace{R^T R}_{I}U) = R^T\hat{T}(F)R$$

$$\boxed{R\hat{T}(U)R^T = \hat{T}(F)}$$

REDUCED FORM OF THE RESPONSE FUNCTION $\hat{T}$

Then

$$\boxed{\hat{T}(QF) = Q\hat{T}(F)Q^T \iff R\hat{T}(U)R^T = \hat{T}(F)}$$

$\Rightarrow$) proved

$\Leftarrow$) $\quad QF = \underbrace{QR}_{\text{ROTATIONS}}\underbrace{U}_{\text{STRETCH}}$

$$\hat{T}(QF) = QR\,\hat{T}(U)(QR)^T = Q\,R\hat{T}(U)R^T\,Q^T$$
$$= Q\,\hat{T}(F)\,Q^T$$

- The value of $\hat{T}$ corresponding to F depends just on U. Different functions get different values of $\hat{T}(F)$ and $\hat{T}(U)$ since R and U are the same.

If we want to perform an experiment to check the material, how do we evaluate the stress?

$$\frac{M}{V_R} = \hat{T}(F)$$

What are the interesting values of F? $\to \hat{T}(U)$

How can we get Piola stress?

$$\boxed{S = T(\operatorname{cof} F) = \hat{T}(F)\operatorname{cof} F = \hat{S}(F)}$$

$$T = \hat{T}(F)$$

Responsive function for the Piola stress $S = \hat{S}(F)$.

Consider the power of the stress

$$T \cdot L = SF^T \frac{1}{\det F} \cdot L$$

$$S = TF^{-}\det F \Rightarrow T = SF^T \frac{1}{\det F}$$

$$T \cdot L = \frac{1}{\det F} S \cdot LF$$

$$(T \cdot L)V_R = \frac{V_R}{\det F} S \cdot LF = V_{\bar{R}}\, S \cdot LF$$

32

What is LF? $L = \nabla v$

in the ref. shape $\nabla \bar{v}$ and $(\nabla \bar{v})\bar{u} = (\nabla v)F\bar{u}$ $\quad \bar{L} = LF$

$$(T \cdot L)V_R = S \cdot \underbrace{LF}_{\bar{L}} V_{\bar{R}}$$

$$\int_R T \cdot \nabla v \, dV = \int_{\bar{R}} S \cdot \nabla \bar{v} \, dV$$

$$\nabla v = \dot{F}F^{-1} \Rightarrow L = \dot{F}F^{-1}$$

$$(T \cdot \dot{F}F^{-1})V_R = S \cdot \dot{F}F^{-1}F V_{\bar{R}} = (S\dot{F})V_{\bar{R}}$$

density for unit current volume — density for unit reference volume.

is useful for the response of the material.

Note $\boxed{S \cdot \dot{F} = \frac{d}{dt}\varphi(t)}$ ELASTIC ENERGY (a potential)

If there is a potential, the material is said to be hyperelastic

30.03.2011

$$(\underset{\hat{T}(F)}{T} \cdot \dot{F}F^{-1})V_R = (\underset{\hat{S}(F)}{S} \cdot \dot{F})V_{\bar{R}}$$

there is a funct whose time derivative is $\hat{S} \cdot \dot{F}$?

$$\hat{S}(F) \cdot \dot{F} = \frac{d}{dt}\varphi(F)$$

STORED ENERGY FUNCTION

If this function $\exists \Rightarrow \varphi(F)$ is the POTENTIAL function of the stress called ELASTIC ENERGY or STRAIN ENERGY. In continuum mechanics we don't call F deform gradient but strain.

$$F^TF - I$$

$(U^2 - I) \rightarrow$ this function measure how much U^2 is close to the identity

Usual hyperelastic materials are not so special.
Thus the elastic energy depends on the response function

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(F)$$

is still density, per unit volume.
wrt reference shape

By the CAUCHY CONTINUUM

$$\mathcal{W}^{ext}(v) = \int_R b\cdot v\,dV + \int_{\partial R} t\cdot v\,dA - \underbrace{\int_R (z\cdot v + T\cdot\nabla v)\,dV}$$

$$\begin{cases} z = 0 \\ \text{skw}\,T = 0 \end{cases} \leftarrow$$

when we assume that it is frame different we get the same result for affine deform.

and by the balance eq:

$$\begin{cases} \operatorname{div} T + b = 0 \\ Tn = t \end{cases}$$

Also in this case we can consider $T = \hat{T}(F)$ with the characterization $\hat{T}(F) = R\,\hat{T}(U)\,R^T$.

Then

$$\int_R (T\cdot\dot{F}F^{-1})\,dV = \int_{\bar{R}} (S\cdot\dot{F})\,dV$$

this density is not uniform in general. Also in this case we can define a potential energy an elastic energy.

Why comparing T^* and T?
Because we've found that F^* and F were related by $F^* = QF$. Have a look to the energy φ, which is the relation with φ^*? Note that the reference shape is the same, is the observer that it is going to change

$$\varphi^*(F^*) = \varphi(F)$$
$$\forall\bar{R} \qquad \forall\bar{R}$$

77

About energy we should assume that we use the same function energy. How can we compare $\varphi(F^*)$ and $\varphi(F)$? ($\varphi^*(\cdot) = \varphi(\cdot)$ it holds since in the principle there is an equality between powers)

$$\boxed{\varphi(F^*) = \varphi(F)}$$

FRAME INVARIANCE PROPERTY

then for any Q it has to be true

$$\varphi(QF) = \varphi(F) \quad \forall Q, \forall F$$

we can derive a consequence: if we consider $\boxed{Q = R^T}$ then

$$\varphi(R^TF) = \varphi(F) \underset{F=RU}{\Rightarrow} \boxed{\varphi(U) = \varphi(F)}$$

on the other way ($\varphi(U) = \varphi(F) \Rightarrow \varphi(F^*) = \varphi(F)$)

$$\varphi(F^*) = \varphi(U^*)$$

$$F^* = R^* U^* \quad \Rightarrow (U^*)^2 = F^{*T}F^*$$

$$F = RU \quad \Rightarrow U^2 = F^TF$$

$$(U^*)^2 = F^TQ^TQF = F^TF$$

$F^* = QF$

the Cauchy-Green tensor is the same

the stretch does not change in the change of observer.

Thus

$$\varphi(F^*) = \varphi(U^*) = \varphi(U) = \varphi(F)$$

$$\Rightarrow \varphi(F^*) = \varphi(F)$$

Is an important result since it is saying that the strain energy depends only on the stretch.

is defined on this symmetric, def positive tensor

So it is reasonable to write $\varphi(U) = \hat{\varphi}(\overset{=U^2}{C})$

We can replace

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(U)$$

even if is useful to leave $S(F)\cdot\dot{F} = \frac{d}{dt}\varphi(F)$ to underline the dependence on F.

Now we have to characterize the elastic material i.e. when we use the elastic energy, a material is not just elastic but hyperelastic.

Some special properties:

$$\tilde{Q}\ \begin{cases}\hat{T}(F) = R\,\hat{T}(U)R^T\\ \varphi(F) = \varphi(U)\end{cases}$$

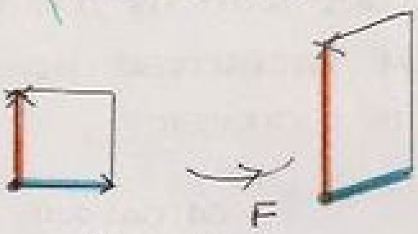
now we rotate this ref

If I overlap this two then the material parts are different

But we can compare the energy

Since $\varphi(\cdot)$ depends on U it can happen that the stretch can be ≠

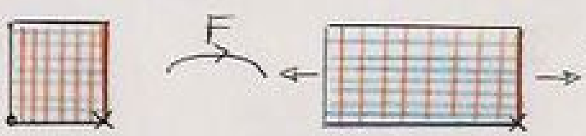
→ It can be that the energies are ≠



$$\hat{T}(F) \neq \hat{T}(F\tilde{Q})$$

$$\varphi(F) \neq \varphi(F\tilde{Q}).$$

Let us consider another case (equal but easier)



blue lines are stretched

$\tilde{Q}$ note that the rotation does not affect the structure of the material

orange lines are stretched

Comparing the stress

$$T = \frac{M}{V} \qquad T^* = \frac{M}{V} \qquad \text{then } T \neq T^*$$

$$\Rightarrow \hat{T}(F) \neq \hat{T}(F\tilde{Q})$$
$$\varphi(F) \neq \varphi(F\tilde{Q})$$

If I consider the case where they have all the same stretches (i.e. all lines are blue) then the images will be same and

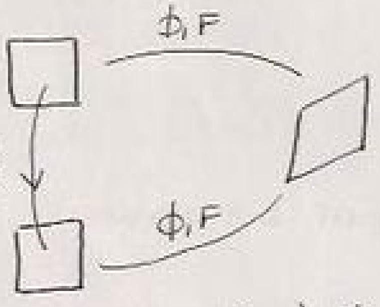
$$T = T^*$$
$$\hat{T}(F) = \hat{T}(F\tilde{Q}) \Rightarrow \tilde{Q} \text{ leaves the response unchanged}$$
$$\varphi(F) = \varphi(F\tilde{Q})$$

$\tilde{Q} \in$ SYMMETRY GROUP OF THE MATERIA

$$\tilde{Q}^T\tilde{Q} = I$$

If this symmetry group coincides with the rotation group $\Rightarrow$ $\hat{T}(F) = \hat{T}(F\tilde{Q})$ is left unchanged by the rotations i.e. is ISOTROPIC

04.04.2011



$$\rightarrow \hat{T}(F) = \hat{T}(F\tilde{Q})$$
$$\hat{T}(I) = 0$$

SYMMETRY GROUP $\mathcal{G} = \{\tilde{Q}_1, \tilde{Q}_2, I\}$

when $\mathcal{G} \equiv Orth^+$ ISOTROPY

$$\varphi(F) = \varphi(F\tilde{Q})$$

Let us consider the case where 04.04.2011

$$\varphi(F) = \varphi(F\tilde{Q}) \quad \forall \tilde{Q} \text{ is an isotropic material}$$

$\tilde{Q} = R^T$

$$\varphi(F) = \varphi(FR^T)$$

$$\varphi(U) = \varphi(RUR^T) \quad \forall R \qquad \text{by } \varphi(F) = \varphi(U)$$

using the spectral decomposition of U

$$RUR^T = R(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3)R^T$$

$$= \lambda_1 RP_1R^T + \lambda_2 RP_2R^T + \lambda_3 RP_3R^T$$

Remember that P_i is the projector and that

$$P_1 u \in \ker(U - \lambda_1 I)$$

$$U a_1 = \lambda_1 a_1$$

$$RU a_1 = \lambda_1 R a_1$$

$$(RUR^T)(Ra_1) = \lambda_1 R a_1$$

i.e. Ra_1 is an eigenvector wrt the tensor RUR^T

Then $RP_1R^T + RP_2R^T + RP_3R^T = R(P_1+P_2+P_3)R^T$

$$= RR^T = I$$

and $RP_1R^T RP_2R^T = R \underbrace{P_1P_2}_{0} R^T = 0$

In general

$$\varphi(F) = \varphi(U) = \hat{\varphi}(\lambda_1, \lambda_2, \lambda_3)$$

if the material is isotropic

It's interesting to observe U^2 instead of U

OGDEN

$$\varphi(F) = \underbrace{\varphi(U)}_{\text{by frame indifference}} = \hat{\varphi}(U^2) = \hat{\varphi}(C) = \underbrace{\check{\varphi}(\lambda_1^2, \lambda_2^2, \lambda_3^2)}_{\text{isotropic}}$$

35

Then $p(\eta)$ of C

$$p(\eta) = \eta^3 - \iota_1\eta^2 - \iota_2\eta - \iota_3 = 0$$

is indep on the basis since

$$\det(C - \eta I) = 0$$

$$\text{vol}\,((C-\eta I)e_1,\ (C-\eta I)e_2,\ (\quad)e_3)$$

$$\boxed{\begin{cases} \iota_3 = \det C \\ \iota_1 = \operatorname{tr} C \\ \iota_2 = \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right) \end{cases}}$$

PRINCIPAL INVARIANTS

thus $\hat{\varphi}(C) = \tilde{\varphi}(\iota_1, \iota_2, \iota_3)$

Now consider

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(\iota_1, \iota_2, \iota_3)$$

Using the RIVLIN FORMULA

$$\dot{\varphi}(\iota_1, \iota_2, \iota_3) = \frac{\partial\varphi}{\partial\iota_1}\dot{\iota}_1 + \frac{\partial\varphi}{\partial\iota_2}\dot{\iota}_2 + \frac{\partial\varphi}{\partial\iota_3}\dot{\iota}_3$$

What we would do do is to get an expression for the response function as $(\ )\cdot\dot{F}$

$$\dot{\iota}_1 = \frac{d}{dt}\iota_1 = \frac{d}{dt}\operatorname{tr} C = \frac{d}{dt}(F\cdot F) = \dot{F}\cdot F + F\cdot\dot{F}$$

$$\iota_1 = \operatorname{tr} C = \operatorname{tr} F^T F = F\cdot F$$

remember that

$$A\cdot B = \operatorname{tr}(A^T B)$$

$$\dot{\iota}_1 = 2F\cdot\dot{F}$$

$$\dot{\iota}_3 = \frac{d}{dt}\iota_3 = \frac{d}{dt}$$

$$\iota_3 = \det C = \frac{\text{vol}(Ce_1, Ce_2, Ce_3)}{\text{vol}(e_1, e_2, e_3)}$$

$$\frac{d}{dt}\det C = \frac{\text{vol}(\dot{C}e_1, Ce_2, Ce_3) + \text{vol}(Ce_1, \dot{C}e_2, Ce_3) + \text{vol}(\ ,\ , \dot{C}e_3)}{\text{vol}(e_1, e_2, e_3)}$$

NON LINEAR ELASTIC DEFORM OGDEN

$$\frac{\mathrm{vol}\,((\dot{C}C^{-1})Ce_1, Ce_2, Ce_3)}{\mathrm{vol}\,(Ce_1, Ce_2, Ce_3)} \quad \det C$$

since $\mathrm{vol}\,(Ce_1, Ce_2, Ce_3) = \det C \,\mathrm{vol}\,(e_1, e_2, e_3)$

CONTINUUM MECHANICS

$$\Rightarrow \boxed{\frac{d}{dt}\det C = \mathrm{tr}\,(\dot{C}C^{-1})\det C}$$

$$\frac{d}{dt} L_3 = L_3\,\mathrm{tr}(\dot{C}C^{-1}) = L_3\,(C^{-1}\cdot\dot{C}) \quad *$$

using the prevous formula for $\det(F)$

$$\boxed{\begin{aligned}\frac{d}{dt}(\det F) = \mathrm{tr}\,(\dot{F}F^{-1})\det F &= \mathrm{tr}\,\nabla v \det F\\ &= \mathrm{div}\, v \det F\end{aligned}}$$

If the volume does not change in time $\rightarrow$ incompressible in time what can we say?

$$\frac{d}{dt}(\det F) = 0 \quad \Rightarrow \boxed{\mathrm{div}\, v \cdot 1 = 0}$$

$$\frac{V_t}{V_R}$$

$$* \quad \dot{C} = \frac{d}{dt}(F^TF) = \dot{F}^TF + F^T\dot{F}$$

$$C = F^TF \Rightarrow C^{-1} = F^{-1}F^{-T}$$

$$C^{-1}\cdot\dot{C} = F^{-1}F^{-T}\cdot\dot{F}^TF + F^{-1}F^{-T}\cdot F^T\dot{F}$$

$$\overset{!}{=} F^{-1}\underbrace{F^{-T}F^T}_{=I}\cdot\dot{F}^T + F^{-T}\dot{F}$$

$$= F^{-1}\dot{F}^T + F^{-T}\dot{F} = 2F^{-T}\cdot\dot{F}$$

$$A\cdot B = B^TA\cdot I = B^T\cdot A^T$$

$$\frac{d}{dt}L_3 = 2L_3\,F^{-T}\cdot\dot{F}$$

$$\det C = (\det F)^2.$$

It means that we can write

$$\hat{S}(F)\cdot\dot{F} = \underbrace{(\qquad)}_{\frac{d}{dt}\varphi(L_1, L_2, L_3)}\cdot\dot{F}$$

36

Calling

$$\hat{S}(F) = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

then we can say $S_{11} = \dfrac{\partial \varphi}{\partial f_{11}}$

in general

$$\boxed{S_{ij} = \frac{\partial \varphi}{\partial f_{ij}}}$$

5/04/2011

INERTIAL FORCES

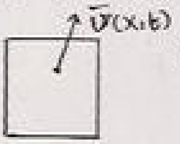
$$\int_R b \cdot v \, dV = \int_{\bar{R}} \bar{b} \cdot \underbrace{\bar{v}}_{\text{test velocity field}} \, dV$$

$x \in R$

$$\bar{v}(x) = \dot{\phi}(x,t)$$

$$\bar{b} \cdot \dot{\phi} = \frac{d}{dt} \, \underline{\varphi}(x,t)$$ this velocity field has a potential described by the function φ

Let us consider the kinetic energy velocity of a fixed domain



$$\frac{1}{2} \rho \, \bar{v} \cdot \bar{v}$$

$$\bar{v}(x,t)$$

$$\frac{d}{dt} \frac{1}{2} \rho \, \bar{v} \cdot \bar{v} = \frac{1}{2} \rho \left(\dot{\bar{v}} \cdot \bar{v} + \bar{v} \cdot \dot{\bar{v}} \right) = \rho \, \dot{\bar{v}} \cdot \bar{v}$$

From this energy we found the power

So $\varphi(x) = \frac{1}{2} \rho(x) \, \bar{v}(x,t) \cdot \bar{v}(x,t)$

if $-\varphi(x) = \frac{1}{2} \rho(x) \, \bar{v}(x,t) \cdot \bar{v}(x,t)$

$$-\frac{d}{dt} \frac{1}{2} \rho \, \bar{v} \cdot \bar{v} = \underbrace{-\rho \dot{\bar{v}}} \cdot \bar{v}$$

$$\boxed{\bar{b} = -\rho \dot{\bar{v}} = -\rho \ddot{\phi} = -\rho \bar{a}}$$

INERTIAL FORCE (DENSITY)

vector field of the ref. shape

Then bulk force - force per unit volume

$$\boxed{\begin{array}{l} b + \operatorname{div} T = 0 \\ Tn = t \end{array}}$$

since

$$\bar{b} = b \det F$$

$$\Rightarrow b = \frac{\bar{b}}{\det F} = -\frac{\rho \bar{a}}{\det F} \Rightarrow \boxed{-\frac{\rho}{\det F}\bar{a} + \operatorname{div} T = 0}$$

Cauchy equation of motion

$$\bar{a}(x,t) = \dot{\bar{v}}(x,t)$$

$$\bar{v}(x,t) = v(\overbrace{\phi(x,t)}^{\mathcal{E}}, t)$$

Referential description   spatial description

$$\bar{a} = (\nabla v)v + \frac{d}{dt}v$$

Thus

$$\bar{b} = -\frac{\rho}{\det F}\bar{a} = -\frac{\rho}{\det F}\left[(\nabla v)v + \frac{d}{dt}v\right]$$

$$\boxed{-\frac{\rho}{\det F}\left((\nabla v)v + v'\right) + \operatorname{div} T = 0}$$

INCOMPRESSIBILITY

this is the basis of getting the eq. of motion of a fluid.

While we are considering a fluid, $\boxed{\det F = 1}$

i.e. is an incompressible material. the motion is called ISOCHORIC

Now we give a characterization for the stress:

$$\nabla v = \dot{F}F^{-1}$$

$$\det F = 1 \rightarrow \frac{d}{dt}\det F = 0$$

$$\operatorname{tr}(\dot{F}F^{-1})\det F = 0$$

$$\operatorname{tr}(\nabla v) = 0 \rightarrow \boxed{\operatorname{div} v = 0}$$

POWER OF THE STRESS ALONG ANY MOTION

along the trajectory

$$T \cdot \nabla v = T \cdot \dot{F}F^{-1}$$

we know $\operatorname{tr}(\nabla v) = 0$

If we call $\boxed{T = aI}$ $\Rightarrow aI \cdot \nabla v = a\operatorname{tr}(\nabla v) = 0$

SPHERICAL TENSOR

31

considering an isochoric motion

$$S \cdot \dot{F} = \frac{d}{dt}\varphi(F)$$

then as the volume does not change $T \cdot \dot{F}F^{-1} = S \cdot \dot{F}$

$\Rightarrow S = TF^{-T}$

$$TF^{-T} \cdot \dot{F} = \frac{d}{dt}\varphi(F)$$

$$\hat{T}(F) \cdot \dot{F}F^{-1} = \frac{d}{dt}\varphi(F)$$

we can $\hat{T}(F) + aI$

decompose $\boxed{T = T_D + T_{SPH}}$ we assume that $\boxed{tr\, T_D = 0}$

not sph

$T = T_D - pI$ $\quad p$: describes just component of the stress

DEVIATORIC part of the stress

and is characterized by the fact

$$-pI \cdot \nabla v = tr\, \nabla v = 0$$

$$tr\, T = tr\, T_D + tr(-pI) = 0 - 3p$$

given $T \Rightarrow p = -\frac{1}{3}\, tr\, T$

$$T_D = T - \frac{1}{3} p T$$

$$dev = T - \frac{1}{3}(tr\, T)I$$

$$\boxed{\left(dev\, \hat{T}(F) - pI\right) \cdot \nabla v = \frac{d}{dt}\varphi(F)}$$

Reactive stress

i.e. strain energy gives us only the deviatoric part of T

$T = dev\, T - pI$ $\quad$ general decomposition of stress & incompressible material

INNER PRESSURE

$$W^{ext} = -W^{int}$$

$$W^{ext} = \int_R (T \cdot \nabla v)\, dV = \int_{\bar{R}} \det F\, (T \cdot \nabla v)\, dV$$

$$\det F\,(T \cdot \dot{F}F^{-1}) - \frac{d}{dt}\varphi(F)$$

$$\boxed{S \cdot \dot{F} - \frac{d}{dt}\varphi(F) \geq 0}$$

DISSIPATION PRINCIPLE

i.e. the power supplied to the system does not flown inside to the body, is not tranformed into energy.

We can still say that

$$\boxed{\hat{S}(F) \cdot \dot{F} = \frac{d}{dt}\varphi(F)}$$

ENERGETIC PART of the STRESS

and

$$S \cdot \dot{F} - \hat{S}(F) \cdot \dot{F} \geq 0$$

$$(S - \hat{S}(F)) \cdot \dot{F} \geq 0$$

from this we get that the stress is made in different parts

$$S - \hat{S}(F) = S^{+} \rightarrow \boxed{S = \hat{S}(F) + S^{+}}$$

DISSIPATIVE STRESS

$$T = \operatorname{dev} T - p I = \operatorname{dev} \hat{T}(F) + \operatorname{dev} T^{+} - p I$$

since $\operatorname{dev} S^{+}$ is not important we consider

$$S^{+} = T^{+} \operatorname{cof} F$$

$$\boxed{T = \hat{T}(F) + T^{+} - p I}$$

if it is not incompress. $\rightarrow pI = 0$

if it is incompress. $\rightarrow$ all 3 terms

if it is compress $\rightarrow T = \hat{T}(F) + T^{+}$

38

18. 04. 2011

$$\hat{S}(F) \cdot \dot{F} = \frac{d}{dt}\varphi(F)$$

by the principle objectivity

$$\varphi(F) = \varphi(U) = \tilde{\varphi}(C)$$

if the material is ISOTROPIC $\check{\varphi}(L_1, L_2, L_3)$

$\check{\varphi}$ is function of L_1, L_2, L_3.

$$\hat{\varphi}(L_1, L_2, L_3) = \frac{\partial}{\partial L_1}\varphi \frac{\partial L_1}{\partial t} + \frac{\partial}{\partial L_2}\varphi \frac{\partial L_2}{\partial t} + \frac{\partial}{\partial L_3}\varphi \frac{\partial L_3}{\partial t}$$

computing these derivatives we get $L_i \cdot \dot{F}$

$$\boxed{\begin{aligned} L_1 &:= \operatorname{tr} C \\ L_3 &:= \det C \\ L_2 &:= \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right) \end{aligned}}$$

ISOTROPY

$$\boxed{\hat{S}(F) = 2F\left[\left(\frac{\partial\varphi}{\partial L_1} + \frac{\partial\varphi}{\partial L_2} L_1\right) I - \frac{\partial\varphi}{\partial L_2} C + \frac{\partial\tilde{\varphi}}{\partial L_3} L_3 C^{-1}\right]}$$

RIVLIN FORMULA

Example DISSIPATIVE STRESS

$$\underbrace{\left(T - \hat{T}(F)\right)}_{} \cdot \dot{F}F^{-1} \geq 0$$

DISSIPATION PRINCIPLE

$T^+ :=$ dissipating stress

$$T^+ \cdot \dot{F}F^{-1} \geq 0$$

MECHANICAL STRESS

$$T = \hat{T}(F) + \underbrace{T^r}_{-pI} + T^+$$

When T_0^+ is a constant is not a good choice since we get $T_0^+ \cdot \dot{F}F^{-1} \le 0$. Then cannot be constant.

Instead it is a good choice to choose

$$\underbrace{(d\,\dot{F}F^{-1})}_{\text{VISCOUS STRESS}} \cdot \dot{F}F^{-1} \ge 0 \quad \Rightarrow \quad d \ge 0$$

const

VISCOSITY

If $\hat{T}(F)$ comes from the elastic energy we'll call the material "VISCOUS-ELASTIC".

We consider a material and the incompressibility condition

not a hard material but something soft

$\varphi(L_1, L_2, L_3) = C_1 L_1$ NEO - HOOKEAN

$\det F = 1$

Eg $L_1 = 3$ $C = I$

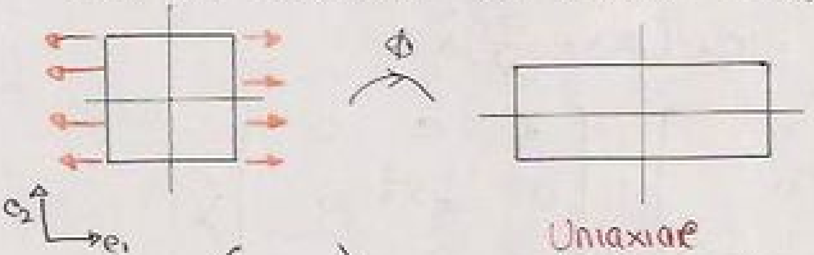
$\varphi = C_1(L_1 - 3) + C_2(L_2 - 3)$ MOONEY RIVLIN

$$L_2 = \frac{1}{2}\left((\mathrm{tr}\,C)^2 - \mathrm{tr}\,C^2\right) = \frac{(9-3)}{2} = 3$$

RUBBER - LIKE MATERIALS

BIO TISSUE

Let us consider an affine body



Uniaxial deformation

This deform. has 3 invariant directions i.e. the eigenvectors of the stretch

$$[F] = \begin{pmatrix} & \\ & \end{pmatrix}$$

$F = U$ symmetric and positive definite

79

$$[F] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$ we assume $\det F = 1$, λ_i principal stretches

$$\det F = \lambda_1 \lambda_2 \lambda_3 = 1$$

then $\lambda_2 = \lambda_3$, $F = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$

$$\lambda_1 \lambda_2^2 = 1 \Rightarrow \lambda_2 = \frac{1}{\sqrt{\lambda_1}} \quad \text{since } \lambda_i > 0$$

$$F = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

Considering $\varphi(\iota_1, \iota_2, \iota_3) = c_1(\iota_1 - 3)$

$$\iota_1 = \operatorname{tr} C = \operatorname{tr} U^2 = \lambda^2 + \frac{2}{\lambda}$$

To get $\hat{S}(F)$ we use

$$\hat{S}(F) = 2F c_1 = 2c_1 F \quad *$$

Let us try to compute $\hat{S}(F) \cdot \dot{F} = \frac{\partial}{\partial t}\varphi$ ⋆

$$\frac{\partial \varphi}{\partial t} = c_1 \frac{d}{dt}\iota_1 = c_1\left(2\lambda\dot{\lambda} - \frac{2}{\lambda^2}\dot{\lambda}\right)$$

$$= c_1\left(2\lambda - \frac{2}{\lambda^2}\right)\dot{\lambda}$$

$$\dot{F} = \begin{bmatrix} \dot{\lambda} & 0 & 0 \\ 0 & -\frac{1}{2}\frac{1}{\sqrt[3]{\lambda^2}}\dot{\lambda} & 0 \\ 0 & 0 & -\frac{1}{2}\frac{1}{\sqrt[3]{\lambda^2}}\dot{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}\lambda^{-\frac{3}{2}} & 0 \\ 0 & 0 & -\frac{1}{2}\lambda^{-\frac{3}{2}} \end{bmatrix}\dot{\lambda}$$

If we call

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$S = \hat{S}(F) + S^{+}$

$S^{+} = 0 \Rightarrow S = \hat{S}(F)$

$$\hat{S}(F) \cdot \dot{F} = S_{11}\dot{\lambda} + S_{22}\left(-\frac{1}{2}\lambda^{-\frac{3}{2}}\right)\dot{\lambda} + S_{33}\left(-\frac{1}{2}\lambda^{-\frac{3}{2}}\right)\dot{\lambda}$$

$$= c_1\left(2\lambda - \frac{2}{\lambda^2}\right)\dot{\lambda}$$

$$\Rightarrow \left(S_{11} + S_{22}\left(-\frac{1}{2}\frac{1}{\lambda^{3/2}}\right) - S_{33}\frac{1}{2}\frac{1}{\lambda^{3/2}}\right)\dot{\lambda} = c_1\left(2\lambda - \frac{2}{\lambda^2}\right)\dot{\lambda}$$

$$S_{11} - \frac{S_{22}}{2}\frac{1}{\lambda^{3/2}} - S_{33}\frac{1}{2}\frac{1}{\lambda^{3/2}} = c_1\left(2\lambda - \frac{2}{\lambda^2}\right)$$

$$S_{11} - \frac{1}{\sqrt{\lambda^3}}\left(\frac{S_{22}+S_{33}}{2}\right) = c_1\left(2\lambda - \frac{2}{\lambda^2}\right) \quad \forall$$

$$T \cdot \underbrace{\dot{F}F^{-1}}$$

$tr(\nabla v) = 0$ since

$$\frac{d}{dt}\det F = tr(\dot{F}F^{-1})\det F$$

$$\det F = 1 \Rightarrow tr(\dot{F}F^{-1}) = 0$$

$\boxed{T = \text{dev}\,T + \text{sph}\,T}$ $*$

the second part will be zero

$$\Rightarrow \text{dev}\,T \cdot \dot{F}F^{-1} = T \cdot \dot{F}F^{-1}$$

$$\boxed{\text{sph}\,T := \frac{1}{3}(\text{Tr}\,T)I}$$

by $*$ $\cancel{tr}\,T = tr(\text{dev}\,T) + \cancel{tr}\,T \Rightarrow tr(\text{dev}\,T) = 0$

$$\boxed{\text{dev}\,T = T - \text{sph}\,T}$$

40

$$\frac{2}{3}\sigma_{11}-\frac{1}{3}\sigma_{22}-\frac{1}{3}\sigma_{33}=\frac{2}{3}\left(\sigma_{11}-\frac{1}{2}(\sigma_{22}+\sigma_{33})\right)$$

$$-\frac{1}{3}\sigma_{11}+\frac{2}{3}\sigma_{22}-\sigma_{33} \qquad \frac{2}{3}\left(\sigma_{33}-\frac{1}{2}(\sigma_{11}+\sigma_{22})\right)$$

$$\begin{bmatrix} \sigma_{11}-\frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) & \sigma_{12} & \sigma_{23} \\ \sigma_{12} & \sigma_{22}-\frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) & \sigma_{23} \\ \sigma_{23} & \sigma_{23} & \sigma_{33}-\frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \end{bmatrix}$$

deviatoric part of the stress

Which is the relation between T and S?

$$T = SF^T \frac{1}{\det F} \qquad S = T \operatorname{cof} F$$

$$T = SF^T \Rightarrow S = TF^{-T}$$

$$[F]^{-1} = \begin{bmatrix} 1/\lambda & 0 & 0 \\ 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & \sqrt{\lambda} \end{bmatrix}$$

$$S_{11} = \frac{1}{\lambda}\sigma_{11} \qquad S_{22} = \sqrt{\lambda}\,\sigma_{22} \qquad S_{33} = \sqrt{\lambda}\,\sigma_{33}$$

substituting in $\boxtimes$

$$\frac{\sigma_{11}}{\lambda} - \frac{1}{\lambda^{\frac{3}{2}}}\lambda^{\frac{1}{2}}\frac{(\sigma_{22}+\sigma_{33})}{2} = c_1\left(2\lambda - \frac{2}{\lambda^2}\right)$$

$$\frac{1}{\lambda}\left(\sigma_{11} - \frac{1}{2}(\sigma_{22}+\sigma_{33})\right) = c_1\left(2\lambda - \frac{2}{\lambda^2}\right)$$

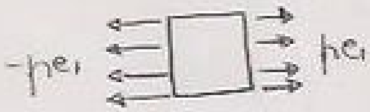
i.e. we realize that we compute the deviatoric part of the Cauchy stress

$$\underbrace{\frac{2}{3}\left(\sigma_{11} - \frac{1}{2}(\sigma_{22}+\sigma_{33})\right)}_{[\operatorname{dev} T]_{11}} = \frac{2}{3}c_1\left(2\lambda - \frac{2}{\lambda^2}\right)\lambda$$

By $\hat{S}(F)\cdot\dot{F} = \frac{\partial}{\partial t}\varphi \Rightarrow$ we cannot identify explicitly $\hat{S}(F)$

We can compute the elastic response by $\Rightarrow$ getting a linear combination of σ_{11}, σ_{22} and σ_{33}

Let us consider



$$p = 0$$
$$\operatorname{skw} M = 0$$
$$\operatorname{sym} M = T V_R$$

$$M = \bar{M} F^T$$

$$M = F\bar{u}_1 \otimes \bar{t}_1 A_{\bar{y}_1} = \underbrace{\left(\bar{u}_1 \otimes \bar{t}_1 A_{\bar{y}_1}\right)}_{\bar{M}} F^T$$

$$T = \frac{M}{V_R}$$

$$M = F(l e_1) \otimes (t_1 A_{y_1}) = l \lambda e_1 \otimes (p e_1 A_{y_1})$$

area stretched by λ_2, λ_3

$$A_{y_1} = \lambda_2 \lambda_3 A_{\bar{y}_1} = \frac{1}{\lambda} A_{\bar{y}_1} = \frac{1}{\lambda} l^2$$

$$M = l_1 (e_1 \otimes e_1)\, p\, l^2$$

$$[M] = p l^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \checkmark\ p = 0 \\ \checkmark\ \operatorname{skw} M = 0 \\ \operatorname{sym} M = T V_R \end{array}$$

$$\frac{M}{V_R} = T \qquad V_R = \det F = 1 \Rightarrow l^3 = V_R = V_{\bar{R}}$$

$$\begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \Rightarrow \boxed{\begin{array}{l} \sigma_{11} = p \\ \sigma_{ij} = 0 \end{array}} \quad i, j \neq 1$$

The sol'n of the balance eqn is $p = \sigma_{11}$ → stress should be able to balance of the applied force

p: applied force

General decomposition of the stress

$$\boxed{T = \hat{T}(F) - pI + T^+}$$

$$T^+ = 0$$

$$\sigma_{11} = \hat{\sigma}_{11} - pI \qquad \sigma_{12} = \sigma_{13} = 0$$

$$\sigma_{22} = \hat{\sigma}_{22} - pI \qquad \sigma_{21} = \sigma_{23} = 0 \Rightarrow \boxed{\begin{array}{l} p = \hat{\sigma}_{11} - p \\ 0 = \hat{\sigma}_{22} - p \\ 0 = \hat{\sigma}_{33} - p \end{array}}$$

$$\sigma_{33} = \hat{\sigma}_{33} - pI \qquad \sigma_{31} = \sigma_{32} = 0$$

41

$$\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = c_1\left(2\lambda - \frac{2}{\lambda^2}\right)\lambda$$

$$p + \cancel{p} - \cancel{p} = c_1\left(2\lambda^2 - \frac{2}{\lambda}\right) = 2c_1\lambda\left(\lambda - \frac{2}{\lambda^2}\right)$$

$$p = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$$

we can find a soln in terms of λ

19.04.2011

$$\begin{cases} p = \sigma_{11} \\ \sigma_{ij} = 0 \quad i,j \neq 1 \end{cases}$$ balance equations

$$\begin{cases} p = \hat{\sigma}_{11} - p \\ 0 = \hat{\sigma}_{22} - p \\ 0 = \hat{\sigma}_{33} - p \end{cases}$$ characteristic of the stress

$$p = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$$

$$\left(\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}\right) - 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$$

$$[F] = \begin{bmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$$

Piola stress $S_{11} \to \sigma_{11}$ Cauchy stress

$$[M] = \underbrace{\begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{T} V_R \quad \Rightarrow \quad T = \frac{M}{V_R}$$

using characterization of the stress

$$T = \hat{T}(F) - pI + T^+$$

but we supposed $T^+ = 0$

$\sigma_{11} = \hat{\sigma}_{11} - p$ $\qquad$ $\sigma_{12} = \hat{\sigma}_{12}$

$\sigma_{22} = \hat{\sigma}_{22} - p$ $\qquad$ $\sigma_{13} = \hat{\sigma}_{13}$

$\sigma_{33} = \hat{\sigma}_{33} - p$ $\qquad$ $\sigma_{23} = \hat{\sigma}_{23}$

then

$$\begin{cases} p = \hat{\sigma}_{11} - p \\ 0 = \hat{\sigma}_{22} - p \\ 0 = \hat{\sigma}_{33} - p \end{cases} \qquad \begin{cases} 0 = \hat{\sigma}_{12} \\ 0 = \hat{\sigma}_{13} \\ 0 = \hat{\sigma}_{23} \end{cases}$$

by repeacing theseexpressions in

$$(\hat{\sigma}_{11} - (\hat{\sigma}_{22} + \hat{\sigma}_{33})) = 2c_1\lambda(\lambda - \frac{1}{\lambda^2})$$

we get

$$\boxed{p = 2c_1\lambda(\lambda - \frac{1}{\lambda^2})}$$

everything is known so we can compute λ.

For $\lambda \neq 0$

$$\lambda p = 2c_1(\lambda^3 - 1)$$

$$2c_1\lambda^3 - \lambda p - 2c_1 = 0$$

$$\boxed{\lambda^3 - \frac{p}{2c_1}\lambda - 1 = 0}$$

$\lambda = 1$ when the deformation is the identity

(p, λ)

$(\hat{p}(\beta), \hat{\lambda}(\beta))$ $\qquad \beta = 0$

$\lambda = 1 + \varepsilon$

$\hat{\lambda}(\beta) = 1 + \hat{\varepsilon}(\beta)$

$$\cancel{1} + 3\lambda^2\Big|_{\beta=0}\left(\frac{d}{d\beta}\lambda\right)\beta - \frac{1}{2c_1}\lambda\Big|_{\beta=0}\left(\frac{d}{d\beta}p\right)\beta - \frac{p}{2c_1}\Big|_{\beta=0}\frac{d\lambda}{d\beta}\beta$$

$$\cancel{1} = 0$$

$$3\left(\frac{d}{d\beta}\hat{\varepsilon}\right)\beta - \frac{1}{2c_1}\left(\frac{d}{d\beta}p\right)\beta = 0$$

$$\varepsilon = \hat{\varepsilon}(\beta) = 0 + \frac{d}{d\beta}\hat{\varepsilon}\Big|_{\beta=0}\beta$$

42

$$p = \hat{p}(\beta) = 0 + \frac{d}{d\beta}\tilde{p}\Big|_{\beta=0}\beta$$

$$\Rightarrow \quad 3\varepsilon - \frac{1}{2C_1}p = 0$$

$$\varepsilon = \frac{p}{6C_1}$$

$$\Rightarrow \quad \lambda = 1 + \frac{p}{6C_1}$$

useful in describing small deformations

Let's consider some dissipation

$$* \quad T = \hat{T}(F) - pI + T^+$$

det F = 1
volume does not change

$$T^+ \cdot \dot{F}F^{-1} \geq 0$$

$$T^+ = d\,\dot{F}F^{-1}$$

$$[\dot{F}] = \begin{bmatrix} \dot{\lambda} & 0 & 0 \\ 0 & -\frac{\dot{\lambda}}{2\lambda^{3/2}} & 0 \\ 0 & 0 & -\frac{\dot{\lambda}}{2\lambda^{3/2}} \end{bmatrix} \qquad [\dot{F}F^{-1}] = \begin{bmatrix} \frac{\dot{\lambda}}{\lambda} & 0 & 0 \\ 0 & -\frac{\dot{\lambda}}{2\lambda} & 0 \\ 0 & 0 & -\frac{\dot{\lambda}}{2\lambda} \end{bmatrix}$$

$$\mathrm{tr}(\dot{F}F^{-1}) = 0$$

$$[\dot{F}F^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \frac{\dot{\lambda}}{\lambda}$$

$$[T^+] = d \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \frac{\dot{\lambda}}{\lambda}$$

Thus by $*$

$$p = \hat{\sigma}_{11} - p + d\,\frac{\dot{\lambda}}{\lambda}$$

$$0 = \hat{\sigma}_{22} - p + \left(-\frac{d}{2}\right)\frac{\dot{\lambda}}{\lambda}$$

$$0 = \hat{\sigma}_{33} - p - \frac{d}{2}\,\frac{\dot{\lambda}}{\lambda}$$

then $\hat{\sigma}_{11} = p + \rho - d\,\frac{\dot{\lambda}}{\lambda}$

$$\hat{\sigma}_{22} = \hat{\sigma}_{33} = \rho + \frac{d}{2}\,\frac{\dot{\lambda}}{\lambda}$$

$$\left(\hat{\sigma}_{11} - \tfrac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33})\right) = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$$

$$\left(p + \rho - d\,\frac{\dot{\lambda}}{\lambda} - \rho - \frac{d}{2}\,\frac{\dot{\lambda}}{\lambda}\right) = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$$

$$p - \frac{3}{2}d\,\frac{\dot{\lambda}}{\lambda} = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$$

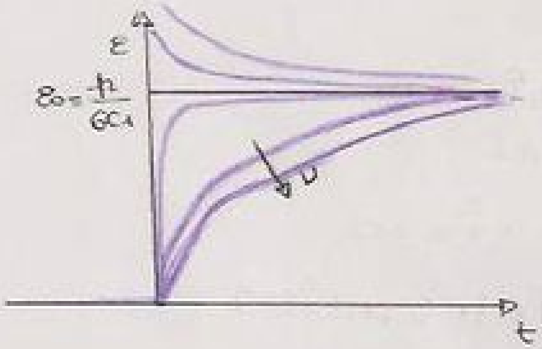
since $\lambda > 0$

$$2c_1\lambda^3 - 2c_1 - p\lambda + \frac{3}{2}d\,\dot{\lambda} = 0$$

$$\boxed{\lambda^3 - \frac{p}{2c_1}\lambda + \frac{3d}{4c_1}\dot{\lambda} - 1 = 0}$$

Checking where $\dot{\lambda} = 0 \Rightarrow$ steady sol^n

$$\varepsilon - \frac{p}{6c_1} + \frac{3d}{4c_1}\dot{\varepsilon} = 0$$



i.e. is a STABLE SOLUTION

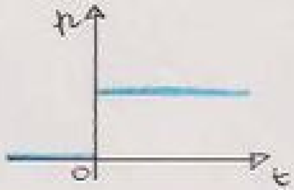
$$\varepsilon - \varepsilon_0 + \nu\dot{\varepsilon} = 0 \qquad \nu = \frac{3d}{4c_1} > 0$$

$$\varepsilon = r + \varepsilon_0 \qquad \dot{\varepsilon} = \dot{r}$$

$$\nu\dot{r} + r = 0 \Rightarrow \dot{r} = -\frac{1}{\nu}r \Rightarrow r(t) = r_0 e^{-\frac{1}{\nu}t}$$

$$\Rightarrow \boxed{\varepsilon(t) = r_0 e^{-\frac{1}{\nu}t} + \varepsilon_0}$$

If we start from $\varepsilon(0) = 0$



$$r_0 + \varepsilon_0 = 0 \Rightarrow r_0 = -\varepsilon_0$$

ELONGATION

$$\boxed{\varepsilon(t) = \varepsilon_0\left(1 - e^{-\frac{1}{\nu}t}\right)}$$

$$\dot{\varepsilon} = \varepsilon_0 \frac{1}{\nu} e^{-\frac{t}{\nu}} \qquad \dot{\varepsilon} > 0 \ \forall t$$

43

$\nu = \frac{3d}{4c}$ d is the viscosity $\Rightarrow$ ν is a kind of viscosity

If ν is very low we get a kind of jump
If there is the dissipation the body does not change suddenly but follows the force

If we consider $\frac{t}{\nu}$ we have a TIME SCALE wrt the dissipation. Time is strongly related to the viscosity.

→ closer to the origin will be smaller

$$h = \hat{\sigma}_{11} - p + d\frac{\dot{\lambda}}{\lambda}$$

We want to calculate p

on the equilibrium point

$$\boxed{\begin{aligned} h &= \hat{\sigma}_{11} - p \\ 0 &= \hat{\sigma}_{22} - p \\ 0 &= \hat{\sigma}_{33} - p \end{aligned}}$$

until now we have considered the deviatoric part of the stress, but we have also the spherical part

$$\hat{T}(F) = \mathrm{dev}\,\hat{T}(F) + \mathrm{sph}\,\hat{T}(F)$$
$$-pI$$

$$tr(\mathrm{dev}\,\hat{T}(F)) = \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} = 0$$

$$tr(\hat{T}(F)) = tr(\mathrm{dev}\,\hat{T}(F)) + tr(\mathrm{sph}(\hat{T}(F))$$
$$= tr(\mathrm{sph}\,\hat{T}(F))$$

I want to do the same in $\boxed{\ }$

$$\Rightarrow h = -3p$$

Let us consider a case where there is not an applied force but a body with mass density and kinetic energy

$$b = \int_R -\rho\,\ddot{x}\,dV$$

$$f = \int_R -\rho \ddot{x}\, dV = \int_{\bar{R}} -(\bar{\rho} \det F)\bar{a}\, dV$$

in the cut shape $\rho = \bar{\rho}$ ieff
since $\det F = 1$
$\Rightarrow \bar{\rho} = \rho$

$$a = \ddot{x} = \frac{d}{dt}\bar{v} \qquad v_A = v_0 + L(P_A - P_0)$$

$$= v_0 + \underbrace{LF}_{\bar{L}}(\bar{P}_A - \bar{P}_0) \qquad \nabla v F = \nabla \bar{v}$$

$\bar{v}(x) = \bar{v}_0 + \bar{L}(x - \bar{P}_0)$ then

$$\bar{a} = \bar{a}_0 + \dot{\bar{L}}(x - \bar{P}_0)$$

$$f = \int_{\bar{R}} -\rho \bar{a}\, dV = -\rho \int_{\bar{R}} (\bar{a}_0 + \dot{\bar{L}}(x - P_0))\, dV$$

Total force of the INERTIAL FORCES at any time

$$= -\rho \bar{a}_0 V\bar{R} - \rho \dot{\bar{L}} \int_{\bar{R}} x - P_0)\, dV$$

$$\rho V = m$$

$$f = -m\bar{a}_0 - \rho \dot{\bar{L}} \int_{\bar{R}} (x - \bar{P}_0)\, dV.$$

The baricenter of a cube

$$(C_0 - \bar{P}_0) = \frac{1}{V\bar{R}} \int_{\bar{R}} (x - P_0)\, dV$$

The CENTER OF THE MASS of a body

$$(C_0 - \bar{P}_0) = \frac{1}{V\bar{R}} \int_{\bar{R}} (x - P_0)\, dV$$

if $C_0 - \bar{P}_0 = 0 \Rightarrow f = -m\bar{a}_0$

The total moment

$$M_{P_0} = \int_R (x - P_0) \otimes (-\rho a)\, dV = \int_{\bar{R}} F(x - \bar{P}_0) \otimes (-\rho \bar{a})\, dV$$

$$= -\rho \int_{\bar{R}} F(x - \bar{P}_0) \otimes \bar{a}_0\, dV = -\rho \int_{\bar{R}} ((x - \bar{P}_0) \otimes \bar{a}_0)\, dV\, F^T$$

$$((x - \bar{P}_0) \otimes \bar{a}_0) F^T$$

if P_0 center of the cube
$\int = 0$

41

20. 04. 2011

$$\lambda^3 - \frac{h}{2C_1}\lambda - 1 = 0$$

λ, h linearizing $\lambda = 1 + \varepsilon \quad \varepsilon = \frac{h}{6C_1}$

$$\downarrow$$

$$\lambda - \frac{h}{6C_1} - 1 = 0 \Rightarrow \lambda = 1 + \frac{h}{6C_1}$$

What are the solution of the original equation?
Using Mathematica:

$$a = \frac{h}{2C_1}$$

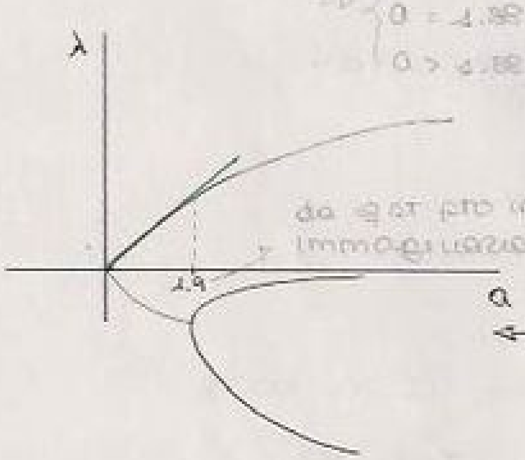
$$\lambda_1 = \frac{2 \times 3^{1/3} a + 2^{2/3}\left(9 + \sqrt{81 - 12a^3}\right)^{2/3}}{6^{2/3}\left(9 + \sqrt{81 - 12a^3}\right)^{1/3}}$$

$81 - 12a^3 > 0$, $a^3 < \frac{81}{12}$

for the values of a s.t. $12a^3 > 81 \Rightarrow$ imaginary roots

$\Rightarrow \begin{cases} a < 1.88 & \text{real} \\ a = 1.88 & \text{but} \\ a > 1.88 & \text{imag} \end{cases}$

is not linear but a very smooth function

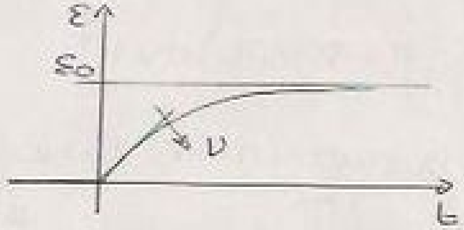


← there is a bifurcation

Linearizing we get an approximation / of the solution.

$$\lambda^3 - \frac{h}{2C_1}\lambda + \frac{3d}{4C_1}\dot{\lambda} - 1 = 0$$

smoothness depends on the viscosity



$$\frac{V_R}{V_{\bar{R}}} = \det F = 1 \quad \Rightarrow \quad R = \bar{R}$$

the volume is the same for the shapes

$$\int_R -\rho a = -\int_{\bar{R}} \rho a = -m a_0$$

For the moment

$$-\int_R \rho (x - P_0) \otimes a \, dV = -\rho \int_{\bar{R}} F(x - \bar{P}_0) \otimes \bar{a} \underset{1}{\det F} \, dV$$

generic point in $\bar{R}$

$$= -\rho \int_{\bar{R}} ((x - \bar{P}_0) \otimes \bar{a}) dV F^T \qquad \bar{a} = \bar{a}_0 + \dot{\bar{L}}(x - \bar{P}_0)$$

$$= -\rho \left(\cancel{\int_{\bar{R}} (x - \bar{P}_0) dV} \otimes a_0 \right) F^T + $$

$$-\rho \int_{\bar{R}} (x - \bar{P}_0) \otimes \dot{\bar{L}} (x - P_0) dV F^T$$

if we choose $\bar{P}_0$ as the center of the cube

$0 \leftarrow (C_0 - \bar{P}_0) V_R$

$$M_{P_0} = -\rho \int_{\bar{R}} (x - \bar{P}_0) \otimes \dot{\bar{L}} (x - P_0) dV F^T$$

By the def of tensor product $(u \otimes Av)e = (u \cdot e)Av$

$$= A((u \cdot e)v) = A(u \otimes v)e$$

$$u \otimes Av = A(u \otimes v)$$

$$(Fu \otimes v)e = (Fu \cdot e)v = (u \cdot F^T e)v = (u \otimes v)F^T e$$

i.e. if it is moving on the right $\rightarrow F^T$

does not depend on the position " left $\rightarrow F$

$$M_{P_0} = -\rho \dot{\bar{L}} \boxed{\int_{\bar{R}} \rho (x - \bar{P}_0) \otimes (x - \bar{P}_0) \, dV} F^T$$

EULER TENSOR

since $\det F = 1 \Rightarrow \bar{E} = J$

Then the moment will be made in two parts

$$\boxed{M_{P_0} = -\dot{\bar{L}} \underbrace{\bar{E} F^T}_{E}}$$

Important quantity while considering inertial forces

45

Let us consider

$$\rho\int_R (x-\bar{P}_0)\otimes(x-\bar{P}_0)\,dV \qquad x-\bar{P}_0 = z_1 e_1 + z_2 e_2 + z_3 e_3$$

Substituting and computing we will get 9 terms.
In matrix form

$$\begin{bmatrix} z_1^2 & z_2 z_1 & z_3 z_1 \\ z_1 z_2 & z_2^2 & z_3 z_2 \\ z_1 z_3 & z_2 z_3 & z_3^2 \end{bmatrix}$$

thus we have to compute

$$\iiint_{-\frac{l}{2}}^{\frac{l}{2}} \Big[\qquad \Big]\, dz_1\, dz_2\, dz_3$$

$$\left[\frac{1}{3} z_1^3\right]_{-\frac{l}{2}}^{\frac{l}{2}} l\, l = \frac{l^3}{12} l^2 = \frac{l^5}{12} = \left[\frac{1}{3} z_3^3\right]_{-\frac{l}{2}}^{\frac{l}{2}} - \left[\frac{1}{3} z_3^3\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$\iiint z_1 z_2\, dz_1 dz_2 dz_3 = 0 = \iiint z_1 z_3 = \iiint z_2 z_3$$

Then

$$[\bar{E}] = \rho \begin{bmatrix} \frac{l^5}{12} & 0 & 0 \\ 0 & \frac{l^5}{12} & 0 \\ 0 & 0 & \frac{l^5}{12} \end{bmatrix}$$

is symmetric since we chose P_0 as the center of the cube

whenever we consider P_0 the center of the shape we get the same matrix

$$[\bar{E}] = \rho \frac{l^5}{12} [I]$$

$$\Rightarrow \boxed{\bar{E} = \rho \frac{l^5}{12} I}$$

~~reference shape~~

$$\boxed{M_{P_0} = -\rho \frac{\ell^5}{12} \dot{L} F^T = -\rho \frac{\ell^5}{12} \ddot{F} F^T}$$ INERTIAL FORCES

for the current shape $L = \dot{F} F^T$

$$[\dot{F}] = \begin{bmatrix} 1 & & \\ & -\frac{1}{\lambda 2\sqrt{\lambda}} & \\ & & -\frac{1}{\lambda 2\sqrt{\lambda}} \end{bmatrix} \dot{\lambda}$$

$$[\ddot{F}] = \begin{bmatrix} 1 & & \\ & -\frac{1}{2\lambda\sqrt{\lambda}} & \\ & & -\frac{1}{2\lambda\sqrt{\lambda}} \end{bmatrix} \ddot{\lambda} + \begin{bmatrix} 0 & & \\ & \frac{3}{4}\lambda^{-\frac{5}{2}}\dot{\lambda} & \\ & & \frac{3}{4}\lambda^{-\frac{5}{2}}\dot{\lambda} \end{bmatrix} \dot{\lambda}$$

$$[\ddot{F} F^T] = \begin{bmatrix} \lambda & & \\ & -\frac{1}{2\lambda^2} & \\ & & -\frac{1}{2\lambda^2} \end{bmatrix} \ddot{\lambda} + \begin{bmatrix} 0 & & \\ & \frac{3}{4}\frac{1}{\lambda^3} & \\ & & \frac{3}{4}\frac{1}{\lambda^3} \end{bmatrix} \dot{\lambda}^2$$

$$\boxed{T = \frac{M}{\ell^3} = -\rho \frac{\ell^2}{12} \ddot{F} F^T}$$

Now let's try to compute $\sigma_{11} = -\rho \frac{\ell^2}{12} \lambda \ddot{\lambda}$

BALANCE EQUATIONS $\rightarrow$ $\sigma_{33} = \sigma_{22} = \rho \frac{\ell^2}{24\lambda^2} \ddot{\lambda} - \rho \frac{\ell^2}{16} \frac{1}{\lambda^3} \dot{\lambda}^2$

the other value are zeros.

Now we'll compare it by the value supplied by the material:

$$\sigma_{11} = \hat{\sigma}_{11} - p$$

p: property reflected by the unknown pressure $\Rightarrow$ INCOMPRESIBILITY

MATERIAL CHARACTERIZATION OF CONSTITUTIVE DESCRIPTION OF THE MATERIAL

$$\sigma_{22} = \hat{\sigma}_{22} - p$$

$$\sigma_{33} = \hat{\sigma}_{33} - p$$

$$\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = 2c_1 \left(\lambda^2 - \frac{1}{\lambda}\right)$$ NEO-HOOKEAN MATERIAL

since it is the deviatoric part of the stress there is no difference of considering $\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})$ (since p is the spherical part) 46

$$\sigma_{11} + p - \frac{1}{2}(\sigma_{22} + \sigma_{33} + 2p) = \sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33}) =$$

$$= -\rho \frac{\ell^2}{12} \lambda \ddot{\lambda} - \left(\rho \frac{\ell^2}{24\lambda^2} \ddot{\lambda} - \rho \frac{\ell^2}{16\lambda^3} \dot{\lambda}^2 \right) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\Rightarrow -\rho \frac{\ell^2}{12}\left(\lambda + \frac{1}{2\lambda^2}\right)\ddot{\lambda} + \rho \frac{\ell^2}{16\lambda^3}\dot{\lambda}^2 = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$-\left(\lambda + \frac{1}{2\lambda^2}\right)\ddot{\lambda} + \frac{3}{4\lambda^3}\dot{\lambda}^2 = \frac{24}{\rho \ell^2}\, c_1 \left(\lambda^2 - \frac{1}{\lambda}\right)$$

they are not dimensionless since they are derivatives

$\frac{1}{\frac{Ns^2}{m^4} m^2}$ $\frac{N}{m^2}$ $\quad \frac{1}{s^2}$

$\rho = \frac{m}{V}$

$F = ma \Rightarrow m = \frac{N s^2}{m}$

02.05.2011

$$(**)\quad \lambda^3 - \frac{\pi}{2c_1}\lambda - 1 = 0$$

$\varphi(F) = c_1(\mathrm{tr}(C) - 3)$ $\rightarrow$ we used this kind of energy only for uncompressible material

$$(*)\quad \hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$$

$$[M] = \pi V_R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the balance equations

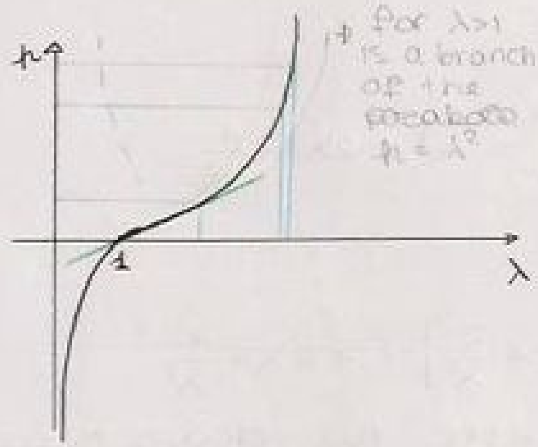
$$\frac{M}{V_R} = T \quad \rightarrow \quad \sigma_{11} = \pi, \quad \sigma_{22} = 0 \;\bullet, \quad \sigma_{33} = 0$$

The characterization for the stress

$$\begin{aligned} \sigma_{11} &= \hat{\sigma}_{11} - pI \\ \sigma_{22} &= \hat{\sigma}_{22} - pI \\ \sigma_{33} &= \hat{\sigma}_{33} - pI \end{aligned} \quad \underset{\text{by } \bullet}{\Longrightarrow} \quad \begin{cases} \pi = \hat{\sigma}_{11} - p \\ 0 = \hat{\sigma}_{22} - p \\ 0 = \hat{\sigma}_{33} - p \end{cases}$$

Substituting them in (*) we get $\boxed{\pi = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)}$ same as (**)

We want to find the relation between π and λ by a drawn of $\pi = 2c_1\lambda\left(\lambda - \frac{1}{\lambda^2}\right)$



$$\frac{p}{2C_1} = \lambda^2 - \frac{1}{\lambda}$$

$\lambda = 1 \quad p = 0$

$\lambda \to 0 \quad p \to -\infty$

$\lambda \to \infty \quad p \to \infty$

$p' = 2\lambda + \frac{1}{\lambda^2} > 0$

This graph is important to describe the material, i.e. the behaviour of the material. When $\lambda < 1$, the value of p is finite i.e. the material is squeezed completely.

Then we can consider the linearization when $\lambda = 1$ or $\lambda \approx 1$.

This graph expresses the relation between p and λ.

Then as we saw in the lecture 20.04.2011 the linearization of (*) is

$$\lambda - 1 = \frac{p}{6C_1}$$

It's obvious that considering a different energy we'll have a different graph.

A different energy is $\boxed{\varphi(F) = C_1(\iota_1 - 3) + C_2(\iota_2 - 3)}$

MOONEY RIVLIN STRAIN ENERGY

using an isotropic material, the strain energy is a function of $\iota_1, \iota_2, \iota_3$, $\varphi(F) = \hat{\varphi}(\iota_1, \iota_2, \iota_3)$.

In this case we'll not have the third invariant since for incompressible material $\det F = 1 \Rightarrow \iota_3 = \det C = 1$.

Instead $\iota_2 = \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right)$

Since

$$[F] = \begin{bmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \qquad [F^T F] = [C] = \begin{bmatrix} \lambda^2 & & \\ & \frac{1}{\lambda} & \\ & & \frac{1}{\lambda} \end{bmatrix}$$

47

$$[C]^2 = \begin{bmatrix} \lambda^4 & & \\ & \frac{1}{\lambda^2} & \\ & & \frac{1}{\lambda^2} \end{bmatrix}$$

Thus $(\operatorname{tr} C)^2 = \left(\lambda^2 + \frac{2}{\lambda}\right)^2 = \lambda^4 + 4\lambda + \frac{4}{\lambda^2}$

$$\operatorname{tr} C^2 = \lambda^4 + \frac{2}{\lambda^2}$$

$$\Rightarrow \iota_2 = \frac{1}{2}\left[+ 4\lambda + \frac{2}{\lambda^2}\right] = + 2\lambda + \frac{1}{\lambda^2}.$$

Reminding that $\iota_1 = \operatorname{tr} C$ the strain energy is

$$\boxed{\varphi(C) = c_1\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + c_2\left(2\lambda + \frac{1}{\lambda^2} - 3\right)}.$$

We know $\hat{S}(F) \cdot \dot{F} = \frac{\partial}{\partial t}\varphi$

and $\hat{S}(F) = \hat{T}(F) F^{-T} \underset{1}{\det F} \Rightarrow \hat{T}(F) F^{-T} \cdot \dot{F} = \frac{\partial}{\partial t}\varphi$

$$\boxed{\hat{T}(F) \cdot \dot{F} F^{-1} = \frac{\partial}{\partial t}\varphi}$$

$$[\dot{F}] = \begin{bmatrix} \dot{\lambda} & & \\ & -\frac{\dot{\lambda}}{2\lambda^{3/2}} & \\ & & -\frac{\dot{\lambda}}{2\lambda^{3/2}} \end{bmatrix} \Rightarrow [\dot{F}F^{-1}] = \begin{bmatrix} \frac{\dot{\lambda}}{\lambda} & & \\ & -\frac{1}{2}\frac{\dot{\lambda}}{\lambda} & \\ & & -\frac{1}{2}\frac{\dot{\lambda}}{\lambda} \end{bmatrix}$$

We can check that $\operatorname{tr} \nabla v = 0$ (in isochoric motion) because $\operatorname{dev} v = 0$, as in this case.

$$\hat{T}(F) \cdot \dot{F}F^{-1} = \hat{\sigma}_{11}\frac{\dot{\lambda}}{\lambda} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33})\frac{\dot{\lambda}}{\lambda}$$

$$\frac{d}{dt}\varphi = c_1\left(2\lambda\dot{\lambda} - \frac{2}{\lambda^2}\dot{\lambda}\right) + c_2\left(2\dot{\lambda} - \frac{2\dot{\lambda}}{\lambda^3}\right)$$

$$= c_1\left(2\lambda^2 - \frac{2}{\lambda}\right)\frac{\dot{\lambda}}{\lambda} + 2c_2\left(\lambda - \frac{1}{\lambda^2}\right)\frac{\dot{\lambda}}{\lambda}$$

Mooney-Rivlin

$$\Rightarrow \boxed{\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) + 2c_2\left(\lambda - \frac{1}{\lambda^2}\right)}$$

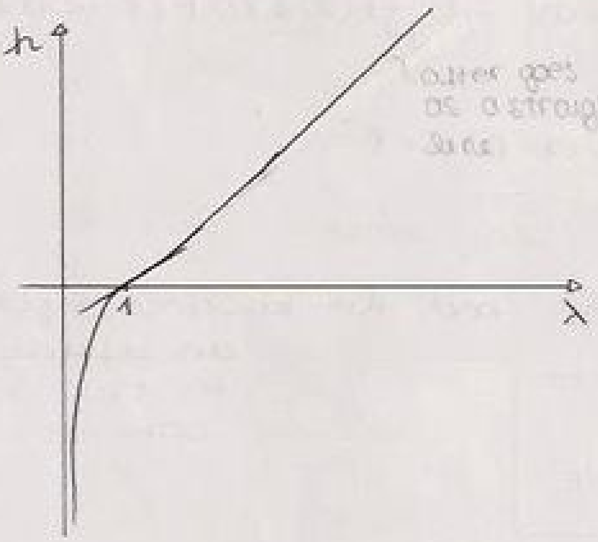
Neo-Hookean

By the characterization of the stress
$$h = \hat{\sigma}_{11} - p$$
$$0 = \hat{\sigma}_{22} - p$$
$$0 = \hat{\sigma}_{33} - p$$

$$\boxed{h = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) + 2c_2\left(\lambda - \frac{1}{\lambda^2}\right)}$$

$\lambda = 1 \quad h \to 0$

$\lambda \to 0 \quad h \to -\infty$



after goes as a straight line

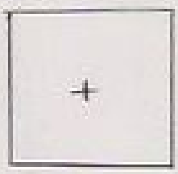
Google Mooney-Rivlin material and find the values for c_1 and c_2

$$\frac{\partial B}{\partial \lambda} = 2c_1\left(2\lambda - \frac{1}{\lambda^2}\right) + 2c_2\left(1 + \frac{2}{\lambda^3}\right) > 0$$

Sometimes the graphs are centered in $\lambda = 1$ 

03.05.2011

Now we consider a material with density ρ, total mass m, a_0 acceleration of the center



$$f = -\int_R \rho a \, dV = -\int_{\bar{R}} (\rho \det F)\bar{a} \, dV \qquad \det F = 1$$

$$\bar{v}(x) = \bar{v}_0 + \bar{L}(x - \bar{p}_0)$$

Since
$$p_A = p_0 + F(\bar{p}_A - \bar{p}_0)$$
$$v_A = v_0 + \dot{F}(\bar{p}_A - \bar{p}_0)$$

then we can consider
$$\bar{v}(x) = \bar{v}_0 + \dot{F}(x - \bar{p}_0)$$
$$\bar{a}(x) = \bar{a}_0 + \ddot{F}(x - \bar{p}_0)$$

Thus

$$f = -\int_{\bar{R}} \rho\,(\bar{a}_0 + \ddot{F}(x - \bar{p}_0))\,dV$$
$$= -\int_{\bar{R}} \rho \, dV \, \bar{a}_0 - \int_{\bar{R}} \rho \ddot{F}(x - \bar{p}_0)\,dV$$
$$= -m\bar{a}_0 - \rho(\bar{p}_G - \bar{p}_0)V_R \qquad \text{if } \bar{p}_0 = \bar{p}_G$$
$$\Rightarrow f = -m\bar{a}_0$$

48

$$M = \int_R \rho(x-P_0)\otimes \bar{a}_0 \, dV - \int_R \rho(x-P_0)\otimes \ddot{F}(x-P_0)\, dV$$

$$= \int_{\bar{R}} \rho F(x-\bar{P}_0)\otimes \bar{a}_0 \, dV - \int_{\bar{R}} \rho F(x-\bar{P}_0)\otimes \ddot{F}(x-P_0)\, dV$$

$$= -\ddot{F}\int_{\bar{R}} \underbrace{\rho(x-\bar{P}_0)\otimes(x-\bar{P}_0)\, dV}_{\bar{E} \quad \text{Euler tensor}} F^T$$

Then $M = -\ddot{F}\bar{E}F^T$ and the balance equations for an affine body in the simple case will be

$$\boxed{\begin{cases} m\ddot{p}_0 = 0 \\ -\ddot{F}\bar{E}F^T = T V_R \end{cases}}$$

The matrix

$$[E] = \int \rho \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_1 x_2 & x_2^2 & x_2 x_3 \\ x_1 x_3 & x_2 x_3 & x_3^2 \end{pmatrix} dV$$

$$\int x_i x_j \, \rho \, dV = 0 \quad i \neq j$$

$$\int x_i^2 \, \rho \, dV = \frac{l^5}{12} \qquad i = 1,2,3$$

$$\bar{E} = \rho \frac{l^5}{12} I$$

$$\begin{cases} m\ddot{p}_0 = 0 \\ -\rho\frac{l^5}{12}\ddot{F}F^T = T V_R \end{cases}$$

$$\frac{N}{m} s^2 \; \frac{1}{m^3} \; m^5 \; \frac{1}{s^2} = N \cdot m \qquad \frac{N\cdot m}{m^3} = \frac{N}{m^2} - T$$

M

$$[\dot{F}] = \begin{bmatrix} 1 & & \\ & -\frac{1}{2\lambda^{3/2}} & \\ & & -\frac{1}{2\lambda^{3/2}} \end{bmatrix} \dot{\lambda}$$

$$[\ddot{F}] = \begin{bmatrix} 1 & & \\ & -\frac{1}{2\lambda^{3/2}} & \\ & & -\frac{1}{2\lambda^{3/2}} \end{bmatrix} \ddot{\lambda} + \begin{bmatrix} 0 & & \\ & \frac{3}{4}\lambda^{-\frac{5}{2}} & \\ & & \frac{3}{4}\lambda^{-\frac{5}{2}} \end{bmatrix} \dot{\lambda}^2$$

$$[\ddot{F}F^T] = \begin{bmatrix} \lambda & & \\ & -\frac{1}{2\lambda^2} & \\ & & -\frac{1}{2\lambda^2} \end{bmatrix} \ddot{\lambda} + \begin{bmatrix} 0 & & \\ & \frac{3}{4}\frac{1}{\lambda^3} & \\ & & \frac{3}{4}\frac{1}{\lambda^3} \end{bmatrix} \dot{\lambda}^2$$

Since $-\frac{\rho l^2}{12}\ddot{F}F^T = T$

$$\begin{cases} \sigma_{11} = -\frac{\rho l^2}{12}\lambda\ddot{\lambda} \\ \sigma_{22} = \frac{\rho l^2}{24}\frac{\ddot{\lambda}}{\lambda^2} - \frac{\rho l^2}{16}\frac{\dot{\lambda}^2}{\lambda^3} = -\frac{\rho l^2}{48}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + 3\frac{\dot{\lambda}^2}{\lambda^3}\right) \\ \qquad = \sigma_{33} \end{cases}$$

Now we have to consider the response function

inner pressure

$$\begin{cases} \sigma_{11} = \hat{\sigma}_{11} - p \\ \sigma_{22} = \hat{\sigma}_{22} - p \\ \sigma_{33} = \hat{\sigma}_{33} - p \end{cases}$$

reactive stress that comes from the incompressibility

spherical part of the stress

constitutive part of the stress

then $\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$

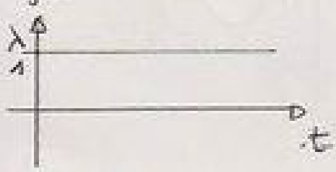
$$\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})$$

We have

$$\boxed{-\frac{\rho l^2}{12}\lambda\ddot{\lambda} + \frac{\rho l^2}{48}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + 3\frac{\dot{\lambda}^2}{\lambda^3}\right) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)}$$

$$-\lambda\ddot{\lambda} + \frac{1}{4}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + 3\frac{\dot{\lambda}^2}{\lambda^3}\right) = \frac{24}{\rho l^2}c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$ SMALL OSCILLATION EQUATION

$$\lambda^2 - \frac{1}{\lambda} = 0 \iff \lambda^3 = 1$$



49

So we can thing of a sol'n near to 1

$$\lambda = 1+\varepsilon$$

We now try to linearize the ODE

$$\lambda = 1+\varepsilon \qquad \lambda^{-1} = 1-\varepsilon$$

$$\lambda^2 = (1+\varepsilon)^2 = 1+2\varepsilon + o(\varepsilon)$$

$$\lambda^{-2} = 1-2\varepsilon + o(\varepsilon)$$

$$\lambda^{3} = 1 - 3\lambda^{2}\Big|_{\lambda=1}(\lambda-1) + o(\varepsilon) = 1-3\varepsilon+o(\varepsilon)$$

and

$$(1+\varepsilon)\ddot{\varepsilon} = \ddot{\varepsilon}+o(\varepsilon) \qquad \ddot{\varepsilon}(1-2\varepsilon) = \ddot{\varepsilon} \qquad \dot{\varepsilon}^2\,(1-3\varepsilon) = o(\varepsilon)$$

$$-\ddot{\varepsilon} + \frac{1}{4}\left(-2\ddot{\varepsilon}\right) = \frac{24c_1}{l^2\rho}\,(1+2\varepsilon-1+\varepsilon)$$

$$-\frac{3}{2}\ddot{\varepsilon} = \frac{24c_1}{\rho l^2}\,3\varepsilon$$

$$\boxed{\ddot{\varepsilon} = -48\,\frac{c_1}{\rho l^2}\,\varepsilon}$$

We remember that from $\ddot{\varepsilon} = -a^2\varepsilon$

$$\Rightarrow \varepsilon = \varepsilon_0 e^{iat}$$

$$\ddot{\varepsilon} = -\omega^2\varepsilon \qquad \omega\text{ - frequency}$$

$$\varepsilon = \varepsilon_0 e^{\pm i\omega t}$$

Thus in our case $\omega = \sqrt{48\,\frac{c_1}{\rho l^2}} = \frac{4}{l}\sqrt{\frac{3c_1}{\rho}}$

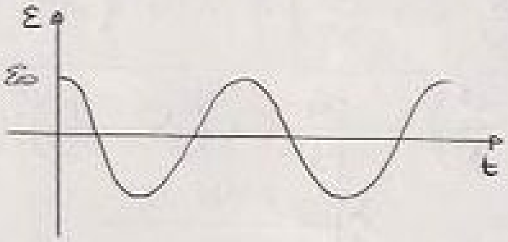
$$\varepsilon(0) = \varepsilon_0 \qquad \varepsilon(t) = A\sin\omega t + B\cos\omega t$$

$$\dot{\varepsilon}(0) = 0$$

$$\varepsilon(0) \Rightarrow B = \varepsilon_0$$

$$\dot{\varepsilon}(0) = \omega A = 0 \Rightarrow A = 0$$

Then $\varepsilon(t) = \varepsilon_0\cos\omega t$ description of the motion of this body



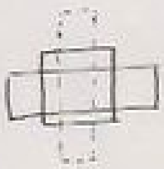
If $\varepsilon(0)=0$, $\dot{\varepsilon}(0)=0$ $\quad \varepsilon(t)=0$ the trivial sol'n

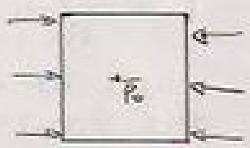
Thus the inertial forces give rise to oscillations that depend on the initial conditions

Eg. Spring-mass system

Note that if ℓ is very large, ω will be very small, and if $\ell \ll 0 \Rightarrow \omega \gg 0$.

These are only some of the oscillations that we can get since it was a consequence of the matrix of F that we have chosen.





$-p e_1$

What we have computed be4

$$M = -\ddot{F}\bar{E}F^T$$

If we consider these forces we have an additional term

$$M = \frac{\ell}{2} F e_1 \otimes (-p e_1) A_{y_1} + \frac{\ell}{2}(-F e_1) \otimes (+p e_1) A_{y_1}$$

$$M = -p \ell F e_1 \otimes e_1 A_{y_1} = -p \ell \lambda e_1 \otimes e_1 \ell^2 \frac{1}{\lambda} = -p \ell^3 e_1 \otimes e_1$$

$$A_{y_1} = A_{\bar{y}_1} \| \mathrm{cof} F e_1 \|$$

$$\frac{M}{V_R} = -p \, e_1 \otimes e_1 \qquad V_R = V_{\bar{R}} \text{ isochoric motion}$$

Finally

$$M = -\ddot{F}\bar{E}F^T - p \ell^3 e_1 \otimes e_1$$

50

The balance equation

$$-\ddot{F}\bar{E}F^T - p\, l^3\, e_1 \otimes e_1 = T l^3$$

$$-\frac{1}{l^3}\ddot{F}\bar{E}F^T - p\, e_1 \otimes e_1 = T$$

$$\sigma_{11} = -\frac{\rho l^2}{12}\lambda\ddot{\lambda} - p = \frac{\rho l^2}{12}\left(-\lambda\ddot{\lambda} - \frac{12}{\rho l^2}p\right)$$

$$\sigma_{22} = \sigma_{33} = -\frac{\rho l^2}{48}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + 3\frac{\dot{\lambda}^2}{\lambda^3}\right)$$

Then $\sigma_{11} - \frac{1}{2}(\sigma_{22}+\sigma_{33}) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$

$$\boxed{-\lambda\ddot{\lambda} - \frac{12}{\rho l^2}p + \frac{1}{4}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + 3\frac{\dot{\lambda}^2}{\lambda^3}\right) = c_1\frac{24}{\rho l^2}\left(\lambda^2 - \frac{1}{\lambda}\right)}$$

Linearizing

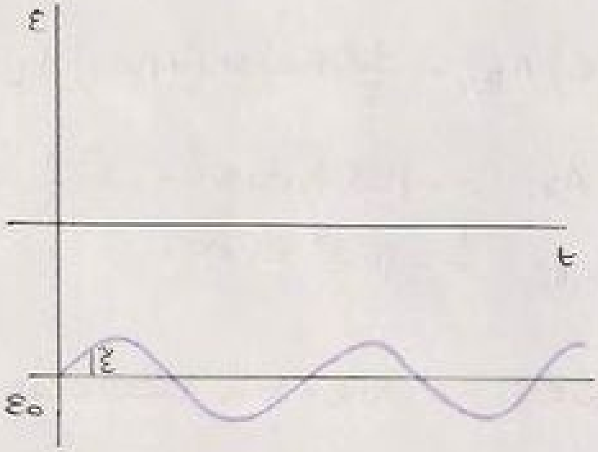
$$\boxed{-\frac{3}{2}\ddot{\varepsilon} = \frac{24}{\rho l^2}c_1 \cdot 3\varepsilon + \frac{12}{\rho l^2}p}$$

When $\ddot{\varepsilon} = 0 \Rightarrow 3 \cdot \frac{24}{\rho l^2}c_1\varepsilon = -\frac{12}{\rho l^2}p$

$$6c_1\varepsilon = -p \Rightarrow \varepsilon = -\frac{p}{6c_1}$$

what we get the first time

If $\varepsilon_0 = -\frac{p}{6c_1}$



If we consider $\varepsilon = \varepsilon_0 + \tilde{\varepsilon}$

$\ddot{\varepsilon} = \ddot{\tilde{\varepsilon}}$

We get

$$-\frac{3}{2}\ddot{\tilde{\varepsilon}} = 72\frac{c_1}{\rho l^2}(\varepsilon_0+\tilde{\varepsilon}) + \frac{12}{\rho l^2}p$$

$$-\frac{3}{2}\ddot{\tilde{\varepsilon}} = 72\frac{c_1}{\rho l^2}\varepsilon_0 + \frac{12}{\rho l^2}p + 72\frac{c_1}{\rho l^2}\tilde{\varepsilon}$$

since $\varepsilon_0 = -\frac{p}{6c_1}$ $\Rightarrow$ $-\frac{3}{2}\ddot{\tilde{\varepsilon}} = 72\frac{c_1}{\rho l^2}\tilde{\varepsilon}$

$$\boxed{\ddot{\tilde{\varepsilon}} = -48\frac{c_1}{\rho l^2}\tilde{\varepsilon}}$$

Same as before!

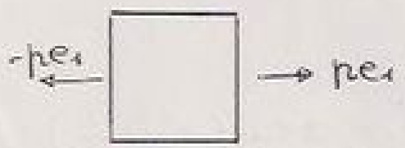
By the way this is not the right analysis, we have to press the body, let it oscillates around ε_0 (we get a stretch shape oscillating) Then we'll get that also the frequency will depend on ε_0.

We have to linearize these eqs around ε_0

$$\lambda = 1+\varepsilon\Big|_{\varepsilon=\varepsilon_0}$$

04.05.2011

<u>Review</u>



Neo-Hookean material $\quad p = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$ ⊛

$$T = \hat{T}(F) - pI + T^+ \qquad T^+ = d\dot{F}F^{-1} \quad d>0$$

(dissipation coefficient)

$$p - \frac{3}{2}d\frac{\dot{\lambda}}{\lambda} = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$-p - \frac{\rho l^2}{12}\lambda\ddot{\lambda} + \frac{\rho l^2}{48}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + 3\frac{\dot{\lambda}^2}{\lambda^3}\right) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

if $p=0$ we linearized the eqn $\quad \lambda = 1+\varepsilon,\ \dot{\lambda} = \dot{\varepsilon},\ \ddot{\lambda} = \ddot{\varepsilon}$

Then we found more useful to use $\quad \lambda = (1+\varepsilon_0)+\tilde{\varepsilon},\ \dot{\lambda} = \dot{\tilde{\varepsilon}},\ \ddot{\lambda} = \ddot{\tilde{\varepsilon}}$

51

set $\lambda_0 = 1 + \varepsilon_0 \qquad \lambda = \lambda_0 + \tilde{\varepsilon}$

$$\lambda\ddot{\lambda} = (\lambda_0 + \tilde{\varepsilon})\ddot{\tilde{\varepsilon}} = \lambda_0 \ddot{\tilde{\varepsilon}} + o(\tilde{\varepsilon})$$

$$\dot{\lambda}^2 = \dot{\tilde{\varepsilon}}^2 + o(\tilde{\varepsilon})$$

we assumed that $\tilde{\varepsilon}$ is a small quantity

$$\lambda^2 = (\lambda_0 + \tilde{\varepsilon})^2 = \lambda_0^2 + 2\lambda_0\tilde{\varepsilon} + o(\tilde{\varepsilon})$$

$$\lambda^{-1} = \lambda_0^{-1} + (\lambda^{-1})'\big|_{\lambda=\lambda_0}(\lambda - \lambda_0) + o(\tilde{\varepsilon})$$

$$= \lambda_0^{-1} - \frac{1}{\lambda_0^2}\tilde{\varepsilon} + o(\tilde{\varepsilon})$$

$$\lambda^{-2} = \lambda_0^{-2} - \frac{2}{\lambda_0^{3}}\tilde{\varepsilon} + o(\tilde{\varepsilon})$$

$$\ddot{\lambda}(\lambda^{-2}) = \lambda_0^{-2}\ddot{\tilde{\varepsilon}} + o(\tilde{\varepsilon})$$

$$-p - \frac{\rho l^2}{12}\lambda_0\ddot{\tilde{\varepsilon}} - \frac{\rho l^2}{24}\ddot{\tilde{\varepsilon}}\lambda_0^{-2} = 2c_1\left(\lambda_0^2 + 2\lambda_0\tilde{\varepsilon} - \lambda_0^{-1} + \frac{1}{\lambda_0^2}\tilde{\varepsilon}\right)$$

$$\boxed{\frac{\rho l^2}{12}\left(\lambda_0 + \frac{1}{2\lambda_0^2}\right)\ddot{\tilde{\varepsilon}} = -p - 2c_1\left(\lambda_0^2 - \frac{1}{\lambda_0}\right) - 2c_1\left(2\lambda_0 + \frac{1}{\lambda_0^2}\right)\tilde{\varepsilon}}$$

Note: we assumed that λ_0 is the sol'n of ⊛ i.e.

$$p = 2c_1\left(\lambda_0^2 - \frac{1}{\lambda_0}\right)$$

It is convenient to replace $p = 2c_1\left(\lambda_0^2 - \frac{1}{\lambda_0}\right)$

$$-2c_1\left(\lambda_0^2 - \frac{1}{\lambda_0}\right) - \frac{\rho l^2}{12}\lambda\ddot{\lambda} + \frac{\rho l^2}{48}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + 3\frac{\dot{\lambda}^2}{\lambda^3}\right) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$-\frac{\rho l^2}{12}\lambda\ddot{\lambda} + \frac{\rho l^2}{48}\left(-2\frac{\ddot{\lambda}}{\lambda^2} + \frac{\dot{\lambda}^2}{\lambda^3}\right) = 2c_1\left[\left(\lambda^2 - \frac{1}{\lambda}\right) + \left(\lambda_0^2 - \frac{1}{\lambda_0}\right)\right]$$

working on the RHS

$$(\lambda^2 + \lambda_0^2) - \left(\frac{1}{\lambda} + \frac{1}{\lambda_0}\right)$$

$$= \lambda_0^2 + 2\lambda_0\tilde{\varepsilon} + \lambda_0^2 - \left(\lambda_0^{-1} + \frac{1}{\lambda_0^2}\tilde{\varepsilon} + \frac{1}{\lambda_0}\right)$$

$$\Rightarrow \quad = 2\lambda_0^2 + 2\lambda_0\tilde{\varepsilon} - \left(\frac{2}{\lambda_0} + \frac{1}{\lambda_0^2}\tilde{\varepsilon}\right)$$

Then

$$\frac{\rho l^2}{12}\left(\lambda_0+\frac{1}{2\lambda_0^2}\right)\ddot{\varepsilon} = -2p - 2C_1\left(2\lambda_0+\frac{1}{\lambda_0^2}\right)\tilde{\varepsilon}$$

in a neighborhood of 1 $\quad 2p = 0$

and

$$\frac{3}{2}\ddot{\varepsilon} = -\frac{24C_1 3\varepsilon}{\rho l^2}$$

we should get that the frequency will depend on λ_0

SMALL OSCILLATIONS EQUATION $\quad \ddot{\varepsilon} = -48\frac{C_1}{\rho l^2}\varepsilon$

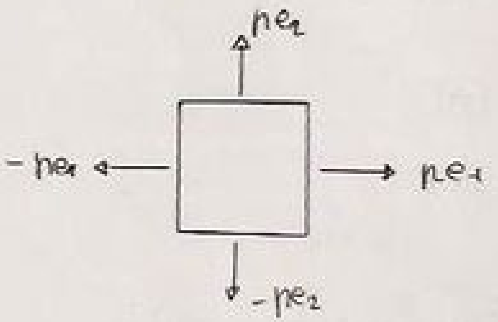
with frequency $\quad \omega = \sqrt{\frac{48}{\rho l^2}C_1} = \frac{4\sqrt{3}}{l}\sqrt{\frac{C_1}{\rho}}$

since we get something like

$$\ddot{\varepsilon} = -\omega^2\varepsilon$$

09.05.2011

Now we want to find another result when a force is applied to each of the six faces



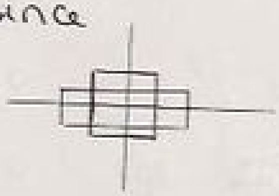
consider also this pressure on y_3 and y_{-3}

Then $\quad M = l^3 p I - \ddot{F}\bar{E}F^T$

$$\bar{E} = \rho\frac{l^5}{12}I$$

$$M = l^3 p I - \rho\frac{l^5}{12}\ddot{F}F^T$$

since



$$M = 2\lambda\frac{l}{2}e_1\otimes pe_1 l^2\frac{1}{\lambda} = l^3 p\, e_1\otimes e_1$$

$$M = l^3 p\,(e_1\otimes e_1 + e_2\otimes e_2 + e_3\otimes e_3)$$

$$\frac{M}{V} = pI - \rho\frac{l^2}{12}\ddot{F}F^T$$

where $[\ddot{F}F^T] = \begin{bmatrix} \lambda\ddot{\lambda} & 0 & 0 \\ 0 & (3\dot{\lambda}^2 - 2\lambda\ddot{\lambda})\frac{1}{4\lambda^3} & 0 \\ 0 & 0 & (3\dot{\lambda}^2 - 2\lambda\ddot{\lambda})\frac{1}{4\lambda^3} \end{bmatrix}$

52

Then

$$\sigma_{11} = p - \lambda\ddot{\lambda}\,\rho\,\frac{\ell^2}{12}$$

$$\sigma_{22} = \sigma_{33} = p - (3\dot{\lambda}^2 - 2\lambda\ddot{\lambda})\frac{1}{4\lambda^3}$$

we consider the characterization of the stress

$$\sigma_{11} = \hat{\sigma}_{11} - p$$

$$\sigma_{22} = \hat{\sigma}_{22} - p$$

$$\sigma_{33} = \hat{\sigma}_{33} - p$$

$$\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) \quad \boxed{}$$

and $\sigma_{11} - \sigma_{22} = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$

we do not have dependence on p! OUTER PRESSURE

If we don't have any acceleration $\ddot{\lambda} = 0$

$$\sigma_{11} - \sigma_{22} = 0 = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) \Rightarrow \lambda = 1$$

for $\boxed{\lambda = 1}$ $\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = 0 \quad \boxed{\star}$

what we know $T = \operatorname{dev} T + \operatorname{sph} T$

we consider this decomposition since we are assuming that $\hat{\sigma}$ has no sph part

$$\operatorname{sph} T := \frac{1}{3}(\operatorname{tr} T)I$$

$$\operatorname{sph} T = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} - 3p)I$$

$$\operatorname{dev} T := T - \frac{1}{3}(\operatorname{tr} T)I$$

$$[\operatorname{dev} T] = \begin{bmatrix} \sigma_{11} - \frac{1}{3}(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} - 3p) & 0 & 0 \\ 0 & \sigma_{22} - \frac{1}{3}(\Sigma\hat{\sigma}_{ii} - 3p) & 0 \\ 0 & 0 & \sigma_{33} - \frac{1}{3}(\Sigma\hat{\sigma}_{ii} - 3p) \end{bmatrix}$$

For $\lambda = 1$ there is no deformation thus we should get that $\sigma_{ii} = -p$ and $\hat{\sigma}_{ii} = 0$

$$\mathrm{dev}T_{11} = \hat{\sigma}_{11} - \cancel{p} - \frac{1}{3}(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33}) + \cancel{p}$$

$$= \frac{2}{3}\hat{\sigma}_{11} - \frac{1}{3}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = \frac{2}{3}\left(\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33})\right)$$

$$\boxed{*}$$

$$= 0$$

$$\mathrm{dev}T_{22} = \hat{\sigma}_{22} - \cancel{p} - \frac{1}{3}(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33}) + \cancel{p}$$

$$= \frac{2}{3}\hat{\sigma}_{22} - \frac{1}{3}(\hat{\sigma}_{11} + \hat{\sigma}_{33})$$

$$\mathrm{dev}T_{33} = \frac{2}{3}\hat{\sigma}_{33} - \frac{1}{3}(\hat{\sigma}_{11} + \hat{\sigma}_{22})$$

computing the sum can be helpful

$$\Rightarrow \mathrm{dev}T_{22} + \mathrm{dev}T_{33} = \frac{2}{3}\hat{\sigma}_{22} + \frac{2}{3}\hat{\sigma}_{33} - \frac{1}{3}(2\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33})$$

$$\stackrel{!}{=} \frac{1}{3}(\hat{\sigma}_{22} + \hat{\sigma}_{33} - 2\hat{\sigma}_{11})$$

By the balance eq. $\pi = \sigma_{ii}$ $i = 1,2,3$ and deviatoric part is zero since we are considering only the spherical part. Then

$$\begin{cases} \hat{\sigma}_{11} = \pi + p \\ \hat{\sigma}_{22} = \pi + p \\ \hat{\sigma}_{33} = \pi + p \end{cases} \quad \bullet$$

substituting in $\boxed{*}$

$$\pi + p - \frac{1}{\cancel{2}}\cancel{2}(\pi + p) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\Rightarrow 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) = 0 \quad \Rightarrow \lambda = 1$$

Now we want to go back and for $\boxed{\lambda = 1}$ we want to compute the components of the elastic stress $\hat{\sigma}_{ii}$: By $\bullet$ we get

$$\hat{\sigma}_{11} = \hat{\sigma}_{22} = \hat{\sigma}_{33}$$

this means trivially that $\boxed{\mathrm{dev}T_{11} = \mathrm{dev}T_{22} = \mathrm{dev}T_{33} = 0}$

Finally if $\boxed{\mathrm{dev}T = 0 \Rightarrow \text{the stress is spherical}}$

53

By assumption we assumed

$$T \cdot \dot{F}F^{-1} = \det \hat{T} \cdot \dot{F}F^{-1} = \frac{d}{dt}\varphi$$

thus the previous results come from this assumption!
by the incompressibility of the material

HOMEWORK #3
time dependence of the solution, inertial forces dissipation, p depending on time

$$[F] = \begin{bmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$$

10.05.2011

If we consider a fluid, $\hat{T}(F)$, that comes from the strain energy, vanishes

$$T = \underbrace{\hat{T}(F)}_{0} - pI + T^{+}$$

E.g. A plastic bag in the water. We can scale the bag without any force.

After, $\forall F$ with $\det F = 1$ $\quad \hat{T}(F) = \hat{T}_0$ then
F is said UNIMODULAR GROUP.

On the other way we can say that the strain energy for a fluid is such that

$$\varphi(F) = \varphi(U) = \varphi(I) \quad \text{is a constant}$$

For a fluid we will say that the stress is made of two parts

$$T = -pI + T^{+}$$

about the dissipation we know

$$T^{+} \cdot \dot{F}F^{-1} \geq 0$$

depends on the velocity gradient. Then the simplest choice that we can do for $T^{+} = d\dot{F}F^{-1}$
thus $d(\dot{F}F^{-1})(\dot{F}F^{-1}) \geq 0 \quad d \geq 0$

In the case of a fluid d is said VISCOSITY.
We can consider some bodies with $d=0$ even if in nature it is quite impossible to find it.
Otherwise we can consider a fluid at rest (no velocity), then $T^+=0$.

Consider the balance equation for an affine body

$$M = T V_R$$

For inertial forces

$$-\ddot{F}\bar{E}F^T \frac{1}{V} = T$$

$$-\ddot{F}\bar{E}F^T \frac{1}{V} = -pI + T^+$$

symmetric tensor

$$T^+ := d \operatorname{sym} \dot{F}F^{-1}$$

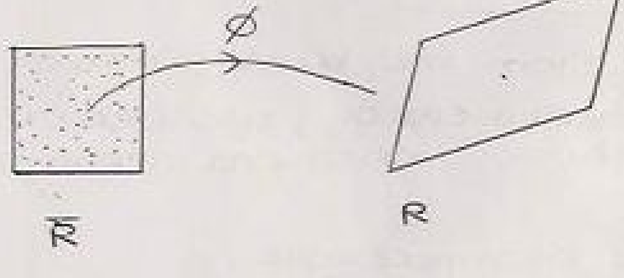
For Cauchy continuum

$$\operatorname{div} T + b = 0 \qquad b = -\rho a$$

$$\operatorname{div} T - \rho a = 0$$

$$\bar{a}(x,t) = \frac{d}{dt} v(\phi(x,t),t)$$

$$\bar{a}(x,t) = a(\phi(x,t),t)$$



$$\frac{d}{dt} v(\phi(x,t),t) = (\nabla v)v + v'$$

gradient times the tangent vector

derivative of v keeping position fixed

$$T = -pI + d \operatorname{sym} \dot{F}F^{-1}$$

this is in a vector field

$$-\operatorname{div}(pI) + d \operatorname{div}(\operatorname{sym} \nabla v) - \rho(\nabla v)v - \rho v' = 0$$

Recall that $\operatorname{div} T \cdot v = \operatorname{div}(T^T v) - T \cdot \nabla v$
restricted to uniform vector field $\operatorname{div} T \cdot e = \operatorname{div}(T^T e)$

Using that definition

$$\operatorname{div}(pI)\cdot e = \operatorname{div}((pI)e) = \operatorname{div}(pe)$$
$$\stackrel{!}{=} \operatorname{tr}(\nabla(pe)) = \nabla p\cdot e$$

usually symbol for viscosity

$$\nabla p - \mu \operatorname{div}(\operatorname{sym}\nabla v) + \rho(\nabla v)v + \rho v' = 0$$

$$\operatorname{div}(\operatorname{sym}\nabla v) = \frac{1}{2}\operatorname{div}(\nabla v + \nabla v^T) = \frac{1}{2}\operatorname{div}\nabla v + \frac{1}{2}\operatorname{div}\nabla v^T$$

$$(\operatorname{div}\nabla v + \operatorname{div}\nabla v^T)\cdot e = \operatorname{tr}\nabla(\nabla^T v\, e) + \operatorname{tr}\nabla(\nabla v\, e)$$

$$\boxed{\Delta v := \operatorname{div}\nabla v} \qquad \boxed{\operatorname{div}\nabla v^T = \nabla \operatorname{div} v}$$

$\nabla v = \dot{F}F^{-1}$ from the incompressibility $\det F = 1$

$\operatorname{tr}(\dot{F}F^{-1}) = 0$

Thus $\operatorname{div} v = \operatorname{tr}\nabla v = 0$ and $\operatorname{div}\nabla v^T = \nabla \operatorname{div} v = 0$.

In general we consider

$$d = 2\mu$$

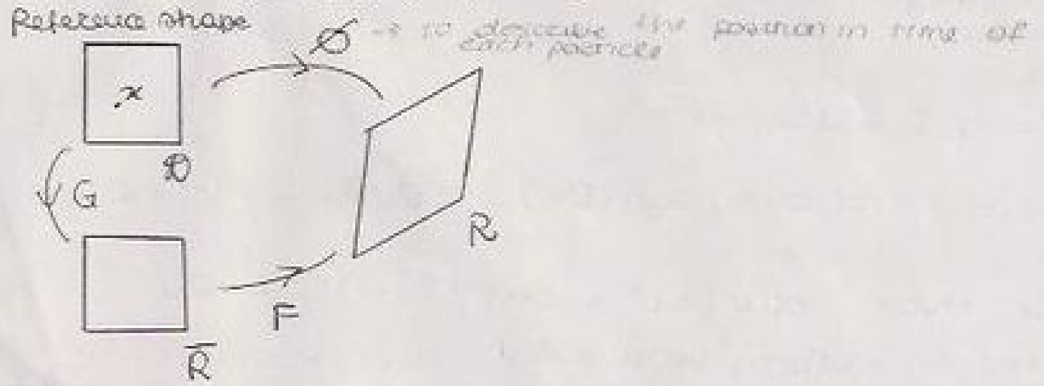
$$\nabla p - 2\mu\frac{1}{2}\Delta v + \rho(\nabla v)v + \rho v' = 0$$

$$\boxed{\nabla p - \mu\Delta v + \rho(\nabla v)v + \rho v' = 0}$$ NAVIER STOKES EQ

what we got comes from the basic principles already seen. We cancelled strain energy, restricted the motion to the isochoric motion and introduced the viscosity.

If the material is incompressible

$$T = \hat{T}(F) - pI + T^+$$



The shape of the body is described by ϕ.

G describes the transformation from $\mathcal{B}$ to $\overline{R}$.

IF $G = I$ $\quad F = \nabla\phi$ $\quad$ then $\quad GF = \nabla\phi$

IF $\nabla\phi = I$ there is just translation

if the shape does not change in time then $FG = I$

ϕ is called VISIBLE DEFORMATION

G describes the MICROSTRUCTURE of the body

IF $F = I$ $\quad \hat{T}(I) = 0$

$\overline{R} = R$ thus $\overline{R}$ is said 0-STRESS RELAXED SHAPE

If we observe a body that does not change in time, then the stress changes if F changes.

$$\boxed{\nabla\phi = FG}$$ KRÖNER - LEE DECOMPOSITION

Then we should be able to write the balance equations

$$W^{out} + W^{int} = 0 \qquad \forall \text{ test velocity field}$$

$$W^{out} = f \cdot v_0 + N \cdot L$$

$$W^{in} = -(z \cdot v_0 + T \cdot L)V_R$$

$G\overline{u} = \tilde{u} \qquad \overline{u} = G^{-1}\tilde{u}$

G is called REMODELING

if G depends on time

$\dot{G}\overline{u} = \dot{\tilde{u}} \qquad \dot{G}G^{-1}\tilde{u} = \dot{\tilde{u}}$

$V = \dot{G}G^{-1}$ defines the velocity of the relaxed shape

$\nabla\phi\,\overline{u} = u \Rightarrow \overline{u} = \nabla\phi^{-1}u$

$\nabla\dot{\phi}\,\overline{u} = \dot{u} \Rightarrow \nabla\dot{\phi}\,\nabla\phi^{-1}u = \dot{u}$

$$\nabla v\, u = \dot{u}$$

Thus $\quad \boxed{L = \nabla\dot{\phi}\,\nabla\phi^{-1}}$ $\quad$ VELOCITY GRADIENT

55

$$\mathbb{V} = \dot{G}G^{-1} \qquad \boxed{\mathbb{V}\tilde{u} = \dot{\tilde{u}}}$$

Then
$$\boxed{\begin{aligned} \mathcal{W}^{out} &:= f\cdot v_0 + M\cdot L + (Q\cdot\mathbb{V})V_{\bar{R}} \\ \mathcal{W}^{in} &:= -(\not{z}\cdot v_0 + T\cdot L)V_R - (A\cdot\mathbb{V})V_{\bar{R}} \end{aligned}} \quad \forall v_0, L, \mathbb{V}$$

and $Q\mathbb{V}$ and $A\cdot\mathbb{V}$ are said REMODELING COUPLES
OUTER INNER

If we assume $W^{out} + W^{in} = 0 \quad \forall v_0, L, \mathbb{V}$ we get

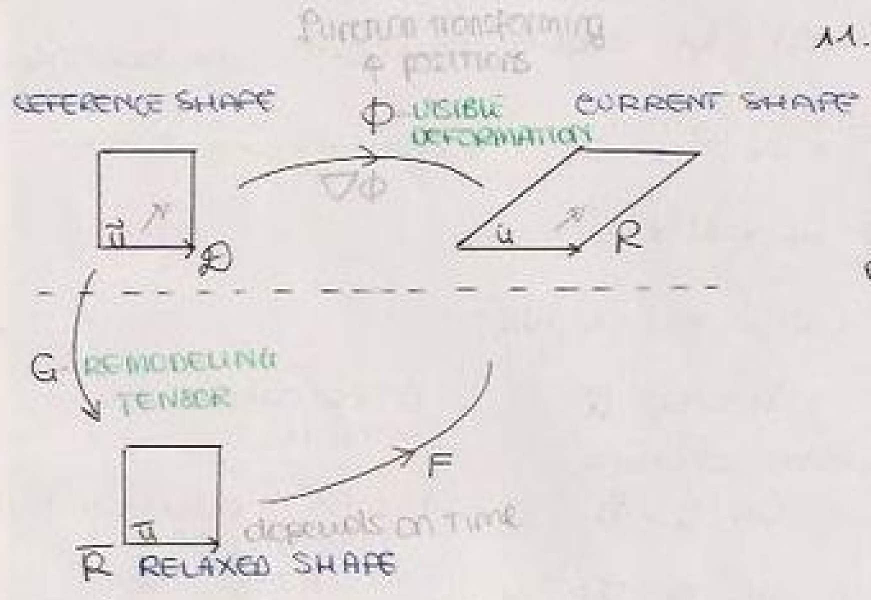
$$\boxed{\begin{aligned} f - \not{z}V_R &= 0 \\ M - TV_R &= 0 \\ Q - A &= 0 \end{aligned}}$$

BALANCE EQUATIONS FOR MODELING MATERIAL

Some characterizations

$$A = A^+ + \varphi(F)I$$
$$T = \hat{T}(F) - pI + T^+$$

11.05.2011



VISIBLE PART OF THE SCHEME
OBJECTIVITY

$$f\cdot v_0 + M\cdot L - T\cdot L\,V_R = 0$$

$$\int_R b\cdot v\,dV + \int_{\partial R} t\cdot v\,dA - \int_R T\cdot\nabla v\,dV = 0$$

$$\int_{\bar{R}} \bar{b}\cdot\bar{v}\,dV + \int_{\partial\bar{R}} \bar{t}\cdot\bar{v}\,dA - \int_{\bar{R}} S\cdot\nabla\bar{v}\,dV = 0$$

When $G = I$ then $F = \nabla\phi$ and we can use $\hat{T}(F)$

Now we can define what we have done for $\mathcal{R}$ also for $\mathcal{D}$.

$$f \cdot v_o + M \cdot L - T \cdot L V_R = 0$$

As in the affine motion: $p_A = p_o + \nabla\phi\,(\bar{p}_A - \bar{p}_o)$

$$\dot{p}_A = \dot{p}_o + \nabla\dot{\phi}\,(\bar{p}_A - \bar{p}_o) \quad \text{since } \phi(\bar{p}_A, t)$$

$$= \dot{p}_o + \underbrace{\nabla\dot{\phi}\,\nabla\phi^{-1}}_{\text{velocity gradient}}(p_A - p_o)$$

$\bar{u} = G\tilde{u}$

$\dot{\bar{u}} = \dot{G}\tilde{u} = \dot{G}G^{-1}\bar{u}$ — difference between velocities

REMODELING VELOCITY $\mathbb{V}$ — $\dot{G}G^{-1}$ velocity gradient of $\mathcal{B}$

action from the outside

$$f \cdot v_o + M \cdot L + (Q \cdot \mathbb{V}) V_{\bar{R}} = 0$$

$$-(z \cdot v_o + T \cdot L) V_R - (A \cdot \mathbb{V}) V_{\bar{R}} = 0 \qquad \forall v_o, \forall L, \forall \mathbb{V}$$

reaction coming from the inside — it is better to consider them as densities

Note that $\mathbb{V}$ can be independent on L.
Then

$$\nabla\phi = FG \quad \Rightarrow \quad \boxed{F := \nabla\phi\, G^{-1}} \quad \text{WARP}$$

Then the balance equations are

$$\boxed{\begin{aligned} f - z V_R &= 0 \\ M - T V_R &= 0 \\ Q - A &= 0 \end{aligned}}$$

Now let's look to the response function

$$T = \hat{T}(F) = R\,\hat{T}(U)\,R^T$$

The main consequences of the objectivity are

$$\begin{cases} z = 0 \\ \text{skw}\,T = 0 \end{cases}$$

56

Thus $f=0$

$\text{skw} M = 0$

$\text{sym} M - T V_R = 0$

$$T = \hat{T}(F) - pI + T^+$$

we saw that we can derive $\hat{T}(F)$ from the strain energy. If we consider that also in this time there is a kind of strain energy, we can say that there is a relation with the responsive function:

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(F).$$

We can consider the sum $(T\cdot L)V_R + (A\cdot\nabla)V_R$ and

$$S\cdot\dot{F} - \frac{d}{dt}\varphi(F) \geq 0$$

we assume that in general is positive

we can see it as the power of the external forces since it is balancing the internal forces

$$\underbrace{Q\cdot\nabla V_R + \dot{F}\cdot R M\cdot L}_{\text{external power}} = \underbrace{T\cdot L\,V_R + A\cdot\nabla V_R}_{\text{power of the stress}}$$

The idea is very simple, in the new scheme

$$T\cdot L V_R + A\cdot\nabla V_R - \frac{d}{dt}\left(\varphi(F)V_R\right) \geq 0$$

$S\cdot\dot{F}\,V_R$

density per unit volume

$$(T\cdot L)V_R - \frac{d}{dt}\left(\varphi(F)V_R\right) \geq 0$$

since it is true then it will be

$\left(S\cdot\dot{F} - \frac{d}{dt}\varphi(F)\right)V_R = 0$

$\Rightarrow T\cdot L\,V_R = S\cdot\dot{F}\,V_R$

in the standard scheme V_R should be the volume of the relaxed shape

$$(T \cdot L) V_R - \frac{d}{dt}(\varphi(F)) V_{\bar{R}} \geq 0$$

$$\boxed{S \cdot \dot{F} V_{\bar{R}} + A \cdot \mathbb{V} V_{\bar{R}} - \left(\frac{d}{dt}\varphi(F)\right) V_{\bar{R}} - \varphi(F)\frac{d}{dt} V_{\bar{R}} \geq 0}$$

Reminding that $V_{\bar{R}} = V_{\mathcal{B}} \det G$, then

$$\frac{d}{dt} V_{\bar{R}} = V_{\mathcal{B}} \frac{d}{dt} \det G \underset{\text{by} *}{=} V_{\mathcal{B}} \det G \, tr(\dot{G}G^{-1})$$

$$\frac{d}{dt} \det G = tr(\dot{G}G^{-1}) \det G \qquad *$$

in the end

$$\frac{d}{dt} V_{\bar{R}} = V_{\bar{R}} \, tr(\dot{G}G^{-1}).$$

this principle is a balance between external & internal power

becomes

$$S \cdot \dot{F} V_{\bar{R}} + A \cdot \mathbb{V} V_{\bar{R}} - \dot{\varphi}(F) V_{\bar{R}} - \varphi(F) V_{\bar{R}} \, tr(\dot{G}G^{-1}) \geq 0$$

$$S \cdot \dot{F} + A \cdot \mathbb{V} - \dot{\varphi}(F) - \varphi(F) \, tr(\dot{G}G^{-1}) \geq 0.$$

Recalling that in our assumptions

$$\hat{S}(F) \cdot \dot{F} = \frac{d}{dt}\varphi(F) \quad , \qquad \mathbb{V} = \dot{G}G^{-1}$$

and

$$tr(AB^T) = A \cdot B$$

We have

$$(S - \hat{S}(F)) \cdot \dot{F} + A \cdot \dot{G}G^{-1} - \varphi(F)\, \dot{G} \cdot G^{-T} \geq 0$$

But $\quad \dot{G} \cdot G^{-T} = \dot{G}G^{-1} \cdot I$

57

Finally we get

$$\underbrace{(S-\hat{S}(F))}_{S^+} \cdot \dot{F} + \underbrace{(A - \varphi(F)I)}_{A^+} \cdot \dot{G}G^{-1} \geq 0$$

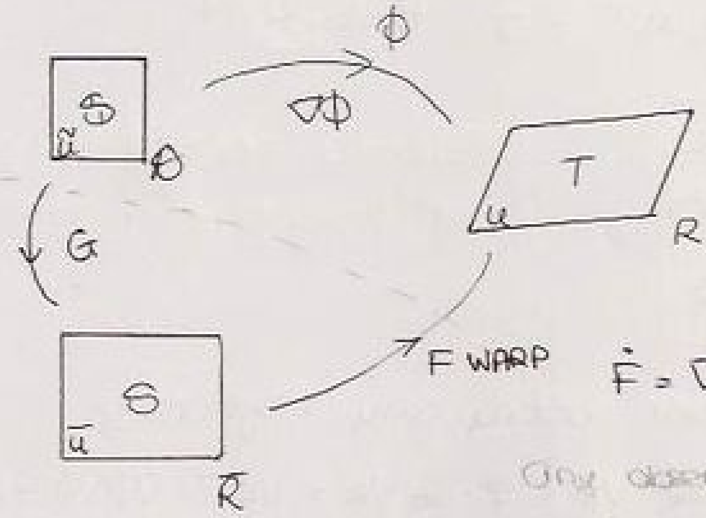
thus

$$S^+ = S - \hat{S}(F)$$

$$A^+ = A - \varphi(F)I$$

$$\boxed{S^+ \cdot \dot{F} + A^+ \cdot \dot{G}G^{-1} \geq 0}$$ For any motion G, F at any time

16.05.2011



KRÖNER-LEE DECOMPOSITION

$$FG = \nabla\Phi$$

$$F = \nabla\Phi\, G^{-1}$$

$$\nabla v = \nabla\dot{\Phi}\, \nabla\Phi^{-1}$$

$$\dot{F} = \nabla\dot{\Phi}\, G^{-1} - \nabla\Phi\, G^{-1}\dot{G}\, G^{-1}$$

any observer does not see anything about this
i.e. we'll not see any change about $\mathbb{V}$: $\mathbb{V}^* = \mathbb{V}$

$$W^{out} + W^{in} = 0$$

$$W^{out} = f \cdot v_0 + M \cdot L$$

$$W^{in} = -(\ell \cdot v_0 + T \cdot L)\, V_R$$

$$\bar{u} = G\tilde{u} \qquad \tilde{u} = G^{-1}\bar{u} \qquad \text{if } G \text{ depends on time}$$

$$\dot{\bar{u}} = \dot{G}\tilde{u} \qquad \dot{\bar{u}} = \dot{G}G^{-1}\bar{u}$$

$$\nabla\Phi\, \tilde{u} = u \quad \rightarrow \quad \tilde{u} = \nabla\Phi^{-1} u$$

$$\nabla\dot{\Phi}\, \tilde{u} = \dot{u} \quad \Rightarrow \quad \nabla\dot{\Phi}\, \nabla\Phi^{-1} u = \dot{u}$$

Thus $\boxed{L = \nabla\dot{\Phi}\, \nabla\Phi^{-1}}$ VELOCITY GRADIENT

$\boxed{\mathbb{V} = \dot{G}G^{-1}}$ REMODELING VELOCITY

Then

$$\boxed{\begin{aligned} W^{out} &:= f \cdot v_0 + M \cdot L + (Q \cdot \mathbb{V})\, V_{\bar{R}} \\ W^{in} &:= -(\ell \cdot v_0 + T \cdot L)\, V_R - (A \cdot \mathbb{V})\, V_{\bar{R}} \end{aligned}}$$

$$\forall v_0, L, \mathbb{V}$$

58

If we assume

$$W^{ext} + W^{in} = 0 \quad \forall v_0, L, \mathbb{V}$$

we get

$$\boxed{\begin{aligned} f - z V_R &= 0 \\ M - T V_R &= 0 \\ Q - A &= 0 \end{aligned}}$$

Balance equations for modeling materials

Computing the power along any trajectory

$$f \cdot v_0 + M \cdot \dot{F}F^{-1} + Q \cdot \dot{Q}Q^{-1} V_{\bar{R}} = z \cdot v_0 V_R + V_R T \cdot \nabla\dot{\Phi}\nabla\Phi^{-1} + A \cdot \dot{G}G^{-1} V_{\bar{R}}$$

by the objectivity $\mathbb{V}^* = \mathbb{V} \Rightarrow z = 0$

then

$$\underbrace{f \cdot v_0 + M \cdot \dot{F}F^{-1} + Q \cdot \dot{G}G^{-1} V_{\bar{R}}}_{\text{external power}} = \underbrace{V_R T \cdot \nabla\dot{\Phi}\nabla\Phi^{-1} + A \cdot \dot{G}G^{-1} V_{\bar{R}}}_{\text{internal stress}}$$

We assume that

$$T \cdot L \, V_R + A \cdot \mathbb{V} \, V_{\bar{R}} - \frac{d}{dt}\left(\varphi(CF) V_{\bar{R}}\right) \geq 0.$$

we can look at this as the power of the internal forces that it is balancing the external forces

Before going on with our computations it is useful to observe:

$$V_R T \cdot \nabla\dot{\Phi}\nabla\Phi^{-1} = V_{\mathcal{B}} \underbrace{(\det\nabla\Phi) T \nabla\Phi^{-T}}_{\mathcal{S}} \cdot \nabla\dot{\Phi}$$

where

$$\boxed{\mathcal{S} := (\det\nabla\Phi) \, T \, \nabla\Phi^{-T}}$$

Piola stress for $\mathcal{B}$

Finally we get

$$\boxed{V_{\mathcal{B}} \mathcal{S} \cdot \nabla\dot{\Phi} + A \cdot \dot{G}G^{-1} V_{\bar{R}} - \frac{d}{dt}\left(\varphi V_{\bar{R}}\right) \geq 0}$$

A RELATION BETWEEN S & T:

The velocity gradient in $\mathcal{R}$ is $\nabla v = \nabla\dot{\phi}\,\nabla\phi^{-1}$ thus

$$\nabla\bar{v} = \nabla v\, F = \nabla\dot{\phi}\,\nabla\phi^{-1} F$$

$$V_{\mathcal{R}}\, T \cdot \nabla\dot{\phi}\,\nabla\phi^{-1} = V_{\bar{\mathcal{R}}}\, S \cdot \nabla\dot{\phi}\,\nabla\phi^{-1} F \quad \bullet$$

i.e. $\quad V_{\mathcal{R}}\, T \cdot \nabla v = V_{\bar{\mathcal{R}}}\, S \cdot \nabla\bar{v}.$

By $\bullet$ using $V_{\mathcal{R}} = V_{\bar{\mathcal{R}}}\,(\det F)$

$$V_{\bar{\mathcal{R}}}\,(\det F)\, T \cdot \nabla v = V_{\bar{\mathcal{R}}}\,(\det F)\, T F^{-T} F^{T} \cdot \nabla v$$

$$\overset{!}{=} V_{\bar{\mathcal{R}}}\,(\det F)\, T F^{-T} \cdot \nabla v\, F$$

$$= V_{\bar{\mathcal{R}}}\,(\det F)\, T F^{-T} \cdot \nabla\bar{v}$$

$$\Rightarrow \quad S = T F^{-T} \det F = T \operatorname{cof} F$$

A RELATION BETWEEN S & $\mathbb{S}$:

$$V_{\mathcal{R}} = V_{\bar{\mathcal{R}}} \det F = V_{\mathcal{B}}(\det G)(\det F) = V_{\mathcal{B}}\,(\det \nabla\phi)$$

then

$$V_{\mathcal{R}}\, T \cdot \nabla v = V_{\mathcal{B}}\,(\det G)(\det F)\, T\, \nabla\phi^{-T} \cdot \nabla\dot{\phi}$$

$$= V_{\mathcal{B}}\,(\det G)\, S G^{-T} \cdot \nabla\dot{\phi} \qquad \text{since } F = \nabla\phi\, G^{-1}$$

$$\Rightarrow \qquad \mathbb{S} = (\det G)\, S G^{-T}.$$

What we attain is that it is correct to consider [illegible] and also to consider S or T. Which is the best choice? A way to make a choice could be how we relate the strain energy with the stress. $\varphi(F)$ is the density per unit volume $V_{\mathcal{B}}$ and if we assume that there is a process that $\bar{\omega}$ is growing then $\varphi(F) V_{\mathcal{B}}$ will not change.

What it is clear is that we have to consider the strain energy as the density per unit volume of the body.

59

Let's start from

$$V_0 \mathbb{S} \cdot \nabla\dot{\phi} + A \cdot \dot{G}G^{-1} V_R - \frac{d}{dt}(\varphi V_R) \geq 0$$

and observe that

$* \; \nabla\dot{\phi} = \dot{F}G + F\dot{G}$

$$* \; V_0 \mathbb{S} \cdot \nabla\dot{\phi} = V_0 \mathbb{S} \cdot \dot{F}G + V_0 \mathbb{S} \cdot F\dot{G}$$
$$= V_0 \mathbb{S} G^T \cdot \dot{F} + V_0 F^T \mathbb{S} \cdot \dot{G}G^{-1}G$$
$$= V_0 \mathbb{S} G^T \cdot \dot{F} + V_0 F^T \mathbb{S} G^T \cdot \dot{G}G^{-1}$$

$$* \; \frac{d}{dt}(\varphi V_0 \det G) = \frac{d}{dt}\varphi \; V_0 \det G + \varphi V_0 \frac{d}{dt} \det G$$
$$= \frac{d}{dt}\varphi \; V_0 \det G + \varphi V_0 \operatorname{tr}(\dot{G}G^{-1}) \det G$$

replacing them in our inequality

$$V_0 \mathbb{S} G^T \cdot \dot{F} + V_0 F^T \mathbb{S} G^T \cdot \dot{G}G^{-1} + A \cdot \dot{G}G^{-1}(V_0 \det G) -$$
$$-\left(\frac{d}{dt}\varphi\right) V_0 \det G - \varphi V_0 \det G \; \dot{G} \cdot G^{-T} \geq 0$$

since $\mathbb{S} = S G^{-T} \det G \Rightarrow \mathbb{S} G^T = S \det G$

$$V_0 \det G \left(S \cdot \dot{F} - \frac{d}{dt}\varphi\right) + V_0 F^T S \det G \cdot \dot{G}G^{-1} + A \cdot \dot{G}G^{-1}(V_0 \det G)$$
$$- \varphi V_0 (\det G) \; \dot{G}G^{-1} \cdot I \geq 0$$

$$V_0 \det G \left(S \cdot \dot{F} - \frac{d}{dt}\varphi\right) + V_0 \det G \left(F^T S + A - \varphi I\right) \cdot \dot{G}G^{-1} \geq 0$$

by our assumption $\hat{S}(F) \cdot \dot{F} - \frac{d}{dt}\varphi = 0$ we have

$$\boxed{(S - \hat{S}(F)) \cdot \dot{F} + (F^T S - \varphi(F) I + A) \cdot \dot{G}G^{-1} \geq 0}$$

$$\boxed{S^+ \cdot \dot{F} + A^+ \cdot \dot{G}G^{-1} \geq 0}$$ for any motion G, F

where

$S^+ = S - \hat{S}(F)$ dissipative stress

$A^+ = F^T S - \varphi(F) I + A$ ESHELBY TENSOR

dissipation related to the branch $\dot{G}$ or $\dot{F}$

A remodeling stress

17.05.2011

$$(S-\hat{S}(F))\cdot\dot{F}+(F^TS-\varphi+A)\cdot\dot{G}G^{-1}\geq 0 \qquad \Delta$$

$$S^+\cdot\dot{F}+A^+\cdot\dot{G}G^{-1}\geq 0$$

S^+ should be something that comes from the material, function of something that is called CONSTITUTIVE PROPERTY

$$S^+=\hat{S}^+(F,\dot{F},G,\dot{G})$$

then $\quad S=\hat{S}(F)+S^+$

Also $A^+=\hat{A}^+(F,\dot{F},G,\dot{G})$ and $A=-F^TS+\varphi I+A^+$

By Δ we can give a characterization for A.

$$A=\underbrace{\overset{\text{ESHELBY TENSOR}}{\left(-F^TS+\varphi I\right)}}_{\substack{\text{comes from}\\ \text{the visible world}}}+A^+$$

Since A^+ and S^+ depend on $F,\dot{F},G,\dot{G}$ then it is not possible to get two kind of conditions from Δ.

We introduced the ESHELBY TENSOR $F^TS-\varphi I$

The standard def'n for the LINEAR VISCOSITY is

$$T^+=\alpha\,\mathrm{sym}\,\dot{F}F^{-1}$$

Let us now have a look to S^+ and try to derive an expression for α. If we make the choice

$$S^+=\hat{S}^+(F,\dot{F})$$

$$A^+=\hat{A}^+(F,\dot{F})$$

then they will be independent on each other. It implies that

$$\boxed{\begin{array}{l} S^+\cdot\dot{F}\geq 0 \\ A^+\cdot\dot{G}G^{-1}\geq 0\end{array}} \quad \Longleftrightarrow \quad T^+\cdot\dot{F}F^{-1}\geq 0$$

60

We can try to write

$\square$ $\boxed{A^+ = a\,\dot{G}G^{-1}}$ $\to$ $a \geq 0$

since $\boxed{T^+ = d\,\dot{F}F^{-1}}$ comes from an <u>ISOTROPIC</u> dissipating stress.

Use $\square$ is limitating thus it is useful to consider

$$A^+ = \underset{\text{4 tensor}}{D}(\dot{G}G^{-1})$$

BTW
For our studies we'll consider $\square$

Let's consider a cube

with $F = \begin{bmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$

Now let's have a look to the balance equations

$$\boxed{\begin{array}{l} f = 0 \\ M = TV_R \\ Q = A \end{array}}$$

Then • $f = 0$ is satisfied.
• $M = TV_R$ is satisfied by

$$T = h\, e_1 \otimes e_1$$
$$A = Q$$

In general $Q = q\, e_1 \otimes e_1$ take the same expression of T, then if there are not interactions we can say that $Q = q\,I$. Thus

$$\boxed{\begin{cases} T = h\, e_1 \otimes e_1 \\ A = q\,I \end{cases}}$$

What we know is

$$T = \hat{T}(F) - pI + T^+$$

$$A = -F^T S + \varphi I + A^+$$

and

$$T^+ = d \operatorname{sym} \dot{F}F^{-1}$$

$$A^+ = a \dot{G}G^{-1}$$

We assume

$G = \gamma I$ and that γ behaves like a tensor

$\gamma > 0$

$$\dot{G}G^{-1} = \frac{\dot{\gamma}}{\gamma} I$$

$$Q \cdot \dot{G}G^{-1} = (qI) \cdot \left(\frac{\dot{\gamma}}{\gamma} I\right) = q \frac{\dot{\gamma}}{\gamma} (I \cdot I) = 3q \frac{\dot{\gamma}}{\gamma}$$

$$[G] = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

since $\nabla \phi = FG$

$$\det \nabla\phi = \det F \det G = \gamma^3$$

then the determinant of $\nabla\phi$ changes since it is not 1

Now we derive some expressions:

- $A^+ = a\dot{G}G^{-1} = a \frac{\dot{\gamma}}{\gamma} I$ a spherical tensor

$$[\dot{G}G^{-1}] = \frac{\dot{\gamma}}{\gamma} [I]$$

- $\varphi(F) = \varphi(C) = c_1 \operatorname{tr} C$ since $C = F^T F \Rightarrow \boxed{\varphi(F) = c_1\left(\lambda^2 + \frac{2}{\lambda} - 3\right)}$

- $\begin{cases} p\, e_1 \otimes e_1 = \hat{T}(F) - pI + T^+ \\ qI = -F^T S + \varphi I + A^+ \end{cases}$

$\hookrightarrow$ $S = \hat{S}(F) + S^+$

as 1st step we can neglect S^+ and T^+

inner pressure

$p\, e_1 \otimes e_1 = \hat{T}(F) - pI$

we consider dev since we have infos about dev $\hat{T}(F)$

$\mathrm{dev}(p\, e_1 \otimes e_1) = \mathrm{dev}(\hat{T}(F)) - \underbrace{\mathrm{dev}(-pI)}_{0}$

since it is spherical

since $\mathrm{dev}\, A := A - \frac{1}{3}\mathrm{tr}(A)\, I$

Considering the deviatoric part

$$p\, e_1 \otimes e_1 - \frac{1}{3} p\, I = \mathrm{dev}\, \hat{T}(F)$$

the matrix of this tensor is

$$\begin{pmatrix} \frac{2}{3}p & 0 & 0 \\ 0 & -\frac{1}{3}p & 0 \\ 0 & 0 & -\frac{1}{3}p \end{pmatrix} = \begin{pmatrix} \frac{2}{3}\sigma_{11} - \frac{1}{3}(\sigma_{22}+\sigma_{33}) & & \\ & \frac{2}{3}\sigma_{22} - \frac{1}{3}(\sigma_{11}+\sigma_{33}) & \\ & & \frac{2}{3}\sigma_{33} - \frac{1}{3}(\sigma_{11}+\sigma_{22}) \end{pmatrix}$$

$$[\hat{T}(F)] = \begin{bmatrix} \sigma_{11} & & \\ & \sigma_{22} & \\ & & \sigma_{33} \end{bmatrix}$$

$$\mathrm{dev}\, \hat{T}(F) = \hat{T}(F) - \frac{1}{3} \underbrace{\sum_{i=1}^{3} \sigma_{ii}}_{\mathrm{tr}\, \hat{T}(F)}$$

Then

$$\frac{2}{3} p = \frac{2}{3}\left(\sigma_{11} - \frac{1}{2}(\sigma_{22}+\sigma_{33})\right)$$

$$\Rightarrow p = \sigma_{11} - \frac{1}{2}(\sigma_{22}+\sigma_{33}) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

Now let's have a look to $qI = -F^T S + \varphi(F) I + A^+$

$S = T F^{-T} \det F \qquad \det F = 1 \Rightarrow S = TF^{-T}$

in general it is not symmetric $\Rightarrow F^T S = F^T T F^{-T}$

Question: Is $F^T S$ symmetric?

since qI is a spherical tensor as $\varphi(F)I$

thus we'll consider only the spherical part

$$qI = -\mathrm{sph}(F^T T F^{-T}) + \varphi(F) I + a \frac{\dot{\gamma}}{\gamma} I$$

In general

$$\operatorname{tr} A = I \cdot A$$

$$\operatorname{tr}(F^T T F^{-T}) = I \cdot F^T T F^{-T} = F \cdot T F^{-T}$$
$$= F F^{-1} \cdot T = \operatorname{tr} T$$

$$\operatorname{sph}(F^T T F^{-T}) = \operatorname{sph} T = -pI \quad \to \operatorname{tr} T = 0 - 3p$$
$$T = \hat{T}(F) - pI$$

Then

$$qI = pI + \varphi(F) I + a \frac{\dot{\gamma}}{\gamma} I$$

$$\boxed{q = p + \varphi(F) + a \frac{\dot{\gamma}}{\gamma}}$$

Now we should compute the value for p.

$h\, e_1 \otimes e_1 = \hat{T}(F) - pI$ if we consider the spherical part we got

trace trace $\operatorname{tr} \hat{T}(F) = 0$

$$\boxed{h = -3p}$$

$$q = -\frac{h}{3} + c_1\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + a \frac{\dot{\gamma}}{\gamma}$$

$$\begin{cases} h = 2c_1\left(\lambda^2 - \dfrac{1}{\lambda}\right) \\ a \dfrac{\dot{\gamma}}{\gamma} = \dfrac{1}{3} h - \underbrace{c_1\left(\lambda^2 + \dfrac{2}{\lambda} - 3\right)}_{\hat{\lambda}(h)} + q \end{cases}$$

What we are describing:
if q is independent of time then $\frac{1}{3}h - \hat{\lambda}(h)$ is constant wrt time

$$\frac{\dot{\gamma}}{\gamma} = \frac{1}{a} b(h, q)$$

solving $\quad \dot{\gamma} = \frac{b}{a}\gamma \quad \Rightarrow \gamma = \gamma_0 e^{\frac{b}{a}t}$

Note that we know that $a > 0$, but we don't know anything about b.

62

Substituting in ●

$$a\frac{\dot r}{r} = \underbrace{\frac{2}{3}c_1\left(\lambda^2-\frac{1}{\lambda}\right) - c_1\left(\lambda^2+\frac{2}{\lambda}\right)}_{\frac{2}{3}c_1\lambda^2-\frac{2}{3}c_1\frac{1}{\lambda}-c_1\lambda^2-\frac{2c_1}{3}+3c_1} + q$$

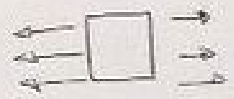
$$a\frac{\dot r}{r} = 2\cdot\frac{1}{3}c_1\left(-\frac{\lambda^2}{2}-\frac{4}{\lambda}+\frac{9}{2}\right)+q$$

if $q=0$ $(c_1>0)$ passive remodeling

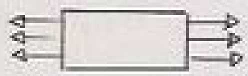
$$r = r_0 e^{\frac{b}{a}t}$$

where $b(\lambda) = \frac{2}{3}c_1\left(-\frac{\lambda^2}{2}-\frac{4}{\lambda}+\frac{9}{2}\right)$

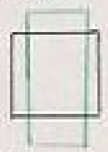
In standard case



the body will deform in this way

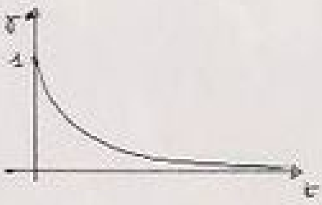


then the relaxed shape at time 0



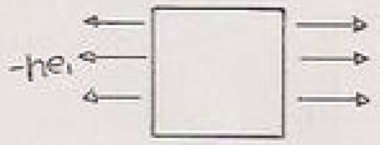
if $\lambda<1$ it will change in the green shape

if $r(0)=1$ if $\frac{b}{a}<0$



18.05.2011

BASIC PROBLEM



$$T = h\, e_1 \otimes e_1 \qquad T = \hat{T}(F) - pI$$

$$\operatorname{sph} T = \frac{1}{3} h I \qquad \operatorname{sph} T = 0 - pI$$

$$\varphi(F) = c \operatorname{tr} C = c\left(\lambda^2 + \frac{2}{\lambda}\right)$$

$$\varphi(F) = c(\operatorname{tr} C - 3)$$

$$h = 2c\left(\lambda^2 - \frac{1}{\lambda}\right) \qquad p = -\frac{1}{3} h$$

$$a\frac{\dot\gamma}{\gamma} = -p - \varphi(F) + q$$

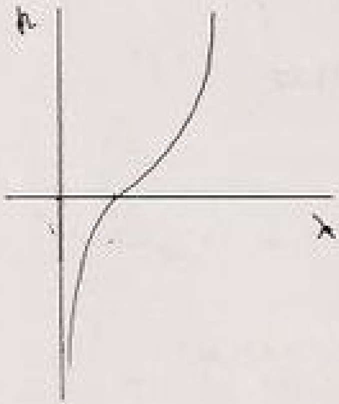
$$a\frac{\dot\gamma}{\gamma} = \frac{1}{3} h - \varphi(F) + q$$

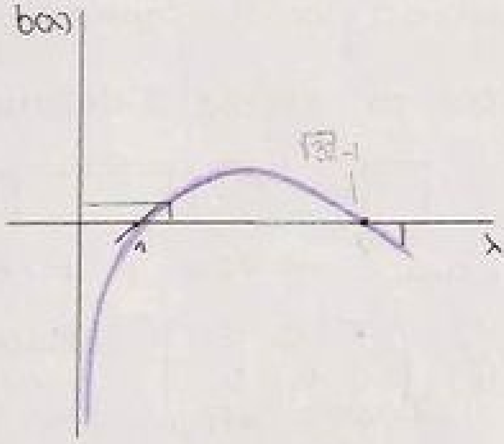
$$a\frac{\dot\gamma}{\gamma} = \frac{2}{3} c\left(\lambda^2 - \frac{1}{\lambda}\right) - c\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + q$$

$$= \frac{2}{3} c\left(\lambda^2 - \frac{1}{\lambda} - \frac{3}{2}\lambda^2 - \frac{3}{\lambda} + \frac{9}{2}\right) + q$$

$$a\frac{\dot\gamma}{\gamma} = \underbrace{\frac{2}{3} c\left(-\frac{\lambda^2}{2} - \frac{4}{\lambda} + \frac{9}{2}\right)}_{b} + q$$

$$a\frac{\dot\gamma}{\gamma} = b + q \qquad \frac{\dot\gamma}{\gamma} = \frac{b(\lambda)}{a} + \frac{q}{a}$$





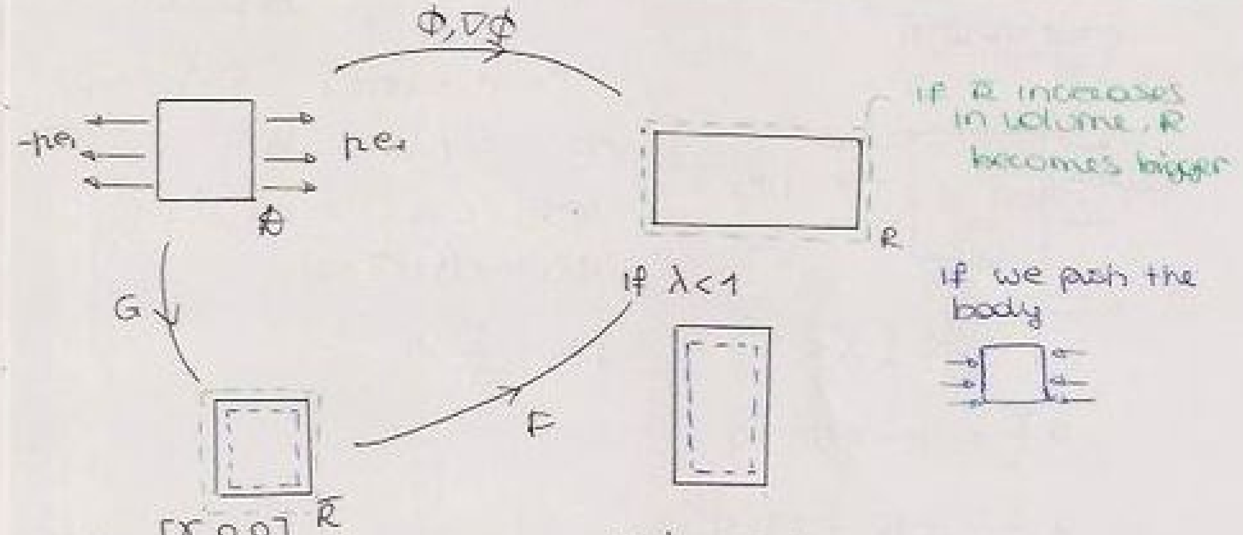
If we consider $\frac{q}{a} = 0$

$$b'(\lambda) = \frac{2}{3} c\left(-\lambda + \frac{4}{\lambda^2}\right)$$

then

$$\gamma(t) = \gamma_0 e^{\frac{b}{a} t}$$

63



$$[G] = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$$\nabla\phi = FG$$

$$[\nabla\phi] = \begin{pmatrix} \gamma\lambda & & \\ & \frac{\gamma}{\sqrt{\lambda}} & \\ & & \frac{\gamma}{\sqrt{\lambda}} \end{pmatrix}$$

$\lambda > 0$

γ starting from 1 is $\gamma > 1$

What happens if $\lambda > (\sqrt{2}-1)$

If instead we'll consider the force q

We want to describe a deformation such that

we want to have p fixed. Thus we try with

$$[G] = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1/\sqrt{\gamma} & 0 \\ 0 & 0 & 1/\sqrt{\gamma} \end{pmatrix}$$

ISOCHORIC
now we can consider the deviatoric part of the stress

USEFUL FOR:
change in description material but not in the volume of the material

$$[\dot{G}G^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\gamma}}{\gamma}$$

$$[\dot{F}F^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\lambda}}{\lambda}$$

$T = h\, e_1 \otimes e_1$

$T = \hat{T}(F) - pI + T^+$ we consider the case $T^+ = 0$

$\operatorname{dev}(T) = \operatorname{dev}(\hat{T}(F) - pI)$

$h\, e_1 \otimes e_1 - \frac{1}{3} h I = \operatorname{dev} \hat{T}(F)$

$h = \frac{2}{3}\left(\hat{\sigma}_{11} - \frac{1}{2}\left(\hat{\sigma}_{22} + \hat{\sigma}_{33}\right)\right)$

From the balance eqn $Q = A$, it is still true that $Q \cdot \dot{G}G^{-1} - A \cdot \dot{G}G^{-1} = 0$ where

$$Q = q\, \dot{G}G^{-1} = q \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \frac{\dot{\gamma}}{\gamma}$$

and $A = -F^T S + \varphi I + A^+$, $A^+ = a\, \dot{G}G^{-1}$

We see that $F^T S = F^T T F^{-T} = F^T(\hat{T}(F) - pI)F^{-T}$

$\overset{!}{=} F^T \hat{T}(F) F^{-T} - pI$

since $[F^T \hat{T}(F) F^{-T}] = \begin{bmatrix} \hat{\sigma}_{11} & & \\ & \hat{\sigma}_{22} & \\ & & \hat{\sigma}_{33} \end{bmatrix}$

then $[F^T S] = \begin{bmatrix} \hat{\sigma}_{11} - p & & \\ & \hat{\sigma}_{22} - p & \\ & & \hat{\sigma}_{33} - p \end{bmatrix}$

and $F^T S \cdot \dot{G}G^{-1} = (\hat{\sigma}_{11} - p)\frac{\dot{\gamma}}{\gamma} - \frac{1}{2}(\hat{\sigma}_{22} - p)\frac{\dot{\gamma}}{\gamma} - \frac{1}{2}(\hat{\sigma}_{33} - p)\frac{\dot{\gamma}}{\gamma}$

$= \left(\hat{\sigma}_{11} - \frac{1}{2}\left(\hat{\sigma}_{22} + \hat{\sigma}_{33}\right)\right)\frac{\dot{\gamma}}{\gamma}$.

Considering the deviatoric part of $Q \cdot \dot{G}G^{-1} - A \cdot \dot{G}G^{-1}$ we have

$$\left(q + \frac{q}{2}\right)\left(\frac{\dot{\gamma}}{\gamma}\right)^2 + \left(\hat{\sigma}_{11} - \frac{1}{2}\left(\hat{\sigma}_{22} + \hat{\sigma}_{33}\right)\right)\frac{\dot{\gamma}}{\gamma} - \left(a + \frac{q}{2}\right)\left(\frac{\dot{\gamma}}{\gamma}\right)^2 = 0$$

$$\frac{3}{2} q \frac{\dot{\gamma}}{\gamma} + \underbrace{\left(\hat{\sigma}_{11} - \frac{1}{2}\left(\hat{\sigma}_{22} + \hat{\sigma}_{33}\right)\right)}_{h} - \frac{3}{2} q \frac{\dot{\gamma}}{\gamma} = 0$$

64

i.e.

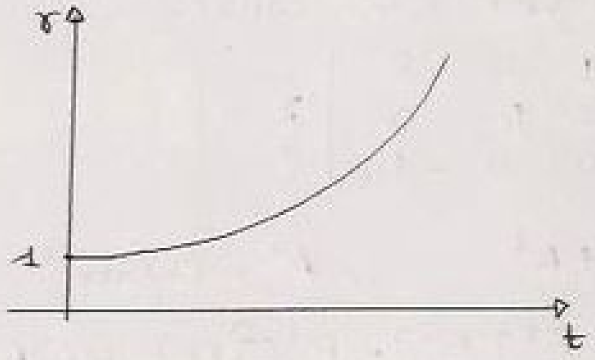
$$\frac{3}{2}(q-a)\frac{\dot r}{r} + p = 0$$

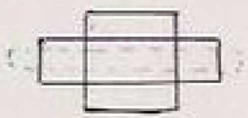
passive remodeling $\Rightarrow$ $q = 0$

$$\frac{\dot r}{r} = \frac{2}{3}\frac{p}{a}$$

$$r(t) = r_0 e^{\frac{2p}{3a}t}$$

if $r(0) = 1$





$$\nabla\phi = FG$$

$$[\nabla\phi] = \begin{bmatrix} \lambda r & & \\ & \frac{1}{\sqrt{r\lambda}} & \\ & & \frac{1}{\sqrt{r\lambda}} \end{bmatrix}$$

Slipping
SHEAR RELAXATION

23.05.2011

HOMEWORK 3

CHARACTERIZATION OF THE STRESS

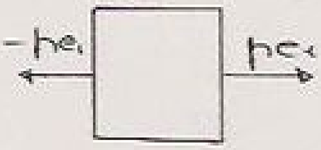
Subject:

Cylinder deformations

Incompressible Neo-Hookean material

Characterization of the stress: energetic, reactive and dissipative parts

Response of a visco-elastic material



$$[F] = \begin{bmatrix} \lambda & & \\ & 1/\sqrt{\lambda} & \\ & & 1/\sqrt{\lambda} \end{bmatrix}$$

Tasks:

Describe the evolution of the affine deformation of a visco-elastic body with a simple force distribution on the boundary independent on time.

Describe the small oscillations of an elastic body around un undeformed shape in the presence of inertial force, with and without viscosity.

Describe the small oscillations of an elastic body around a deformed shape (under the action of a fixed force distribution), in the presence of inertial force.

HOMEWORK 4

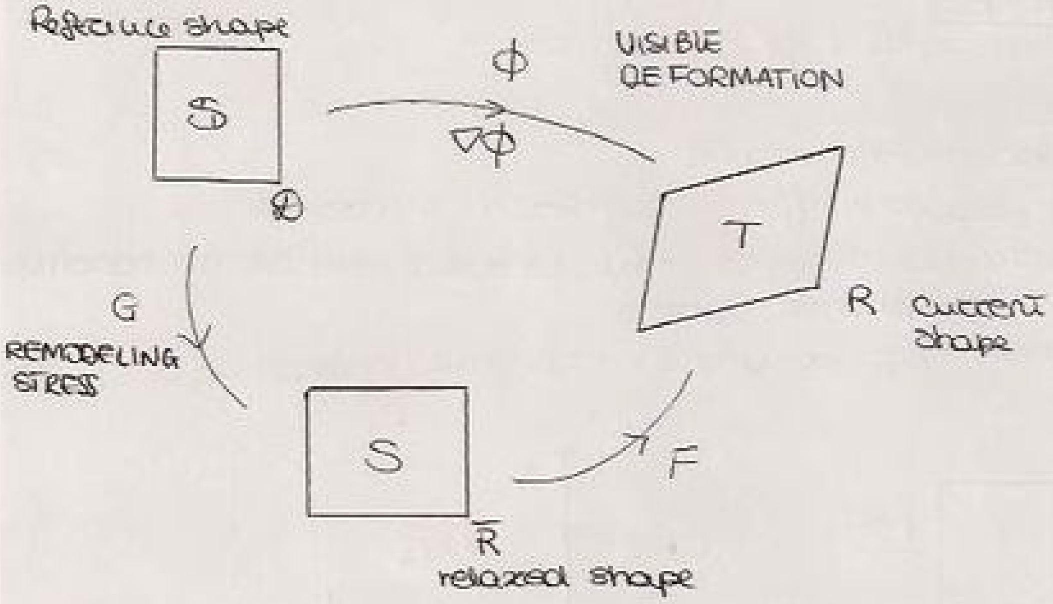
REMODELING

Subject:

Cylinder deformations and remodeling

Incompressible Neo-Hookean material

65

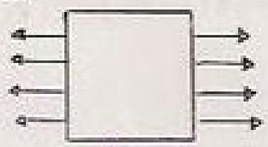


Tasks

1. Describe the relaxation of an elastic body generated by a passive isochoric remodeling
2. Describe the relaxation of an elastic body generated by a passive spherical remodeling
3. Compute the stress in an elastic body, in the shape of a parallelepiped with two opposite faces constrained to be a fixed distance, under the action of an increasing spherical remodeling.

HINTS

1.



$$[F] = \begin{bmatrix} \lambda & & \\ & 1/\sqrt{\lambda} & \\ & & 1/\sqrt{\lambda} \end{bmatrix}$$

$$[G] = \begin{bmatrix} \gamma & & \\ & 1/\sqrt{\gamma} & \\ & & 1/\sqrt{\gamma} \end{bmatrix} \quad \text{SHEAR RELAXATION}$$

From $\begin{cases} f = 0 \\ M = TVR \\ Q = A \end{cases}$

Remember that $T^+ = 0 \Rightarrow T = \hat{T}(F) - pI$

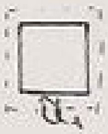
PASSIVE REMODELING $\Rightarrow Q = 0$

2. Spherical remodeling $[G] = \gamma [I]$

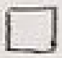
3. $[G] = \gamma [I]$ and we assume we can control the value of G.

A cube is constrained in two rigid walls

I don't use any force but $\bar{R}$ is changed



If we apply $G = \gamma I$ and l is the initial length, then γl is  and the volume $\det G = \gamma^3 l^3$.

Once we've stretched  we want to shrink  in order to put it in the two walls, i.e. we have to find which is the G s.t.

$$G\tilde{u}_1 = \bar{u}_1$$

CONSTRAINT $\boxed{FG\tilde{u}_1 = \tilde{u}_1}$

$$\tilde{u}_1 = l e_1 \rightarrow G(l e_1) = \gamma l e_1$$

$$F(\gamma l e_1) = \gamma \lambda l e_1$$

$$\Rightarrow \gamma\lambda l e_1 = l e_1 \Rightarrow \boxed{\gamma\lambda = 1} \; ※$$

Thus

$$FG\tilde{u}_2 = FG(l e_2) = l\, FG e_2 = l \gamma F e_2$$

$$\overset{!}{=} l \frac{\gamma}{\sqrt{\lambda}} e_2$$

$$\text{by ※} \overset{!}{=} l \gamma \sqrt{\gamma}\, e_2$$

and

$$F G \tilde{u}_3 = F G (l \ \ _3) = l F (r e_3) = l \, r \sqrt{\gamma} \, e_3$$

$$\text{Volume} = (l \, r \sqrt{\gamma})^2 l = \gamma^3 l^3 \overset{!}{=} \det G$$

What about the stress?

$$T = \hat{T}(F) - pI + T^+$$

$$\operatorname{dev} T = \operatorname{dev} \hat{T}(F)$$

$$\operatorname{dev} \hat{T}(F) \cdot \dot{F} F^{-1} = \frac{d}{dt} \varphi(F)$$

For Neo-Hookean material we know

$$\hat{\sigma}_{11} - \frac{1}{2}(\hat{\sigma}_{22} + \hat{\sigma}_{33}) = 2c \left(\lambda^2 - \frac{1}{\lambda}\right)$$

knowing the values of λ we calculate $\operatorname{dev} \hat{T}(F)$

since

$$\operatorname{dev} T = \sigma_{11} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{2}{3}\left(\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})\right) \overset{!}{=} \frac{4}{3} c \left(\lambda^2 - \frac{1}{\lambda}\right)$$

Thus we can use $\frac{M}{V_e} = T$ to know M!

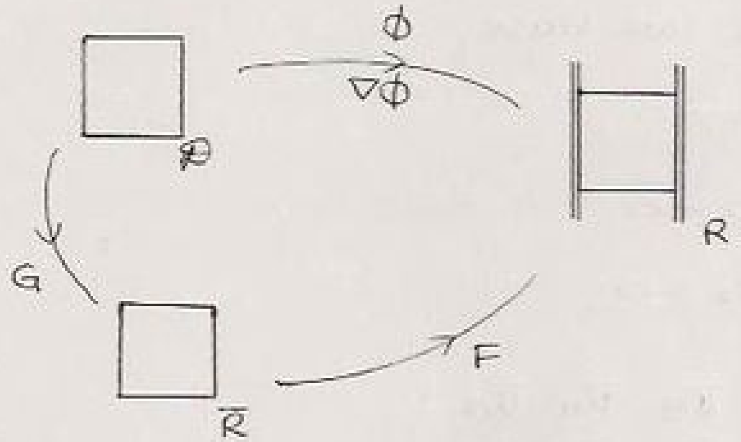
$$f = f^a + f^r$$

$$M = M^a + M^r$$

$$f^a \cdot v_0 + M^a \cdot L^v - T \cdot L^v V_e = 0$$

$$f^r \cdot v_0^v + M^r \cdot L^v = 0$$

24.05.2011



$T^+=0$

$A^+=0$

$$\begin{cases} f=0 \\ M=TV_R \\ Q=A \end{cases}$$

$G=\gamma I$

REACTIVE FORCES

Since $f=f^a+f^r$ then we can consider

$$M=M^a+M^r$$

By the balance equations
$$f^a+f^r=0$$
$$M^a+M^r=TV_R$$

<u>AIM</u>: characterize the reactive forces.

<u>Example</u>

A body on a surface

In the power the f and the M are related by (v_0, L)

$$v(x)=v_0+L(x-p_0)$$
$$v_{\mathcal{F}_{-2}}(x)=v_0+L\left(-\frac{\ell}{2}e_2+z_1e_1+z_3e_3\right)$$

Since the body cannot move along the face $\mathcal{F}_2$

$$\boxed{v_{\mathcal{F}_{-2}}(x)\cdot e_2=0}$$

$$\left(v_0+L\nabla\phi\left(-\frac{\ell}{2}e_2+z_1e_1+z_3e_3\right)\right)\cdot e_2=0$$

$$v_0\cdot e_2-\frac{\ell}{2}\underbrace{L\nabla\phi}_{\bar{L}}\,e_2\cdot e_2+z_1\underbrace{L\nabla\phi}_{\bar{L}}\,e_1\cdot e_2+z_3\underbrace{L\nabla\phi}_{\bar{L}}\,e_3\cdot e_2=0 \qquad \forall z_1,z_3$$

67

Note that $\nabla v = L$

$\nabla \bar{v} = L \nabla \phi$

letting $v_0 \cdot e_2 = v_{02}$ we have

$$\begin{cases} v_{02} - \frac{\ell}{2} \bar{L} e_2 \cdot e_2 = 0 \\ \bar{L} e_1 \cdot e_2 = 0 \\ \bar{L} e_3 \cdot e_2 = 0 . \end{cases}$$

Considering

$$[\bar{L}] = [\nabla \bar{v}] = \begin{pmatrix} \bar{v}_{11} & \bar{v}_{12} & \bar{v}_{13} \\ \bar{v}_{21} & \bar{v}_{22} & \bar{v}_{23} \\ \bar{v}_{31} & \bar{v}_{32} & \bar{v}_{33} \end{pmatrix}$$

we write

$$\begin{cases} v_{02} - \frac{\ell}{2} \bar{v}_{22} = 0 \\ \bar{v}_{21} = 0 \\ \bar{v}_{23} = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \overset{\vee}{v}_{02} = \frac{\ell}{2} \overset{\vee}{\bar{v}}_{22} \\ \overset{\vee}{\bar{v}}_{21} = 0 \\ \overset{\vee}{\bar{v}}_{23} = 0 \end{array}} \quad *$$

OBS: We have the object on the table, if:

- it stays on it $\Rightarrow$ reactive forces upward
- it moves $\Rightarrow$ the velocity field will be described by horizontal vectors, the forces will be vertical
  $\Downarrow$
  the power will be zero

CONCLUSION:

The characterization that we can give for reactive forces is that for a compatible vector field satisfying $*$ the power is zero

$$\boxed{f^r \cdot \overset{\vee}{v}_0 + M^r \cdot \overset{\vee}{L} = 0} \quad \bullet$$

After $f \cdot v_o + M \cdot L - T L V_R = 0 \qquad \forall v_o, \forall L$

$$(f^a + f^r) \cdot v_o^v + (M^a + M^r) \cdot L^v - T \cdot L^v V_R = 0$$

Then according to ● we get

$$f^a \cdot v_o^v + M^a \cdot L^v - T \cdot L^v V_R = 0 \qquad \forall v_o^v, \forall L^v$$

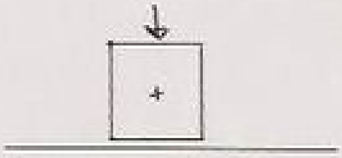
but I cannot say from this that

$$\begin{cases} f^a = 0 \\ M^a = T V_R \end{cases}$$

since there is a restriction on v_o^v and L^v.

Just to understand what we are doing

"If we consider the case



$$f^a = -p e_2 A_{y_2}$$

$$\frac{M^a}{V_R} = -p \frac{l}{2} e_2 \otimes e_2$$

and $f^a \cdot v_o^v = f^a \cdot \bar{v}_o^v = -p A_{y_2} e_2 \cdot \bar{v}_o^v = -p A_{y_2} \frac{l}{2} \bar{v}_{22}^v$

$$M^a \cdot L^v = M^a \cdot \bar{L}^v \nabla\phi^{-1} = \underbrace{M^a \nabla\phi^T}_{\bar{M}^a} \cdot \bar{L}^v$$

$$\bar{L} = L \nabla\phi$$

$$L = \bar{L} \nabla\phi^{-1}$$

$$[L^v] = \begin{bmatrix} \bar{v}_{11} & \bar{v}_{12} & \bar{v}_{13} \\ 0 & \bar{v}_{22} & 0 \\ \bar{v}_{31} & \bar{v}_{32} & \bar{v}_{33} \end{bmatrix}$$

$$M^a = (\nabla\phi (\tfrac{l}{2} e_2)) \otimes (-p e_2) A_{y_2}$$

$$M^a = \frac{l}{2} (\nabla\phi e_2 \otimes (-p e_2)) A_{y_2} = -p \frac{l}{2} A_{y_2} (\nabla\phi e_2 \otimes e_2)$$

by the rules of the tensor product $A u \otimes v = (u \otimes v) A^T$

$$M^a = -p A_{y_2} \frac{l}{2} (e_2 \otimes e_2) \nabla\phi^T$$

Thus

$$M^a \cdot L^v = -p A_{y_2} \frac{l}{2} (e_2 \otimes e_2) \cdot \bar{L}^v = -p A_{y_2} \frac{l}{2} \bar{v}_{22}$$

68

Now

$$T \cdot L^v V_R = (T \nabla\phi^{-T}) \cdot \bar{L}^v \det \nabla\phi V_{\mathcal{B}} = S \cdot \bar{L}^v V_{\mathcal{B}}$$

$$T \cdot L^v V_R = S \bar{L}^v V_{\mathcal{B}} = (\sigma_{11} \bar{v}_{11} + \sigma_{12} \bar{v}_{12} + \sigma_{13} \bar{v}_{13} + \sigma_{22} \bar{v}_{22} + \sigma_{31} \bar{v}_{31} + \sigma_{32} \bar{v}_{32} + \sigma_{33} \bar{v}_{33}) V_{\mathcal{B}}$$

We are considering the case

$$f^a \cdot v_0^v + M^a \cdot L^v - T \cdot L^v V_R = 0$$

collecting the same elts we get

$$\sigma_{11} = 0$$
$$\sigma_{12} = 0 = \sigma_{13} = \sigma_{31} = \sigma_{32} = \sigma_{33}$$

$$\bar{v}_{2} \left(-\rho A y_2 l - \sigma_{22} V_{\mathcal{B}} \right) = 0$$

Since $S = T \nabla\phi^{-T} \det F$

$$\frac{1}{\det F} S \nabla\phi^T = T$$

We could try to apply the balance principle

$$f^a \cdot v_0^v + M^a \cdot L^v - T \cdot L^v V_R = 0 \qquad L = \nabla v$$

we should assume that there are reactive forces

and that $f^r \cdot v_0^v + M^r \cdot L^v = 0$

The only condition that we have is $-T \cdot L^v V_R = 0$

by $\bar{L} = L \nabla\phi$

$$T \cdot \bar{L}^v \nabla\phi^{-1} V_R = 0$$

$$\det \nabla\phi \, T \nabla\phi^{-T} \cdot \bar{L}^v V_{\mathcal{B}} = 0$$

$$\Rightarrow \boxed{S \cdot \bar{L}^v = 0}.$$

Back to the pb p. 67 if $M^a = 0 \Rightarrow T \cdot L^v V_R = 0$

How is made L^v in this case? we know that the parallelepiped is constrained between two rigid walls $\Rightarrow L^v e_1 = 0$

$$v_{y_1} = v_0 + \bar{L}(x - \bar{p}_0) = v_0 + \bar{L}\left(\frac{\ell}{2}e_1 + z_2 e_2 + z_3 e_3\right)$$

$$v_{y_{-1}} = v_0 + \bar{L}\left(-\frac{\ell}{2}e_1 + z_2 e_2 + z_3 e_3\right)$$

If $v_{y_1} \cdot e_1 = 0 \Rightarrow 0 = v_{01} + \frac{\ell}{2}\bar{L}e_1 \cdot e_1 + z_2 \bar{L}e_2 \cdot e_1 + z_3 \bar{L}e_3 \cdot e_1$

$v_{y_{-1}} \cdot e_1 = 0 \Rightarrow v_{01} - \frac{\ell}{2}\bar{L}e_1 \cdot e_1 + z_2 \bar{L}e_2 \cdot e_1 + z_3 \bar{L}e_3 \cdot e_1 = 0$

$$v_{01} + \frac{\ell}{2}\bar{v}_{11} + z_2 \bar{v}_{12} + z_3 \bar{v}_{13} = 0$$

$$v_{01} - \frac{\ell}{2}\bar{v}_{11} + z_2 \bar{v}_{12} + z_3 \bar{v}_{13} = 0$$

$$\begin{cases} v_{01} + \frac{\ell}{2}\bar{v}_{11} = 0 \\ v_{01} - \frac{\ell}{2}\bar{v}_{11} = 0 \end{cases} \qquad \begin{cases} \bar{v}_{12} = 0 \\ \bar{v}_{13} = 0 \end{cases} \qquad \Rightarrow v_{01} = \bar{v}_{11} = \bar{v}_{12} = \bar{v}_{13} = 0$$

Thus

$$[\bar{L}] = \begin{pmatrix} 0 & 0 & 0 \\ \bar{v}_{21} & \bar{v}_{22} & \bar{v}_{23} \\ \bar{v}_{31} & \bar{v}_{32} & \bar{v}_{33} \end{pmatrix}$$

From ▲

$$[S] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S = T \nabla\phi^{-T} \det \nabla\phi$$

if we want to compute T starting from S

$$T = S \nabla\phi^T \frac{1}{\det \nabla\phi}$$

$$\nabla\phi = FG = F(\gamma I) \qquad \det \nabla\phi = \gamma^3$$

$$[T] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma\lambda & 0 & 0 \\ 0 & \frac{\gamma}{\sqrt{\lambda}} & 0 \\ 0 & 0 & \frac{\gamma}{\sqrt{\lambda}} \end{pmatrix} \frac{1}{\gamma^3} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

69

$\sigma_{11} = \sigma_{11} \dfrac{\lambda}{\gamma^2}$ $\quad \sigma_{ij} = 0 \quad \forall\, ij \neq 1$ since $[T]$ is symm

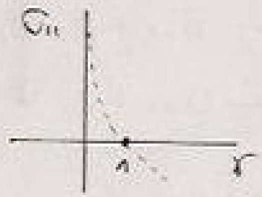
$T = \sigma_{11}\, e_1 \otimes e_1$

what we have found the last time is

$$\gamma\lambda = 1 \Rightarrow \lambda = \frac{1}{\gamma}$$

then $\sigma_{11} = 2c\left(\lambda^2 - \dfrac{1}{\lambda}\right)$

$= 2c\left(\dfrac{1}{\gamma^2} - \gamma\right)$



Now the moment will be made only on reactive forces.

↓

could be on the boundary on the right and left faces

1° interpretation of γ ; T increasing
2° " : biological tissue increasing volume

25.05.2011

Growth $G = \gamma I$

$T = \sigma_{11}\, e_1 \otimes e_1$

By the balance equation $A = Q$

$A = -F^T S + \varphi + A^+$

if $A^+ = 0$ we can compute A since we have all the unknowns

Now we can reverse the problem and start from $Q = A$ and deduce a value for γ and T.

$$Q = A$$

$$A = -F^T S + \varphi + A^+$$

$$\begin{cases} Q = -F^T S + \varphi + A^+ \\ M = T V_R \end{cases}$$

where $T = \hat{T}(F) - pI + T^+$.

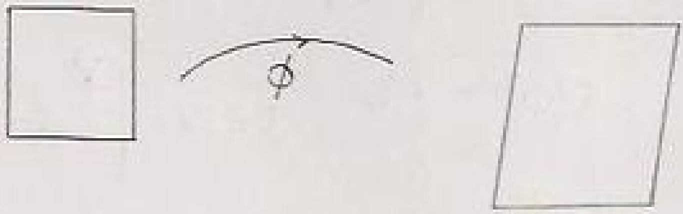
It is applicable to a current shape without constraints.

Considering the dissipations A^+, T^+ the sol'ns will be time dependent, something like

$$\begin{cases} \dot{r} = f(r, \lambda) \\ \dot{\lambda} = g(r, \lambda). \end{cases}$$

If we consider a time-dependent Q then $\dot{r} = f(r, \lambda, t)$.

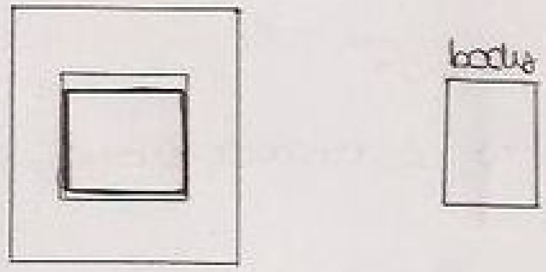
Consider the standard continuum model

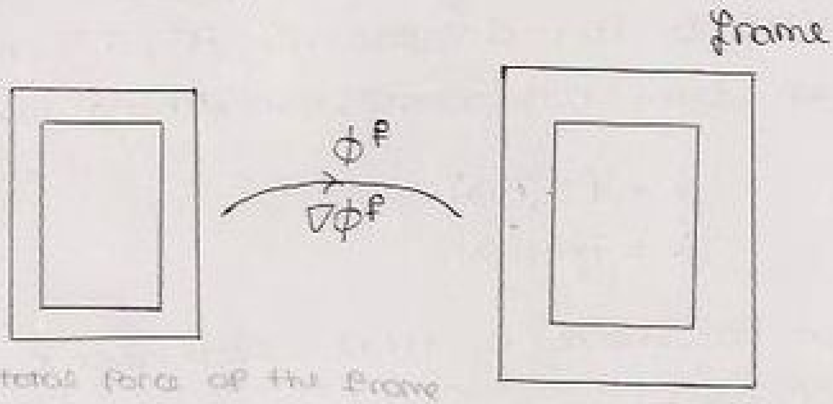


$\dot{\lambda} = g(\lambda, p)$

with inertial forces we have $\ddot{\lambda}, \dot{\lambda}$

We now consider another problem (an extension)
A frame containing a body that it is growing on time, if it is not rigid what does it happen?





total force of the frame

$f^f = 0$

$M^f = T^f V_{R^f}$

We suppose that there is no remodeling

$\nabla\phi^f = F^f \qquad [F^f]$

total force of the body

$f^b = 0$

$M^b = T^b V_{R^b}$

$\nabla\phi^b = F^b G$

$$[F^b] = \begin{bmatrix} \lambda^b & & \\ & \frac{1}{\sqrt{\lambda^b}} & \\ & & \frac{1}{\sqrt{\lambda^b}} \end{bmatrix}$$

Note that in the previous case by $G = \gamma I$

$\nabla\phi e_1 = e_1$

$\to \lambda\gamma = 1$

In this way $\nabla\phi^f e_1 = \nabla\phi^b e_1$

$$F^f e_1 = F^b G e_1$$

$$\boxed{F^f e_1 = F^b(\gamma I) e_1}$$

$$\boxed{\lambda^f = r\lambda^b} \quad \text{if } r=1 \Rightarrow \lambda^f = \lambda^b$$

The balance equation is

$$f\cdot v_o + M\cdot L - T\cdot L\, V_R + (Q\cdot\mathbb{V} - A\cdot\mathbb{V})\, V_{\bar{R}} = 0$$

now

$$\begin{cases} f^f\cdot v_o^f + M^f\cdot L^f - T^f\cdot L^f\, V_{Rf} = 0 \\ f^b\cdot v_o^b + M^b\cdot L^b - T^b\cdot L^b\, V_{Rb} + (Q-A)\cdot\mathbb{V}\, V_{\bar{R}b} = 0 \end{cases} \quad \boxed{*}$$

$$f^f\cdot v_o^f + f^b\cdot v_o^b + M^f\cdot L^f + M^b\cdot L^b - T^f\cdot L^f\, V_{Rf} - T^b\cdot L^b\, V_{Rb} + (Q-A)\cdot\mathbb{V}\, V_{\bar{R}b} = 0$$

If the velocities are indep we get $\boxed{*}$

O/W they are related and the body which is inside remains inside.

We consider the case $v_o = v_o^f = v_o^b$

$L = L^f = L^b$

$$(f^f + f^b)\cdot v_o + (M^f + M^b)\cdot L - (T^f V_{Rf} + T^b V_{Rb})\cdot L + (Q-A)\cdot\mathbb{V}\, V_{\bar{R}b} = 0$$

Then

$$\boxed{f^f + f^b = 0}$$

from the point of view of:

<u>frame</u>: total force of the reactive forces

<u>body</u>: reactive forces

$$\boxed{M^b + M^f = T^b V_{Rb} + T^f V_{Rf}}$$

71

QUESTION!

$$\underbrace{f^b + f^f}_{} = 0 \quad ?$$

force we applied from the outside — since we are not applying any forces thus condition is satisfied

thus $M^b + M^f = 0$

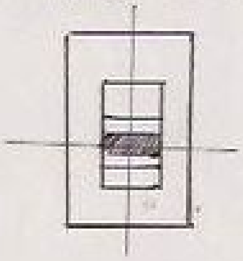
$$\Rightarrow T^b V_{R^b} + T^f V_{R^f} = 0$$

$$T^b V_{R^b} = -T^f V_{R^f}$$

T will depend on the elastic part i.e. on λ^b

$\lambda^f = \lambda^b r$ in this problem we assume that we know the value of r.

Calculate the stretch



$$\nabla\phi^b e_2 = F^b G e_2 = r \frac{1}{\sqrt{\lambda^b}} e_2$$

$$\nabla\phi^f e_2 = F^f e_2 = \frac{1}{\sqrt{\lambda^f}} e_2 = \frac{1}{\sqrt{r\lambda^b}} e_2$$

$$\Rightarrow \nabla\phi^b e_2 \neq \nabla\phi^f e_2$$

In which way the frame is pulled by the body inside?

if $r > 1$, $\lambda^b > 1$ $\Rightarrow$ the height of the frame will decrease

IF $\frac{r}{\sqrt{\lambda^b}} = \frac{1}{\sqrt{r\lambda^b}} \Rightarrow r = 1$

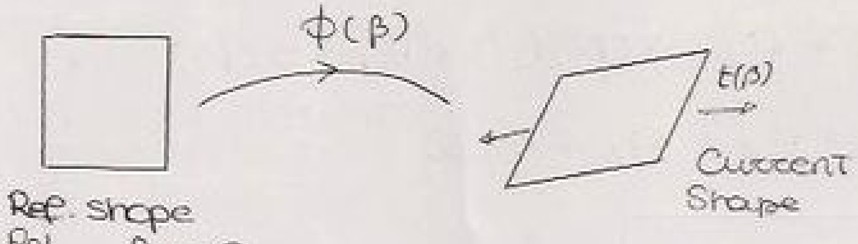
i.e. I can't find another sol'n for $r \neq 1$.

30.05.2011

LINEAR ELASTICITY

β measures the distance between the current shape and the reference shape

is not depending on time anymore



Rel $\quad f_\beta = 0$

Then $\quad M_\beta = T_\beta V_{R_\beta}$.

If $\beta = 0 \Rightarrow \phi = I \Rightarrow$ $F = I$, $U = I$ (closed to), $R = I$

$t(\beta)\big|_{\beta=0} = 0$

Then I want to write the Taylor expansion of F_β:

$$F_\beta = R_\beta U_\beta$$

$$R_\beta = I + \left(\frac{d}{d\beta} R_\beta\Big|_{\beta=0} \beta\right) + o(\beta) = I + \Theta_\beta + o(\beta)$$

$$U_\beta = I + \frac{d}{d\beta} U_\beta \Big|_{\beta=0} \beta + o(\beta) = I + E_\beta + o(\beta)$$

depends linearly on β

$$V_{R_\beta} = V_{\bar{R}} \det F_\beta$$

OBS: 1. Since U_β is symmetric then also $\frac{d}{d\beta} U_\beta$ will be symmetric i.e. $\boxed{E_\beta^T = E_\beta}$ symmetric tensor

- $R_\beta R_\beta^T = I \quad \forall \beta$

$$\left(\frac{d}{d\beta} R_\beta\right) R_\beta^T + R_\beta \left(\frac{d}{d\beta} R_\beta^T\right) = 0$$

$$\beta\left(\frac{d}{d\beta} R_\beta\right) R_\beta^T\Big|_{\beta=0} + R_\beta \left(\frac{d}{d\beta} R_\beta^T\right)\beta\Big|_{\beta=0} = 0$$

$$\Theta_\beta + \Theta_\beta^T = 0 \quad \Rightarrow \quad \boxed{\Theta_\beta = -\Theta_\beta^T} \text{ skew-symmetric tensor}$$

72

Then

- Θ_β is called INFINITESIMAL ROTATION
- E_β is called INFINITESIMAL STRETCH.

→ they have to be small quantities wrt the identity

Thus

$$F_\beta = (I + \Theta_\beta + o(\beta))(I + E_\beta + o(\beta))$$

$$* \quad F_\beta = I + E_\beta + \Theta_\beta + o(\beta)$$

→ here there is $\Theta_\beta E_\beta$ that contains 2nd order elts

$$\boxed{F_\beta - I = E_\beta + \Theta_\beta}$$

Using the fact that E_β is symmetric and Θ_β is skew-symmetric we have

$$\boxed{\begin{aligned} \text{sym}(F_\beta - I) &= \frac{1}{2}\left((F_\beta - I) + (F_\beta - I)^T\right) \\ \text{skw}(F_\beta - I) &= \frac{1}{2}\left((F_\beta - I) - (F_\beta - I)^T\right) \end{aligned}}$$

<u>Note</u>: $(F_\beta - I)$ has the same meaning of ∇v

i.e. $\quad \nabla v = E_\beta + \Theta_\beta$

Let's have a look to the quantity $\det F_\beta$;

we expand $\det F_\beta$ by Taylor expansion

$$\det F_\beta = 1 + \left(\frac{d}{d\beta} \det F_\beta\right)\beta + o(\beta)$$

we know

$$\frac{d}{d\beta}(\det F_\beta) = (\det F_\beta)\, tr\left(\left(\frac{d}{d\beta} F_\beta\right) F_\beta^{-1}\right) \beta \Big|_{\beta=0} = tr\left(\left(\frac{d}{d\beta} F_\beta\right)\beta\, F_\beta^{-1}\right)$$

$\left(\frac{d}{d\beta} F_\beta\right)\beta$ is a term of the Taylor expansion of F_β

Since F_β is $*$ then $(E_\beta + \Theta_\beta) = \left(\frac{d}{d\beta} F_\beta\right)\beta$

$$\frac{d}{d\beta}(\det F_\beta) = tr\left((E_\beta + \Theta_\beta) F_\beta^{-1}\right)$$

$$F_\beta^{-1} = I - E_\beta - \Theta_\beta + o(\beta)$$ easy check $F_\beta F_\beta^{-1} = I$

$$(E_\beta + \Theta_\beta)(I - E_\beta - \Theta_\beta + o(\beta)) = E_\beta + \Theta_\beta + o(\beta)$$

$$\frac{d}{d\beta}(\det F_\beta) = tr(E_\beta + \Theta_\beta) = tr\, E_\beta$$ since $tr\,\Theta_\beta = 0$

skew symm tensor

$$\boxed{\det F_\beta = 1 + tr\, E_\beta + o(\beta)}$$

Then we can say

$$\frac{V_R}{V_{\bar{R}}} = 1 + tr\, E_\beta \Rightarrow V_R = V_{\bar{R}}(1 + tr\, E_\beta)$$

$$\boxed{V_R = V_{\bar{R}} + V_{\bar{R}}\, tr\, E_\beta}$$

i.e. the current volume is the original volume plus the increased.

Now let's have a look to the moment tensor of the uniform distributions on each faces

$$M_\beta = V_R\,(u_1 \otimes t_1 + u_2 \otimes t_2 + u_3 \otimes t_3)$$

$$= (V_{\bar{R}} \det F_\beta)(F_\beta \bar{u}_1 \otimes t_1 + F_\beta \bar{u}_2 \otimes t_2 + F_\beta \bar{u}_3 \otimes t_3)$$

$$M_\beta = (V_{\bar{R}} \det F_\beta)(\bar{u}_1 \otimes t_1 + \bar{u}_2 \otimes t_2 + \bar{u}_3 \otimes t_3) F_\beta^T$$

$$M_\beta = \bar{M} F_\beta^T \det F_\beta$$

Replacing it in $M_\beta = T_\beta V_{R_\beta}$ we have

$$\bar{M}_\beta F_\beta^T \det F_\beta = T_\beta \underline{V_{\bar{R}}} \det F_\beta$$

moment on the original shape

does not depend on β

$$\boxed{\bar{M}_\beta F_\beta^T = T_\beta V_{\bar{R}}}$$

73

Note that we assumed that the forces $t_i(\beta)$ depend on β!

We expand $\overline{M}_\beta F_\beta^T = T_\beta V_{\overline{R}}$

$$F_\beta^T = I + E_\beta + \Theta_\beta$$

then $\overline{M}_\beta F_\beta^T = \overline{M}_\beta + \underbrace{\cancel{\overline{M}_\beta E_\beta}}_{\sim\beta^2} + \underbrace{\cancel{\overline{M}_\beta \Theta_\beta}}_{\sim\beta^2}$

thus

$$\boxed{\overline{M}_\beta = T_\beta V_{\overline{R}} + o(\beta)}$$

So we can compute $T_\beta = \dfrac{\overline{M}_\beta}{V_{\overline{R}}}$.

We recall that

$$\int_{R_\beta} T \cdot \nabla v \, dV = \int_{\overline{R}} (T \cdot \nabla v) \det F \, dV = \int_{\overline{R}} S \cdot \nabla \bar{v} \, dV$$

While consider small deformations, what about S_β?

$$S_\beta = T_\beta F_\beta^{-T} \det F_\beta$$
$$= T_\beta (I - E_\beta - \Theta_\beta)^T (1 + \operatorname{tr} E_\beta) + o(\beta)$$

$$\boxed{S_\beta = T_\beta + o(\beta)}$$

then there is no difference between Piola and Cauchy stress in linear elasticity.

We've found that

$$\int_{R_\beta} T \cdot \nabla v \, dV = \int_{\overline{R}} S \cdot \nabla \bar{v} \, dV$$

i.e. it is easier to calculate $\int_{\overline{R}}$ instead of $\int_{R_\beta}$.

After, if

$$T = \hat{T}(F_\beta) = \hat{T}(I + E_\beta + \Theta_\beta)$$

From the objectivity end we recall

$$\hat{T}(F) = R\,\hat{T}(U)R^T$$

$$\hat{T}(F_\beta) = (I + \Theta_\beta)\hat{T}(U_\beta)(I - \Theta_\beta)$$
$$= \hat{T}(U)(I - \Theta_\beta) + \cancel{\hat{T}(U_\beta)\Theta_\beta(I - \Theta_\beta)}$$
$$= \hat{T}(U_\beta) + o(\beta)$$

$$\boxed{\hat{T}(F_\beta) = \hat{T}(U_\beta) + o(\beta)}$$

Then

ELASTIC TENSOR

$$T = \hat{T}(F_\beta) = \hat{T}(I + E_\beta) = \mathbb{C}(E_\beta) + o(\beta)$$

linear function transforming the symm tensor into a symm tensor

The tensor T is a linear function of the infinitesimal stretch.

31.05.2011

$$R_\beta = I + \Theta_\beta + o(\beta)$$

$$U_\beta = I + E_\beta + o(\beta)$$

$$\hat{T}(F_\beta) = \hat{T}(I + E_\beta) = \tilde{T}(I) + \underbrace{\frac{d}{d\beta}\tilde{T}(E_\beta)\beta}_{\mathbb{C}(E_\beta)} + o(\beta)$$

where $\mathbb{C}: \underset{\mathbb{R}^6}{E} \to \underset{\mathbb{R}^6}{T}$ LINEAR ELASTICITY IS DESCRIBED BY THIS FUNCTION

If we look to the strain energy:
we have seen that $S_\beta = T_\beta + o(\beta)$, we know

$$\hat{S}(F) \cdot \dot{F} = \frac{d}{dt}\varphi(F)$$

In linear elasticity $\hat{S}(F) = \hat{T}(F) = \mathbb{C}(E)$,

$F_\beta = I + E_\beta + \Theta_\beta$ $\qquad \dot{F}_\beta = \dot{E}_\beta + \dot{\Theta}_\beta$

$$\mathbb{C}(E) \cdot (\dot{E}_\beta + \dot{\Theta}_\beta) = \frac{d}{dt}\varphi(E)$$

since $\varphi(F) = \varphi(U)$ by objectivity

77

Thus for symmetry

$$\mathbb{C}(E)\cdot\dot{E} = \frac{d}{dt}\varphi(E).$$

Look at $\frac{d}{dt}\varphi(E) = (\quad)\cdot\dot{E} \Rightarrow (\quad) = \mathbb{C}(E).$

Then we can guess that $\varphi(E) = \mathbb{C}(E)\cdot E$ and

$$\frac{d}{dt}\varphi(E) = \left(\frac{d}{dt}\mathbb{C}(E)\right)\cdot E + \mathbb{C}(E)\cdot\dot{E}$$

$$\frac{d}{dt}\mathbb{C}(E) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big(\mathbb{C}(E(t+\Delta t)) - \mathbb{C}(E(t))\Big)$$

by linearity of $\mathbb{C}$ $\overset{!}{=} \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{C}\Big(E(t+\Delta t) - E(t)\Big)$

$$\overset{!}{=} \lim_{\Delta t\to 0}\mathbb{C}\left(\frac{E(t+\Delta t) - E(t)}{\Delta t}\right)$$

$$\frac{d}{dt}\mathbb{C}(E) = \mathbb{C}(\dot{E})$$

$$\frac{d}{dt}\varphi(E) = \mathbb{C}(\dot{E})\cdot E + \mathbb{C}(E)\cdot\dot{E}$$

$$\overset{!}{=} \dot{E}\cdot\mathbb{C}^T(E) + \mathbb{C}(E)\cdot\dot{E}$$

by the symmetry of the scalar product $\overset{!}{=} \mathbb{C}^T(E)\cdot\dot{E} + \mathbb{C}(E)\cdot\dot{E} = (\mathbb{C}^T(E)+\mathbb{C}(E))\cdot\dot{E}$

If we change the guess saying

$$\varphi(E) = \frac{1}{2}\mathbb{C}(E)\cdot E$$

then we have

$$\frac{d}{dt}\varphi(E) = \frac{1}{2}\left(\mathbb{C}^T(E) + \mathbb{C}(E)\right)\cdot\dot{E}$$

OBS $\mathbb{C}(E) = \frac{1}{2}\left(\mathbb{C}^T(E) + \mathbb{C}(E)\right) = \frac{1}{2}(\mathbb{C}^T + \mathbb{C})(E)$

$$\Rightarrow \mathbb{C} = \frac{1}{2}(\mathbb{C}^T + \mathbb{C}) \Rightarrow \boxed{\mathbb{C} = \mathbb{C}^T}$$

i.e. $\mathbb{C}$ is a symmetric tensor (no than F, E, T)

What is the consequence of the symmetry of this tensor?

$e_1 \otimes e_1$, $e_2 \otimes e_2$, $e_3 \otimes e_3$

$$\left\{ \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1), \frac{1}{2}(e_2 \otimes e_3 + e_3 \otimes e_2), \frac{1}{2}(e_1 \otimes e_3 + e_3 \otimes e_1) \right\}$$

this is an orthonormal basis for $\mathbb{C}$.

$\{e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3\}$

since $(e_1 \otimes e_1) \cdot (e_2 \otimes e_2) = tr((e_1 \otimes e_1)(e_2 \otimes e_2))$

$(e_1 \otimes e_1)(e_2 \otimes e_2)u = (e_1 \otimes e_1)(e_2 \cdot u)e_2 = 0$

$(e_1 \otimes e_1) \cdot (e_1 \otimes e_1) = tr((e_1 \otimes e_1)(e_1 \otimes e_1))$

$(e_1 \otimes e_1)(e_1 \otimes e_1)u = (e_1 \otimes e_1)(e_1 \cdot u)e_1 = (e_1 \cdot u)e_1 \stackrel{!}{=} (e_1 \otimes e_1)u$

$\Rightarrow$ also this is an orthonormal basis for $\mathbb{C}$.

$$\underset{6\times 6}{[\mathbb{C}]} = \underset{6\times 6}{[\mathbb{C}]^T} \quad \rightarrow \quad \frac{36-6}{2} + 6 = 21$$

ELASTIC MODULUS

number of scalars we need to describe a body in linear elasticity

$\mathcal{G} = \{I\}$ $\quad \mathcal{G} = orth \rightarrow$ just 2 scalars

we need 21 scalars

$$\varphi(U) = \varphi(\lambda_1, \lambda_2, \lambda_3) = \tilde{\varphi}(l_1, l_2, l_3) \qquad C = U^2$$

Now let's have a look to what happens to the responsive function of an isotropic material

$$* \quad \boxed{\mathbb{C}(E) = \lambda(trE)I + 2\mu E}$$

Guardo nel Gurtin

75

where λ, μ are called LAMÉ MODULI.

scalars

Note that λ is not the stretch!!

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(\iota_1,\iota_2,\iota_3)$$

$$= \frac{\partial}{\partial \iota_1}\varphi\,\frac{\partial \iota_1}{\partial t} + \frac{\partial}{\partial \iota_2}\varphi\,\frac{\partial \iota_2}{\partial t} + \frac{\partial \varphi}{\partial \iota_3}\,\frac{\partial \iota_3}{\partial t}$$

$$\iota_1 = \operatorname{tr} C = F^T F\cdot I = F\cdot F$$

$$\frac{\partial \iota_1}{\partial t} = 2F\cdot\dot{F} = 2(I+E+\Theta)\cdot(\dot{E}+\dot{\Theta})$$

$$= 2(\dot{E}+\dot{\Theta}+E\cdot\dot{E}+\underbrace{\cancel{E\cdot\dot{\Theta}}+\cancel{\Theta\cdot\dot{E}}}_{\text{sym}\cdot\text{skw}=0}+\Theta\cdot\dot{\Theta})$$

$$= 2(I+E)\cdot\dot{E} + 2(I+\Theta)\cdot\dot{\Theta}$$

We will prove $*$ as a property of the material.

Since $\det F = 1 + \operatorname{tr}(E)$ when $\det F = 1 \Rightarrow$

$\operatorname{tr}(E)$ and $\mathbb{C}(E)$ is described only by $2\mu E$

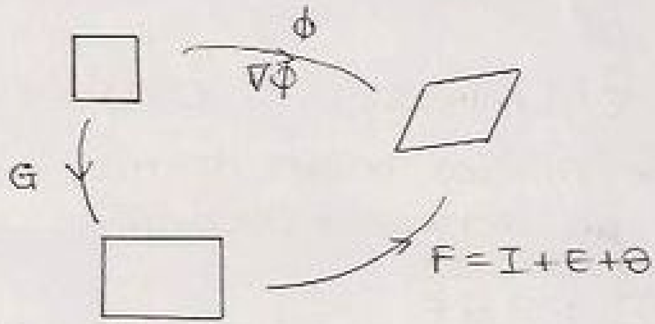
$$\operatorname{tr}T = 3\lambda(\operatorname{tr}E) + 2\mu\operatorname{tr}E = (3\lambda+2\mu)\operatorname{tr}E \quad ▲$$

since $T = \mathbb{C}(E) = \lambda(\operatorname{tr}E)I + 2\mu E$. From this

$$E = \frac{1}{2\mu}\left(T - \lambda(\operatorname{tr}E)I\right)$$

By ▲ $\operatorname{tr}E = \dfrac{\operatorname{tr}T}{3\lambda+2\mu}$ and

$$E = \frac{1}{2\mu}\left(T - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}T)I\right)$$



$$G = I + \tilde{E} + \tilde{\Theta}$$

Then $\nabla\phi = FG = I + E + \Theta + \tilde{E} + \tilde{\Theta}$

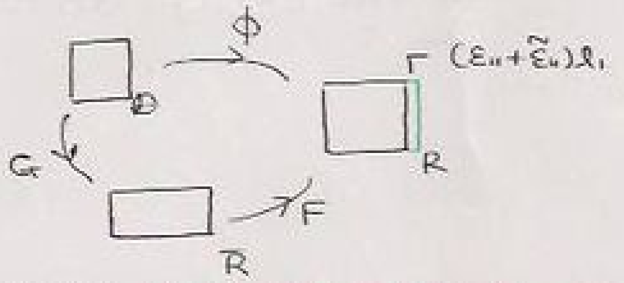
$$\stackrel{!}{=} I + (E + \tilde{E}) + (\Theta + \tilde{\Theta})$$

if they are sufficiently small, we neglect $o(\beta)$

$E + \tilde{E}$: VISIBLE INFINITESIMAL STRETCH

$\Theta + \tilde{\Theta}$: VISIBLE INFINITESIMAL ROTATION

$(E + \tilde{E})e_1 \cdot e_1 = \varepsilon_{11} + \tilde{\varepsilon}_{11}$ i.e. R will deform by $\tilde{\varepsilon}_{11}$



Then the TOTAL INFINITESIMAL STRETCH is $\boxed{E^* = E + \tilde{E}}$

The balance equations are

$$\begin{cases} f = 0 \\ \bar{M} = T\nu_{\mathcal{B}} \\ Q = A \end{cases} \Rightarrow T = \mathbb{C}(E) + T^+$$

$T^+ = \mathrm{d}\,\mathrm{sym}\,\dot{F}F^{-1}$

$\dot{F}F^{-1} = (\dot{E} + \dot{\Theta})(I - E - \Theta) = \dot{E} - \cancel{E\dot{E}} - \cancel{\Theta\dot{\Theta}} + \dot{\Theta}$

$\mathrm{sym}\,\dot{F}F^{-1} = \dot{E}$

$T^+ = \mathrm{d}\dot{E}$

then $T = \lambda(\mathrm{tr}E)I + 2\mu E + \mathrm{d}\dot{E}$

$Q = -F^T S + \varphi I + A^+$

$F^T S = (I + E - \Theta)T = T$

$\varphi = \frac{1}{2}\mathbb{C}(E)\cdot E\, I$

$T \to T \cdot E = o(\beta)$

For this reason in linear elasticity $A = -T + A^+$

76

$$A^+ = a\,\dot{G}G^{-1} = a\,(\dot{\mathring{E}} + \dot{\mathring{\Theta}})$$

When $Q=0$ there is no external remodeling, there is passive remodeling then

$$0 = -T + a\,(\dot{\mathring{E}} + \dot{\mathring{\Theta}}) \Rightarrow \operatorname{sym}(\mathbb{H}A^+) = 0$$

$$\begin{cases} 0 = -T + a\,\dot{\mathring{E}} \\ 0 = a\,\dot{\mathring{\Theta}} \end{cases}$$

there is an evolution of the stretch driven by T

the velocity is 0