$(19\text{-}20)_4$ Monday [2014-03-17] 16:00-18:00
A1.3

Velocity fields (vector fields)

$$v : \mathcal{R} \subset \mathcal{E} \to \mathcal{V}$$
$$\uparrow \phi$$
$$\bar{v} : \bar{\mathcal{R}} \to \mathcal{V}$$
$$\uparrow \kappa$$
$$\mathcal{B}$$

velocity gradient

$$c(h) = o + c_1(h)\, e_1 + c_2(h)\, e_2 + c_3(h)\, e_3$$

$$c'(0) = \lim_{h \to 0} \frac{1}{h}\Big(c(h) - c(0)\Big) = c_1'(0)\, e_1 + c_2'(0)\, e_2 + c_3'(0)\, e_3$$

scalar functions

$$v(c(h)) = v_1(c_1(h), c_2(h), c_3(h))\, e_1 + \cdots$$

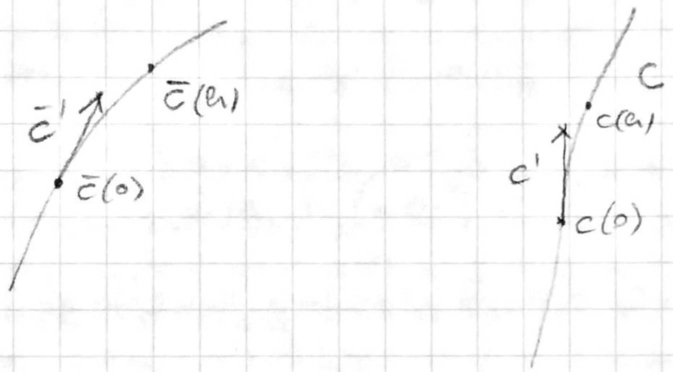
$$\lim_{h \to 0} \frac{1}{h}\Big(v(c(h)) - v(c(0))\Big) =$$

$$\partial_1 v_1\, c_1'\, e_1 + \partial_2 v_1\, c_2'\, e_1 + \partial_3 v_1\, c_3'\, e_1$$
$$+ \partial_1 v_2\, c_1'\, e_2 + \partial_2 v_2\, c_2'\, e_2 + \partial_3 v_2\, c_3'\, e_2$$
$$+ \partial_1 v_3\, c_1'\, e_3 + \partial_2 v_3\, c_2'\, e_3 + \partial_3 v_3\, c_3'\, e_3$$

$$= \nabla v\, c'$$

$$\begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ & & \\ & & \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \\ c_3' \end{pmatrix}$$

$$\uparrow$$
$$[\nabla v]$$



PULL BACK OF VECTOR FIELDS $\quad \bar{v}(\bar{c}(h)) = v(c(h))$

$$\frac{\bar{v}(\bar{c}(h)) - \bar{v}(\bar{c}(0))}{h} = \frac{v(c(h)) - v(c(0))}{h}$$

taking the limit $\quad \nabla\bar{v}\,\bar{c}' = \nabla v\, c'$

$$\nabla\bar{v}\,\bar{c}' = \nabla v\, F\,\bar{c}'$$

$$\Rightarrow \quad \nabla\bar{v} = \nabla v\, F$$

$$v(c(h)) = v(c(0)) + \nabla v\,(c(h) - c(0)) + o(h)$$

$$\frac{v(c(h)) - v(c(0))}{h} = \nabla v\,\frac{c(h) - c(0)}{h} + \frac{o(h)}{h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{o(h)}{h} = 0$$

(21-22)$_4$ Tuesday [2014-03-18] 9:00-11:00
A1.3

velocity field

$$v(c(h)) = v(c(0)) + \overset{L}{\nabla v}\,(c(h) - c(0)) + o(h)$$

$$c(h) = c(0) + F\,(\bar{c}(h) - \bar{c}(0)) + o(h)$$

$$L := \nabla v \; ; \quad F := \nabla \phi$$

let us differentiate w.r.t. time

$$\dot{c}(h) = \dot{c}(0) + \dot{F}\,(\bar{c}(h) - \bar{c}(0)) + o(h)$$

$$F^{-1}(c(h) - c(0)) = (\bar{c}(h) - \bar{c}(0)) + o(h)$$

$$\dot{c}(h) = \dot{c}(0) + \dot{F}\,F^{-1}(c(h) - c(0)) + o(h)$$

$$\Rightarrow \quad \nabla v = \dot{F} F^{-1}$$

$$\nabla \bar{v} = \nabla v\, F = \dot{F}$$

we assumed $v(c(h)) = \dot{c}(h) \mid v(c(h,t)) = \dot{c}(h,t)$

Differentiating again

$$\frac{d}{dt}\, v(c(h,t),t) = \nabla v\, \dot{c} + \frac{\partial}{\partial t} v(x,t)$$

fixed position

fixed trajectory

Acceleration from spatial velocity field

$$\bar{w}(\bar{c}(h)) = w(c(h))$$ pull-back of any vector field w

moving $c(h)$, fixed w

$$\bar{w}(\bar{c}(h)) = w(c(h,t))$$

$$\phi(\bar{c}(h),t)$$

velocity field $v(x,t)$

$$\bar{v}(\bar{c}(h),t) = v(c(h,t),t)$$

acceleration field

$$\bar{a}(\bar{c}(h),t) = \frac{d}{dt}\bar{v}(\bar{c}(h),t)$$

$$= \frac{d}{dt}v(c(h,t),t)$$

velocity field

$$\bar{v}(\bar{p}_A,t) = \frac{d}{dt}p_A(t)$$

$$= \dot{p}_A(t) = v(p_A(t),t)$$

acceleration field

$$\bar{a}(\bar{p}_A,t) = \frac{d}{dt}\bar{v}(\bar{p}_A,t)$$

$$= \frac{d}{dt}v(p_A(t),t)$$

$\bar{v}, \bar{a}$ referential description (Lagrangean)

v, a spatial description (Eulerian)

Time differentiation of the velocity field

$$\frac{d}{dt}\bar{v}(\bar{c}(h),t) = \frac{d}{dt}v(c(h,t),t)$$

$$= \underbrace{\nabla v}_{v \text{ field at time } t}\ \dot{c} + \underbrace{\frac{\partial}{\partial t}}_{\text{at a fixed position } c(h,t)} v$$

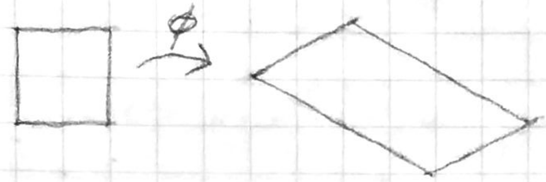
$\dot{c} = v(c(h,t),t)$

$$\bar{a}(\bar{c}(h),t) = a(c(h,t),t) = (\nabla v)v + \frac{\partial}{\partial t}v$$

$$a(x,t) = (\nabla v(x,t))\,v(x,t) + \frac{\partial}{\partial t}v(x,t)$$

$(23-24)_4$ Wednesday [2014-03-19]

Integrals, Jacobian, total volume, mass densities



$$\rho_0 \bar{V} \underset{\substack{\uparrow \\ \text{conservation of mass}}}{=} \rho V = (\rho \det F)\bar{V} \quad \Rightarrow \quad \overset{\substack{\text{initial value} \\ \downarrow}}{\rho_0} = \underbrace{\rho \det F}_{\text{constant quantity}}$$

$$\frac{d}{dt}\rho_0 = 0 \Rightarrow 0 = \dot{\rho} + \rho \operatorname{div} v$$

$$\int_{\mathcal{R}} \rho \, dV = \int_{\bar{\mathcal{R}}} \rho \det F \, dV$$

possibly changing in time

mass conservation

$$\int_{\bar{\mathcal{R}}} \rho_0 \, dV = \int_{\mathcal{R}} \rho \, dV$$

isochoric deformation $\det F = 1$

$$\nabla v = \dot{F} F^{-1}$$

$$\frac{d}{dt}(\det F) = (\det F) \operatorname{tr} \dot{F} F^{-1} = 0 \Rightarrow \operatorname{tr} \nabla v = 0$$

$$\operatorname{div} v = \operatorname{tr} \nabla v \qquad \operatorname{div} v = 0$$

$$\int_{\bar{\mathcal{R}}} dV = \operatorname{vol}(\bar{\mathcal{R}}) \; ; \quad \int_{\mathcal{R}} dV = \operatorname{vol}(\mathcal{R})$$

$$\int_{\mathcal{R}} dV = \int_{\bar{\mathcal{R}}} \det F \, dV$$

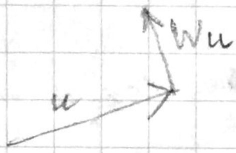
$$\frac{d}{dt}\int_{\mathcal{R}} dV = \frac{d}{dt}\int_{\bar{\mathcal{R}}} \det F \, dV = \int_{\bar{\mathcal{R}}} \frac{d}{dt} \det F \, dV = \int_{\bar{\mathcal{R}}} (\det F) \operatorname{tr} \nabla v \, dV$$

$$= \int_{\mathcal{R}} \operatorname{tr} \nabla v \, dV = \int_{\mathcal{R}} \operatorname{div} v \, dV = \int_{\partial \mathcal{R}} v \cdot n \, dA$$

$\nabla v = \dot{F}F^{-1}$

$\nabla v = \dot{R}R^T \in Skw$ $\qquad \frac{d}{dt}\, RR^T = \dot{R}R^T + R\dot{R}^T = 0$

SPIN $\quad W := \dot{R}R^T \qquad Wu \cdot u = 0 \qquad Wu \cdot u = -u \cdot Wu$



$Wa = \lambda a$

$Wa \cdot a = 0 \Rightarrow \lambda (a \cdot a) = 0$

$\Downarrow$

$\lambda = 0$

$W = -W^T$

$D \in Sym$

STRETCHING