$(25\text{-}26)_5$ Monday [2014-03-24] A1.3
16:00-18:00

$$W a = \lambda a \quad \Rightarrow \lambda = 0$$

$$\omega = a / \|a\|$$

$$W\omega = 0 \qquad \underset{\uparrow\ \mathrm{vol}(e_1, e_2, e_3) = 1}{\mathrm{vol}}(\omega, u, v) = \omega \times u \cdot v \qquad \forall v$$

$$\mathrm{vol}(w, u, v) = (\ast w) u \cdot v \qquad \forall w, \forall u, \forall v$$

Hodge star

$$\mathrm{vol}(\omega, u, v) = W u \cdot v$$

Power as a linear functional on velocity fields

Force $\quad \mathcal{F}(v) \quad$ representation

special form $\quad \mathcal{F}(v) = f_A \cdot v(p_A) + f_B \cdot v(p_B) + f_C \cdot v(p_C)$

more general $\quad \mathcal{F}(v) = \int_{\mathcal{R}} b(x) \cdot v(x)\, dV + \int_{\partial\mathcal{R}} t(x) \cdot v(x)\, dA$

velocity field

$$v(c(h)) = v(c(0)) + \nabla v(c(0)) \big(v(c(h)) - v(c(0))\big) + o(h)$$

$$v(x) = v_0 + \nabla v\,(x - p_0) \qquad \text{affine}$$

$$\nabla v = \dot{R} R^T = W \qquad \text{rigid}$$

$$W u_A \cdot f_A = \omega \times u_A \cdot f_A$$
$$= f_A \cdot \omega \times (p_A - p_0) = (p_A - p_0) \times f_A \cdot \omega$$

$$\mathcal{F}(v) = f_A \cdot v(p_A) + f_B \cdot v(p_B) + f_C \cdot v(p_C)$$
$$= f_A \cdot v_0 + f_A \cdot \omega \times (p_A - p_0) + \ldots$$
$$= f_A \cdot v_0 + (p_A - p_0) \times f_A \cdot \omega + \ldots$$

$$= \ldots = f \cdot v_0 + m \cdot \omega$$

$$\mathcal{F}(v) = \int_{\mathcal{R}} b(x) \cdot v(x) \, dV + \int_{\partial\mathcal{R}} t(x) \cdot v(x) \, dA$$

Bulk force (density with respect to the volume)

Traction (density with respect to the area)

per unit volume

per unit area

$$= \left(\int_{\mathcal{R}} b(x) \, dV + \int_{\partial\mathcal{R}} t(x) \, dA \right) \cdot v_0$$

$$+ \left(\int_{\mathcal{R}} (x - p_0) \times b(x) \, dV + \int_{\partial\mathcal{R}} (x - p_0) \times t(x) \, dA \right) \cdot \omega$$

$$= f \cdot v_0 + m \cdot \omega$$

$(27\text{-}28)_5$ Tuesday [2014-03-25]

example 1

$t(x) = 0$, $b(x) = \rho g$ $\quad$ g is a vector

$$f = \int_{\mathcal{R}} \rho g \, dV = \rho g \, Vol(\mathcal{R}) = \underbrace{\rho \, Vol(\mathcal{R})}_{\text{total mass}} g$$

$$m = \int_{\mathcal{R}} (x - p_0) \times (\rho g) \, dV = \int_{\mathcal{R}} (x - p_0) \, dV \times (\rho g)$$

$$\frac{1}{V} \int_{\mathcal{R}} (x - p_0) \, dV = (x_G - p_0)$$

$\uparrow$ barycenter of $\mathcal{R}$

if $p_0 = x_G$ then $m = 0$

example 2

$b(x) = 0$, $t(x) = \rho g$ $\quad$ ρ "skin" mass density

$$f = \int_{\partial\mathcal{R}} \rho g \, dA = \underbrace{\rho \, Area(\partial\mathcal{R})} g \; ; \; m = \int_{\partial\mathcal{R}} (x - p_0) \, dA \times (\rho g)$$

$$\frac{1}{A} \int_{\partial\mathcal{R}} (x - p_0) \, dA = (x_G - p_0)$$

$\uparrow$ barycenter of $\partial\mathcal{R}$

Balance principle

$$\mathfrak{F}(v) = 0 \quad \forall v$$

$$v(x) = v_0 + W(x - p_0)$$

$$\Rightarrow \quad \mathfrak{F}(v) = f \cdot v_0 + m \cdot \omega$$

$$\mathfrak{F}(v) = 0 \quad \forall v \quad \Leftrightarrow \quad f = 0, \; m = 0$$

v general continuous and differentiable vector field

$$\mathfrak{F}(v) = \int_{\mathcal{R}} b(x) \cdot v(x)\, dV + \int_{\partial\mathcal{R}} t(x) \cdot v(x)\, dV$$

$$\mathfrak{F}(v) = 0 \quad \forall v \quad \Rightarrow \quad b = 0, \; t = 0$$ no force is allowed

A convenient approach consists in setting

$$\mathfrak{F}(v) = \mathfrak{F}^{ext}(v) + \mathfrak{F}^{int}(v)$$

based on a suitable characterization of the two different parts, and still requiring

$$\mathfrak{F}(v) = 0 \quad \forall v$$

In order to have some insight into the way of characterizing the force distribution let us consider just the extension of rigid velocity fields to affine velocity fields:

$$v(x) = v_0 + L\,(x - p_0)$$

where L is the velocity gradient (a tensor)

$$\mathcal{F}^{ext}(v) = f_A \cdot v_A + f_B \cdot v_B + f_C \cdot v_C$$

$$= f_A \cdot v_0 + f_A \cdot L\,(p_A - p_0) + \dots$$

$$= f_A \cdot v_0 + (p_A - p_0) \otimes f_A \cdot L + \dots$$

$$= (f_A + f_B + f_C) \cdot v_0$$
$$+ \big((p_A - p_0) \otimes f_A + (p_B - p_0) \otimes f_B + (p_C - p_0) \otimes f_C \big) \cdot L$$

$$= f \cdot v_0 + M \cdot L$$

$(29\text{-}30)_5$ Wednesday [2014-03-26]

$$f_A \cdot L(p_A - p_0) = \underbrace{(p_A - p_0) \otimes f_A}_{= f_A \otimes (p_A - p_0)} \cdot L \qquad \otimes, \otimes$$

Gurtin

$$M \cdot L = tr(M^T L)$$

$$tr(AB) = \frac{vol(ABe_1, e_2, e_3) + \cdots}{vol(e_1, e_2, e_3)}$$

$$Be_i = b_{1i} e_1 + b_{2i} e_2 + b_{3i} e_3$$

any basis

$$Ae_i = a_{1i} e_1 + a_{2i} e_2 + a_{3i} e_3$$

$$\begin{aligned} ABe_i &= b_{1i} Ae_1 + b_{2i} Ae_2 + b_{3i} Ae_3 \\ &= b_{1i}(a_{11} e_1 + a_{21} e_2 + a_{31} e_3) \\ &\quad + b_{2i}(a_{12} e_1 + a_{22} e_2 + a_{32} e_3) \\ &\quad + b_{3i}(a_{13} e_1 + a_{23} e_2 + a_{33} e_3) \end{aligned}$$

$$vol(ABe_1, e_2, e_3) = (b_{11} a_{11} + b_{21} a_{12} + b_{31} a_{13})$$

$$vol(e_1, ABe_2, e_3) = (b_{12} a_{21} + b_{22} a_{22} + b_{32} a_{23})$$

$$vol(e_1, e_2, ABe_3) = (b_{13} a_{31} + b_{23} a_{32} + b_{33} a_{33})$$

If we exchange A and B the sum above will be left unchanged. So $tr(AB) = tr(BA)$

$$M \cdot L = \mathrm{tr}(M^T L)$$

$$\mathrm{tr}(M^T L) = m^T_{11} v_{11} + m^T_{21} v_{12} + m^T_{31} v_{13}$$

$$+$$

$$+$$

By using an orthonormal basis we realize that

since $m^T_{21} = m_{12}$ then

$$\mathrm{tr}(M^T L) = \sum \sum m_{ij} v_{ij}$$

$$A \cdot W = \mathrm{tr}(A^T W) = \mathrm{tr}(W^T A) = W \cdot A$$

$$\left.\begin{array}{l} A^T = A \\ W^T = -W \end{array}\right\} \Rightarrow \quad = \mathrm{tr}(AW) = -\mathrm{tr}(WA) = -\mathrm{tr}(AW)$$

$$\Rightarrow \quad A \cdot W = 0$$

Properties of "tr"

$$\mathrm{tr}(A) = \mathrm{tr}(A^T)$$

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

$$\mathsf{M}\cdot\mathsf{L} = (\operatorname{skw}\mathsf{M} + \operatorname{sym}\mathsf{M})\cdot(\mathsf{W}+\mathsf{D})$$

$$= \operatorname{skw}\mathsf{M}\cdot\mathsf{W} + \cancel{\operatorname{skw}\mathsf{M}\cdot\mathsf{D}}$$

$$+ \cancel{\operatorname{sym}\mathsf{M}\cdot\mathsf{W}} + \operatorname{sym}\mathsf{M}\cdot\mathsf{D}$$

$$\mathcal{F}^{ext}(v) = f\cdot v_0 + \mathsf{M}\cdot\mathsf{L}$$

$$= f\cdot v_0 + \operatorname{skw}\mathsf{M}\cdot\mathsf{W} + \operatorname{sym}\mathsf{M}\cdot\mathsf{D}$$

v affine velocity field

$$v(x) = v_0 + \mathsf{L}(x - p_0)$$

v rigid velocity field $(\mathsf{L}=\mathsf{W})$

$$\mathcal{F}^{ext}(v) = f\cdot v_0 + \underbrace{\operatorname{skw}\mathsf{M}\cdot\mathsf{W}}_{m\cdot w}$$