MECHANICS OF SOLIDS AND MATERIALS 2013-2014

2° Term. Class started on [2014-02-24], room A1.3

$(1-2)_1$ 16:00-18:00

Very small body: body point and its positions

$$p : \{A\} \times \mathbb{R} \to \mathcal{E}$$

Collection of body points

$$\mathcal{B} = \{A, B, C\}$$

placement $p : \{A, B, C\} \to \mathcal{E}$

motion $p : \{A, B, C\} \times \mathbb{R} \to \mathcal{E}$

one-parameter family of placements

trajectory, trajectories

shapes $\operatorname{imp} \subset \mathcal{E}$

[Notebook page scanned on 2014-06-20]

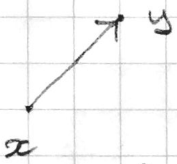
Operations on positions

$$x + u = y$$

$$\{\mathcal{E}, \mathcal{V}\}$$

$\mathcal{V}$ — translation space

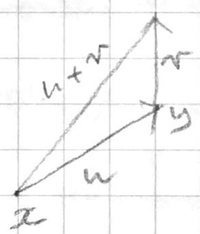
$\mathcal{E}$ — Euclidean space




$$y - x = u$$

for any two positions $(x, y)$ there is a vector $u$ translating $x$ to $y$; this vector is unique

$$(x+u)+v = x+(u+v) \quad \Rightarrow$$

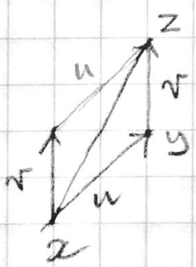



$$x + 0 = x$$

$$x+u = y \qquad \Rightarrow \quad y - x = u$$

$$(x+u)+(-u) = x + 0 = x$$

$$y + (-u) = x \qquad \Rightarrow \quad x - y = -u$$




$$(x+u)+v = x+(u+v)$$

$$= x+(v+u) = (x+v)+u$$

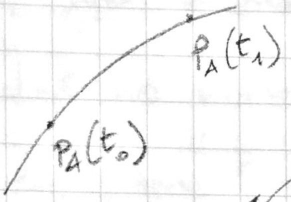
[Notebook page scanned on 2014-06-16]

Tuesday [2014-02-25]

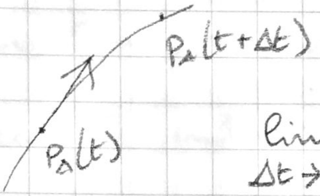
9:00 - 11:00

$(3-4)_1$

motion $p : \mathcal{B} \times \mathbb{R} \to \mathcal{E}$




trajectory parameterized by $t$

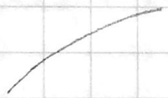



velocity vector

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(p_A(t+\Delta t) - p_A(t)\right)$$

Curve-like shape parameterized by a scalar $s$

Square-like shape parameterized by two scalars $s_1, s_2$

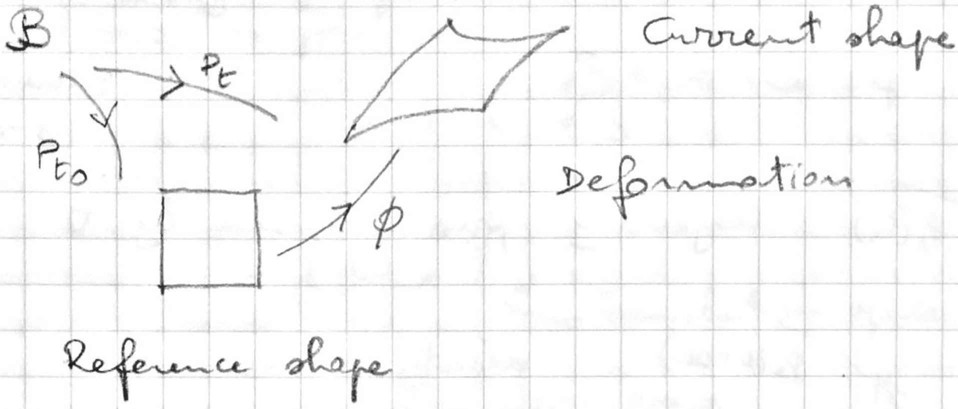


$$\hat{x} : [0, \ell] \to p_t(\mathcal{B})$$

$$\hat{x} : [0, \ell_1] \times [0, \ell_2] \to p_t(\mathcal{B})$$

Dimension of the shape = Dimension of the body

[Notebook page scanned on 2014-06-16]




motion description

$$\phi : p_{t_0}(\mathcal{B}) \times \mathbb{R} \to \mathcal{E}$$

$$\bar{\mathcal{R}}$$

$$\phi(\bar{\mathcal{R}}, t) = \mathcal{R}_t$$

"region" of $\mathcal{E}$

We will discover properties of a deformation by comparing shapes

[Notebook page scanned on 2014-06-16]

Wednesday [2014-02-26]

(5-6)$_1$ 11:00 – 13:00

Rigid deformation (it's not an OXYMORON)

Let us introduce first the "distance" between two positions.

We need to introduce a scalar product in $\mathcal{V}$ and the induced norm.

The distance between any two position is left unchanged by a rigid deformation.

$$\|p_A - p_B\| = \|\bar{p}_A - \bar{p}_B\|$$

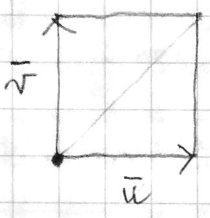
$$\|u + v\|^2 = (u+v)\cdot(u+v) = u\cdot u + 2u\cdot v + v\cdot v$$
$$= \|u\|^2 + 2u\cdot v + \|v\|^2$$

$$\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + 2\bar{u}\cdot\bar{v} + \|\bar{v}\|^2$$

Subtracting we get $\quad 0 = u\cdot v - \bar{u}\cdot\bar{v}$

i.e. the scalar product of any two vectors is left unchanged

[Notebook page scanned on 2014-06-16]




$$\bar{u} \mapsto u \quad , \quad \bar{v} \mapsto v$$

$$\bar{u} \cdot \bar{v} = 0 \quad \Rightarrow \quad u \cdot v = 0$$


$$\bar{u} + \bar{v} \mapsto u + v \qquad \text{linearity}$$

$$u = R\bar{u} \quad , \quad v = R\bar{v}$$

$$u \cdot v = \bar{u} \cdot \bar{v} \quad \Rightarrow \quad R\bar{u} \cdot R\bar{v} = \bar{u} \cdot \bar{v}$$

$$R^T R \bar{u} \cdot \bar{v} = \bar{u} \cdot \bar{v}$$

$$(R^T R - I)\bar{u} \cdot \bar{v} = 0$$

Rigid deformation representation

$$\phi(\bar{p}_A) = \phi(\bar{p}_0) + R(\bar{p}_A - \bar{p}_0)$$

Rigid motion

$$\phi(\bar{p}_A, t) = \phi(\bar{p}_0, t) + R(t)(\bar{p}_A - \bar{p}_0)$$

[Notebook page scanned on 2014-06-16]

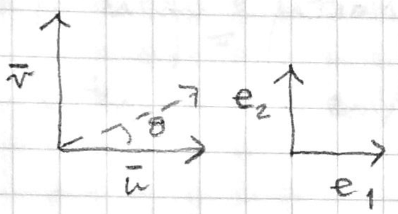
$(7-8)_2$ Monday [2014-03-03]

16-18 A1.3

Matrix representation of the rotation tensor and coordinate representation of a rigid deformation

$$\phi(\bar{p}_1) = \phi(\bar{p}_0) + R\,(\bar{p}_1 - \bar{p}_0)$$

↑ vector




basis vectors

$$R\bar{u} = \cos\vartheta\,\bar{u} + \sin\vartheta\,\bar{v}$$

$$R\bar{v} = -\sin\vartheta\,\bar{u} + \cos\vartheta\,\bar{v}$$

$$R\bar{u}\cdot R\bar{u} = \cos^2\vartheta\,\bar{u}\cdot\bar{u} + \sin^2\vartheta\,\bar{v}\cdot\bar{v} + 2\cos\vartheta\sin\vartheta\,\bar{u}\cdot\bar{v}$$

if $\bar{u}\cdot\bar{v} = 0$, $\bar{u}\cdot\bar{u} = \bar{v}\cdot\bar{v}$ $\Rightarrow$ $R\bar{u}\cdot R\bar{u} = \bar{u}\cdot\bar{u} = \bar{v}\cdot\bar{v}$, $R\bar{v}\cdot R\bar{v} = \bar{v}\cdot\bar{v} = \bar{u}\cdot\bar{u}$

$\Downarrow$

$$R\bar{u}\cdot R\bar{v} = -\sin\vartheta\cos\vartheta\,\bar{u}\cdot\bar{u} + \cos^2\vartheta\,\bar{u}\cdot\bar{v} - \sin^2\vartheta\,\bar{v}\cdot\bar{u} + \sin\vartheta\cos\vartheta\,\bar{v}\cdot\bar{v} = 0$$

It is convenient to choose an orthonormal basis

[Notebook page scanned on 2014-06-16]

# Orthonormal basis

$$R e_1 = \cos\vartheta\, e_1 + \sin\vartheta\, e_2$$

$$R e_2 = -\sin\vartheta\, e_1 + \cos\vartheta\, e_2$$

matrix of $R$ $\quad \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$

we arrange components in columns

$$R e_1 \cdot e_1 = \cos\vartheta$$

$$R e_1 \cdot e_2 = \sin\vartheta$$

...

$$R^T e_1 \cdot e_1 = e_1 \cdot R e_1 = R e_1 \cdot e_1 = \cos\vartheta$$

$$R^T e_1 \cdot e_2 = R e_2 \cdot e_1 = -\sin\vartheta$$

matrix of $R^T$

In general

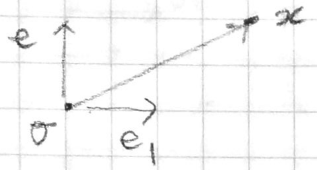
$$A e_1 = a_{11} e_1 + a_{21} e_2$$

$$A e_2 = a_{12} e_1 + a_{22} e_2$$

$$[A] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

[Notebook page scanned on 2014-06-16]

Coordinates (cartesian)




$$(x-o) = x_1 e_1 + x_2 e_2$$

$$p_A = p_o + R(\bar{p}_A - \bar{p}_o)$$

$$\begin{pmatrix} x_{1A} \\ x_{2A} \end{pmatrix} = \begin{pmatrix} x_{1o} \\ x_{2o} \end{pmatrix} + \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \bar{x}_{1A} - \bar{x}_{1o} \\ \bar{x}_{2A} - \bar{x}_{2o} \end{pmatrix}$$

extending rotations to 3D

$$R a = \lambda a \qquad \|R a\| = \|a\|$$

$$\Rightarrow |\lambda| = 1 \qquad \lambda = \alpha \pm \beta i \qquad \lambda = \pm 1$$

[Notebook page scanned on 2014-06-16]

[2014-03-04] Tuesday $(9-10)_2$

9:00 – 11:00 (9:00-10:45, no breaks)

$$Ra = \lambda a$$

$$Ra \cdot Ra = a \cdot a \Rightarrow \lambda^2 = 1$$

extending $V$ to the complex field $\mathbb{C}$

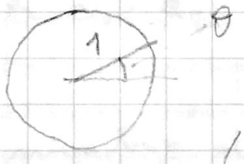
$$u \cdot v = \overline{v \cdot u}$$

$$(\alpha u) \cdot v = \alpha (u \cdot v)$$

$$\Rightarrow \quad u \cdot (\beta v) = \overline{\beta v \cdot u} = \bar{\beta}\, \overline{v \cdot u} = \bar{\beta}\, u \cdot v$$

$$Ra \cdot Ra = \lambda \bar{\lambda} = |\lambda|^2$$

$$\lambda_{1,2} = \alpha \pm i \beta \qquad \alpha^2 + \beta^2 = 1 \qquad \lambda_3 = \pm 1$$

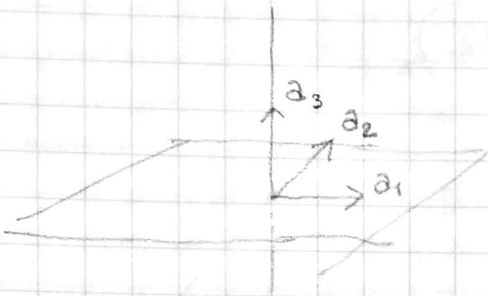



$$\alpha = \cos\theta$$
$$\beta = \sin\theta$$

$$\begin{pmatrix} \cos\theta + i \sin\theta & 0 & 0 \\ 0 & \cos\theta - i \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[Notebook page scanned on 2014-06-16]




orthogonal to each other

$$R a_3 = a_3$$

$$R a_1 \cdot a_3 = 0$$

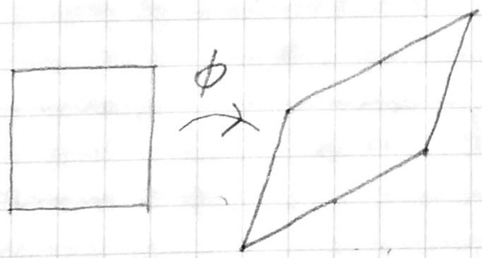
$$R a_2 \cdot a_2 = 0$$

We generalize rigid deformations to <u>affine deformations</u>

$$\phi(\bar{p}_A) = \phi(\bar{p}_o) + F(\bar{p}_A - \bar{p}_o)$$

where $F$ is a non singular tensor $(F(\bar{u}) = 0 \Leftrightarrow u = 0)$

- Straight lines are deformed into straight lines
- parallel segments are deformed into parallel segments
- a square is deformed into a parallelogram
- a cube is deformed into a parallelepiped
- a circle is deformed into an ellipse
- a sphere is deformed into an ellipsoid



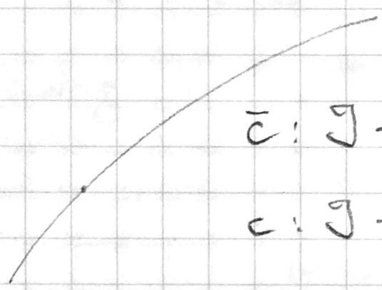

An <u>affine</u> deformation is also called a <u>homogeneous</u> deformation

[Notebook page scanned on 2014-06-20]

[2014-03-05] Wednesday $(11-12)_2$

11:00 - 13:00 A1.3

Tangent vectors



$\bar{c} : \mathcal{I} \to \mathcal{E}$ curve on $\bar{\mathcal{R}} = \operatorname{im} \bar{p}$

$c : \mathcal{I} \to \mathcal{E}$ curve on $\mathcal{R} = \operatorname{im} p$

Parameterization of the <u>reference shape</u>

$$\kappa(s_1, s_2, s_3) \in \bar{\mathcal{R}}$$

in general it is a piecewise parameterization

so $\kappa^{-1}$ does exist locally in general

$$\kappa : \mathcal{D} \to \bar{\mathcal{R}} \subset \mathcal{E}$$

parameterization domain

$(s_1, s_2, s_3)$ are also called "material coordinates"

$$\begin{array}{c} \phi : \bar{\mathcal{R}} \to \mathcal{R} \subset \mathcal{E} \\ \uparrow \bar{p} \\ \mathcal{B} \end{array}$$

[Notebook page scanned on 2014-06-16]

$$\phi : \bar{\mathcal{R}} \to \mathcal{R}$$
$$\uparrow \kappa$$
$$\mathcal{D}$$

$$\phi_\kappa := \phi \circ \kappa \; : \; \mathcal{D} \to \mathcal{R}$$

$$\phi(\kappa(s_1, s_2, s_3)) \in \mathcal{R}$$

parameterization of the current shape (convected coordinates)

By using a coordinate system

$$\phi_\kappa(s_1, s_2, s_3) = o + \phi_{\kappa 1}(s_1, s_2, s_3)\, e_1$$
$$= \phi(\kappa(s_1, s_2, s_3)) = \qquad + \phi_{\kappa 2}(s_1, s_2, s_3)\, e_2$$
$$+ \phi_{\kappa 3}(s_1, s_2, s_3)\, e_3$$

$$\bar{c}_1(h) = \underbrace{\bar{p}_0}_{x \in \bar{\mathcal{R}}} + h\, e_1 = \kappa(s_1, s_2) + h\, e_1$$

Let us define $\kappa$, for example, by using coordinates in $\mathcal{E}$

$$\kappa(s_1, s_2, s_3) = o + s_1 e_1 + s_2 e_2 + s_3 e_3$$

Thus

$$\bar{c}_1(h) = o + s_1 e_1 + \cdots + h\, e_1 = \kappa(s_1 + h, s_2, s_3)$$

[Notebook page scanned on 2014-06-16]

$$c_1(h) = \phi(\bar{c}_1(h)) = \phi(\kappa(s_1+h, s_2, s_3))$$

$$c_1(h) = \phi_\kappa(s_1+h, s_2, s_3)$$

$$\begin{aligned} c_1(h) = o &+ \phi_{\kappa 1}(s_1+h, s_2, s_3)\, e_1 \\ &+ \phi_{\kappa 2}(s_1+h, s_2, s_3)\, e_2 \\ &+ \phi_{\kappa 3}(s_1+h, s_2, s_3)\, e_2 \end{aligned}$$

$$c_1'(0) = \lim_{h\to 0} \frac{1}{h}\left(c_1(h) - c_1(0)\right) = \ldots$$

$$= \partial_1 \phi_{\kappa 1}\, e_1 + \partial_1 \phi_{\kappa 2}\, e_2 + \partial_1 \phi_{\kappa 3}\, e_3$$

[...]

$$c_2'(0) = \ldots$$

$$c_3'(0) = \ldots$$

$$\bar{c}(h) = \kappa(s_1 + h\alpha_1, s_2 + h\alpha_2, s_3 + h\alpha_3)$$

$$c(h) = o + \phi_{\kappa 1}(s_1 + h\alpha_1, s_2 + h\alpha_2, s_3 + h\alpha_3)\, e_1 + \ldots$$

$\Rightarrow$ there exists a linear transformation (tensor) $F : \mathcal{V} \to \mathcal{V}$, called the <u>deformation gradient</u>, such that

$$F\bar{c}_1' = c_1', \quad F\bar{c}_2' = c_2', \quad F(\alpha_1 \bar{c}_1' + \alpha_2 \bar{c}_2') = \alpha_1 c_1' + \alpha_2 c_2'$$

[Notebook page scanned on 2014-06-22]

$(13-14)_3$ Monday [2014-03-10]

16:00 → 18:00

A1.3

Local deformation

$$c_1(h) = \phi(p_0) + F(\bar{p}_0)\left(\bar{c}_1(h) - \bar{c}_1(0)\right) + o(h)$$

$$o(h) := c_1(h) - \big(\underbrace{\phi(p_0)}_{c_1(0)} + F(\bar{p}_0)\left(\bar{c}_1(h) - \bar{c}_1(0)\right)\big)$$

$$\lim_{h \to 0} o(h) = 0$$

$$\lim_{h \to 0} \frac{c_1(h) - c_1(0)}{h} - F(\bar{p}_0)\,\bar{c}_1'(0) = 0$$

$$\lim_{h \to 0} \frac{o(h)}{h} = c_1'(0) - F(\bar{p}_0)\,\bar{c}_1'(0) = 0$$

$o(h) \to 0$ faster than $h$

That is why we study AFFINE DEFORMATIONS

[Notebook page scanned on 2014-06-20]

Polar decomposition of the deformation gradient

$$F = RU \qquad R^TR = I \quad \det R = 1$$

$$U \in Psym$$

$$C := F^TF = (RU)^T(RU) = U^2$$

$$U = \sqrt{F^TF}$$

$$[C] = [A]\,\mathrm{diag}(\eta_1, \eta_2, \eta_3)[A]^T$$

$$[u_1, u_2, u_3]^T[u_1] = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$[u_1, u_2, u_3]\ \mathrm{diag}(\eta_1, \eta_2, \eta_3)\begin{pmatrix}1\\0\\0\end{pmatrix} = [u_1]\,\eta_1$$

$$C = \eta_1\, u_1 \otimes u_1 + \eta_2 u_2 \otimes u_2 + \eta_3 u_3 \otimes u_3$$

$$C u_1 = \eta_1 u_2 \qquad \ldots$$

[Notebook page scanned on 2014-06-20]

$(15-16)_3$ Tuesday [2014-03-11]
A1.3 09:00 – 11:00

$\mathrm{vol}(u_1, u_2, u_3)$ $\qquad \mathrm{vol} : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$

$$\begin{cases} \mathrm{vol}(u_1 + v, u_2, u_3) = \mathrm{vol}(u_1, u_2, u_3) + \mathrm{vol}(v, u_2, u_3) \\ \mathrm{vol}(\alpha u_1, u_2, u_3) = \alpha\, \mathrm{vol}(u_1, u_2, u_3) \\ \mathrm{vol}(u_2, u_1, u_3) = -\mathrm{vol}(u_1, u_2, u_3) \end{cases}$$

$$\Rightarrow \quad \mathrm{vol}(u_1, u_1, u_3) = -\mathrm{vol}(u_1, u_1, u_3) = 0$$

$$\mathrm{vol}(\underbrace{u_1 - u_1}_{0}, u_2, u_3) = \mathrm{vol}(u_1, u_2, u_3) - \mathrm{vol}(u_1, u_2, u_3) = 0$$

$$\mathrm{vol}(\alpha_2 u_2 + \alpha_3 u_3, u_2, u_3) = 0$$

$\mathrm{vol}(u_1, u_2, u_3) = 0$ for $u_1, u_2, u_3$ linearly independent

$\Rightarrow \quad \mathrm{vol} = 0$

$$A_{g_1} := (n_1, u_2, u_3)$$

$$V = \mathrm{vol}(u_1, u_2, u_3) = \mathrm{vol}(\underbrace{w_1 + (u_1 \cdot n_1) n_1}, u_2, u_3)$$

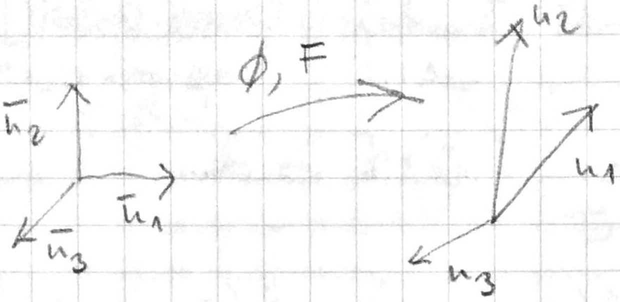
$$w_1 := u_1 - (u_1 \cdot n_1) n_1$$

$n_1$ unit normal vector $\quad \Rightarrow \quad w_1 \cdot n_1 = u_1 \cdot n_1 - (u_1 \cdot n_1)(n_1 \cdot n_1) = 0$

$$w_1 = \alpha_2 u_2 + \alpha_3 u_3$$

$$\Rightarrow \quad V = h_1 A_{g_1} \qquad h_1 := (u_1 \cdot n_1)$$

[Notebook page scanned on 2014-06-20]




How volumes and areas are changed by $F$ ?

$$V = \mathrm{vol}(u_1, u_2, u_3) = \mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)$$

$$\frac{V}{\bar{V}} = \frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

Choose any basis $\{e_1, e_2, e_3\}$

For example $e_1 = \bar{u}_1,\ e_2 = \bar{u}_2,\ e_3 = \bar{u}_3$

In the very simple case of planar deformation

$$F e_1 = f_{11}\, e_1 + f_{21}\, e_2 + 0\, e_3$$

$$F e_2 = f_{12}\, e_1 + f_{22}\, e_2 + 0\, e_3$$

$$F e_3 = e_3$$

we get

$$\frac{\mathrm{vol}(Fe_1, Fe_2, Fe_3)}{\mathrm{vol}(e_1, e_2, e_3)} = f_{11} f_{22} - f_{21} f_{12}$$

[Notebook page scanned on 2014-06-20]

The ratio does not depend on the basis (1)

It does not depend on the volume function

That is why we define

$$\det F = \frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

(1) just because the definition of vol does not rely on any basis

[Notebook page scanned on 2014-06-20]

Wednesday [2014-03-12] 11:00-13:00 A1.3 $(17-18)_3$

$$\frac{V}{\bar{V}} = \det F$$

Let us consider $\quad V = h_1 A_{\mathcal{F}_1} \qquad h_1 := u_1 \cdot n_1$

$\qquad \bar{V} = \bar{h}_1 \bar{A}_{\bar{\mathcal{F}}_1} \qquad \bar{h}_1 := \bar{u}_1 \cdot \bar{n}_1$

$$\frac{A_{\mathcal{F}_1}}{\bar{A}_{\bar{\mathcal{F}}_1}} = \frac{V}{h_1}\,\frac{\bar{h}_1}{\bar{V}} = \frac{\bar{h}_1}{h_1} \det F$$

$$h_1 := u_1 \cdot n_1 \qquad u_2 \cdot n_1 = 0 \qquad u_3 \cdot n_1 = 0$$

$$h_1 = F\bar{u}_1 \cdot n_1 \qquad F\bar{u}_2 \cdot n_1 = 0 \qquad F\bar{u}_3 \cdot n_1 = 0$$

$F\bar{u}_2$ and $F\bar{u}_3$ are the edges of $\mathcal{F}_1$

$$h_1 = \bar{u}_1 \cdot F^T n_1 \qquad \bar{u}_2 \cdot F^T n_1 = 0 \qquad \bar{u}_3 \cdot F^T n_1 = 0$$

$\Rightarrow$ $F^T n_1$ is orthogonal to

$\bar{u}_2$ and $\bar{u}_3$ which are the edges of $\bar{\mathcal{F}}_1$

Hence $F^T$ pulls back $n_1$ to a vector normal to $\bar{\mathcal{F}}_1$

$$F^T n_1 = k_1^{-1} \bar{n}_1$$

[Notebook page scanned on 2014-06-20]

$$n_1 = k_1^{-1} F^{-T} \bar{n}_1 \quad \Rightarrow \quad h_1 = \bar{u}_1 \cdot F^T n_1 = \bar{u}_1 \cdot \bar{n}_1 \, k_1^{-1}$$

$$h_1 = \bar{h}_1 \, k_1^{-1}$$

$$k_1 = \frac{\bar{h}_1}{h_1} \quad \Leftarrow$$

$$F^{-T} \bar{n}_1 = k_1 n_1 = \frac{A_{g_1}}{\bar{A}_{\bar{g}_1}} \frac{1}{\det F} n_1$$

$$\underbrace{(\det F) F^{-T}}_{\operatorname{cof} F} \bar{n}_1 = \frac{A_{g_1}}{\bar{A}_{\bar{g}_1}} n_1$$

Matrix of cof F

vol $(e_1, e_2, e_3) = 1$ orthonormal basis

$$(\operatorname{cof} F) e_1 = \frac{A_{g_1}}{\underbrace{\bar{A}_{\bar{g}_1}}_{\to 1}} n_1 = A_{g_1} n_1$$

$$(\operatorname{cof} F) e_1 \cdot e_i = A_{g_1} n_1 \cdot e_i = (n_1 \cdot e_i) \operatorname{vol}(n_1, F e_2, F e_3)$$

$$e_i = w_{1i} + (e_i \cdot n_1) n_1 \quad \Rightarrow \quad w_{1i} \cdot n_1 = 0$$

$$\begin{aligned}
(\operatorname{cof} F) e_1 \cdot e_i &= \operatorname{vol}((n_1 \cdot e_i) n_1, F e_2, F e_3) \\
&= \operatorname{vol}(e_i - w_{1i}, F e_2, F e_3) \\
&= \operatorname{vol}(e_i, F e_2, F e_3)
\end{aligned}$$

[Notebook page scanned on 2014-06-20]

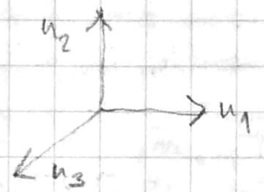
$$\underbrace{(\mathrm{cof}\,F)}_{=\frac{A_{\mathfrak{J}_2}}{\bar{A}_{\bar{\mathfrak{J}}_2}} n_2} e_2 \cdot e_i = A_{\mathfrak{J}_2}\, n_2 \cdot e_i \qquad \bar{A}_{\bar{\mathfrak{J}}_2} = 1$$

$$(\mathrm{cof}\,F)\, e_2 \cdot e_i = A_{\mathfrak{J}_2}(n_2 \cdot e_i) = (n_2 \cdot e_i)\,\mathrm{vol}(Fe_1, n_2, Fe_3)$$

$A_{\mathfrak{J}_2} = \mathrm{vol}(e_1, n_2, e_3)$

$e_i = w_{2i} + (e_i \cdot n_2)\, n_2$

$\Rightarrow w_{2i} \cdot n_2 = 0$




$$(\mathrm{cof}\,F)\, e_2 \cdot e_i = \mathrm{vol}(Fe_1, e_i - w_2, Fe_3)$$

$$= \mathrm{vol}(Fe_1, e_i, Fe_3)$$

recall that $\mathrm{vol}(u_1, u_2, u_3)$ is equal to the det of the matrix made up with the vectors components

$$= \det \begin{pmatrix} f_{11} & \cdot & f_{13} \\ f_{21} & \cdot & f_{23} \\ f_{31} & \cdot & f_{33} \end{pmatrix}$$

$$= Fe_1 \cdot e_i \times Fe_3 = -e_i \cdot Fe_1 \times Fe_3 \qquad [\ldots]$$

[Notebook page scanned on 2014-06-20]

$$\operatorname{tr} A = \frac{\operatorname{vol}(A e_1, e_2, e_3)}{\operatorname{vol}(e_1, e_2, e_3)} + \ldots$$

$$\frac{d}{dt} \det F(t) = \frac{\operatorname{vol}(\dot{F}\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} + \ldots$$

$$= \frac{\operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3)}{\operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} + \ldots$$

$$= \frac{\operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3)}{\operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} \frac{\operatorname{vol}(u_1, u_2, u_3)}{\operatorname{vol}(u_1, u_2, u_3)} + \ldots$$

$$= \frac{\operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3)}{\operatorname{vol}(u_1, u_2, u_3)} \det F + \ldots$$

$$= \det F \operatorname{tr}(\dot{F}F^{-1})$$

$$\det AB = \frac{\operatorname{vol}(ABe_1, ABe_2, Ae_3)}{\operatorname{vol}(e_1, e_2, e_3)}$$

$$\det B = \frac{\operatorname{vol}(Be_1, Be_2, Be_3)}{\operatorname{vol}(e_1, e_2, e_3)}$$

$$\det A = \frac{\operatorname{vol}(ABe_1, ABe_2, ABe_3)}{\operatorname{vol}(Be_1, Be_2, Be_3)}$$

$$\Rightarrow \det A \det B = \det AB$$

[Notebook page scanned on 2014-06-20]

$$F^T n_1 = k_1^{-1}\bar{n}_1 = \frac{h_1}{\bar{h}_1}\bar{n}_1 \qquad h_1 = u_1\cdot n_1\,;\ \bar{h}_1 = \bar{u}_1\cdot\bar{n}_1$$

There are 2 couples of dual bases

$$\begin{cases} F^T n_i = a_{1i}\,u_1 + a_{2i}\,u_2 + a_{3i}\,u_3 \\ n_i = b_{1i}\,\bar{u}_1 + b_{2i}\,\bar{u}_2 + b_{3i}\,\bar{u}_3 \end{cases} \quad \binom{*}{0} \qquad \begin{matrix}\{u_i\},\{n_j\}\\ \{\bar{u}_i\},\{\bar{n}_j\}\end{matrix}$$

$$\begin{cases} F^T n_1\cdot n_2 = a_{21}\,u_2\cdot n_2 = a_{21}\,h_2 \quad ; \quad F^T n_1\cdot u_1 = a_{11}\,h_1 & \binom{*}{1} \\ n_2\cdot\bar{n}_1 = b_{12}\,\bar{u}_1\cdot\bar{n}_1 = b_{12}\,\bar{h}_1 \quad ; \quad n_1\cdot\bar{n}_1 = b_{11}\,\bar{h}_1 & \binom{*}{2} \end{cases}$$

$$\Uparrow$$

$$\left.\begin{matrix} \boxed{\begin{matrix} F^T n_1\cdot\bar{u}_2 = n_1\cdot u_2 = 0\ ;\ F^T n_1\cdot\bar{u}_3 = 0 \\ F^T n_2\cdot\bar{u}_1 = 0\ ;\ F^T n_2\cdot\bar{u}_3 = 0 \\ F^T n_3\cdot\bar{u}_1 = 0\ ;\ F^T n_3\cdot\bar{u}_2 = 0 \end{matrix}} & \begin{matrix} \Rightarrow F^T n_1 = k_1^{-1}\bar{n}_1 \\ \Rightarrow F^T n_2 = k_2^{-1}\bar{n}_2 \\ \Rightarrow F^T n_3 = k_3^{-1}\bar{n}_3 \end{matrix} \end{matrix}\right\}\binom{*}{3}$$

$$\begin{cases} F^T n_1\cdot n_1 \underset{\binom{*}{3}}{=} k_1^{-1}\bar{n}_1\cdot n_1 \underset{\binom{*}{1}}{=} a_{11}\,h_1 \underset{\binom{*}{2}}{\Rightarrow} \dfrac{h_1}{\bar{h}_1}b_{11}\bar{h}_1 = a_{11}h_1 \\ F^T n_1\cdot n_2 = k_1^{-1}\bar{n}_1\cdot n_2 = a_{21}\,h_2 \Rightarrow \dfrac{h_1}{\bar{h}_1}b_{12}\bar{h}_1 = a_{21}h_2 \end{cases} \qquad (*)$$

$$\det F^T = \frac{vol(F^T n_1, F^T n_2, F^T n_3)}{vol(n_1,n_2,n_3)} = \frac{\langle\det a_{ij}\rangle}{\langle\det b_{ij}\rangle}\,\underbrace{\frac{vol(u_1,u_2,u_3)}{vol(\bar{u}_1,\bar{u}_2,\bar{u}_3)}}_{1} = \det F$$

$$\begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3\end{pmatrix}\begin{pmatrix} b_{11} & \cdot & \cdot \\ b_{21} & \cdot & \cdot \\ b_{31} & \cdot & \cdot\end{pmatrix}\begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3\end{pmatrix}^{-1} = \begin{pmatrix} a_{11} & \cdot & \cdot \\ a_{21} & \cdot & \cdot \\ a_{31} & \cdot & \cdot\end{pmatrix}^T \qquad h_i b_{ij} = a_{ji} h_j \qquad (*)$$

It is much easier to rely on the determinant's independence of any basis, and choose an orthonormal basis and exploit the property of the matrices of being the transpose of each other thus resulting in the same expression for the determinant.

[Notebook page scanned on 2014-06-20]

$(19\text{-}20)_4$ Monday [2014-03-17] 16:00-18:00
A1.3

Velocity fields (vector fields)

velocity gradient

$$
\begin{array}{l}
v : \mathcal{R} \subset \mathcal{E} \to \mathcal{V} \\
\quad\ \ \uparrow \phi \\
\bar{v} : \bar{\mathcal{R}} \to \mathcal{V} \\
\quad\ \ \uparrow \kappa \\
\quad\ \ \mathcal{B}
\end{array}
$$

$$c(h) = o + c_1(h)\, e_1 + c_2(h)\, e_2 + c_3(h)\, e_3$$

$$c'(0) = \lim_{h\to 0} \frac{1}{h}\Big(c(h) - c(0)\Big) = c_1'(0)\, e_1 + c_2'(0)\, e_2 + c_3'(0)\, e_3$$

scalar functions

$$v(c(h)) = v_1(c_1(h), c_2(h), c_3(h))\, e_1 + \cdots$$

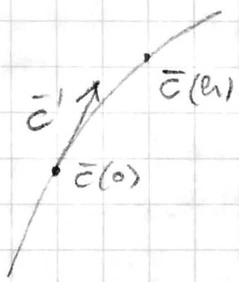
$$\lim_{h\to 0} \frac{1}{h}\Big(v(c(h)) - v(c(0))\Big) =$$

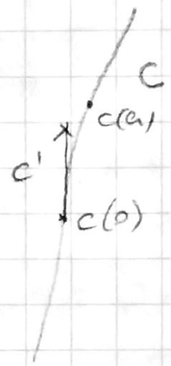
$$
\begin{aligned}
& \partial_1 v_1\, c_1'\, e_1 + \partial_2 v_1\, c_2'\, e_1 + \partial_3 v_1\, c_3'\, e_1 \\
+ & \partial_1 v_2\, c_1'\, e_2 + \partial_2 v_2\, c_2'\, e_2 + \partial_3 v_2\, c_3'\, e_2 \\
+ & \partial_1 v_3\, c_1'\, e_3 + \partial_2 v_3\, c_2'\, e_3 + \partial_3 v_3\, c_3'\, e_3
\end{aligned}
$$

$$= \nabla v\, c'$$

[Notebook page scanned on 2014-06-20]

$$\underbrace{\begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ & & \\ & & \end{pmatrix}}_{[\nabla v]} \begin{pmatrix} c_1' \\ c_2' \\ c_3' \end{pmatrix}$$




PULL BACK OF VECTOR FIELDS $\bar{v}(\bar{c}(h)) = v(c(h))$

$$\frac{\bar{v}(\bar{c}(h)) - \bar{v}(\bar{c}(0))}{h} = \frac{v(c(h)) - v(c(0))}{h}$$

taking the limit $\nabla\bar{v}\,\bar{c}' = \nabla v\, c'$

$$\nabla\bar{v}\,\bar{c}' = \nabla v\, F\,\bar{c}'$$

$$\Rightarrow \quad \nabla\bar{v} = \nabla v\, F$$

$$v(c(h)) = v(c(0)) + \nabla v\,(c(h) - c(0)) + o(h)$$

$$\frac{v(c(h)) - v(c(0))}{h} = \nabla v\,\frac{c(h) - c(0)}{h} + \frac{o(h)}{h}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{o(h)}{h} = 0$$

[Notebook page scanned on 2014-06-20]

$(21-22)_4$ Tuesday [2014-03-18] 9:00-11:00
A1.3

velocity field

$$v(c(h)) = v(c(0)) + \overset{L}{\nabla v}\,(c(h) - c(0)) \quad + o(h)$$

$$c(h) = c(0) + F\,(\bar{c}(h) - \bar{c}(0)) \quad + o(h)$$

$L := \nabla v \; ; \; F := \nabla \phi$

let us differentiate w.r.t. time

$$\dot{c}(h) = \dot{c}(0) + \dot{F}\,(\bar{c}(h) - \bar{c}(0)) + o(h)$$

$$F^{-1}(c(h) - c(0)) = (\bar{c}(h) - \bar{c}(0)) \quad + o(h)$$

$$\dot{c}(h) = \dot{c}(0) + \dot{F}F^{-1}(c(h) - c(0)) \quad + o(h)$$

$$\Rightarrow \quad \nabla v = \dot{F}F^{-1}$$

$$\nabla \bar{v} = \nabla v\, F = \dot{F}$$

we assumed $v(c(h)) = \dot{c}(h) \;|\; v(c(h,t)) = \dot{c}(h,t)$

Differentiating again

$$\frac{d}{dt}\, v(c(\underset{\text{fixed}}{h}, \underset{\text{trajectory}}{t}), t) = \nabla v\, \dot{c} + \frac{\partial}{\partial t}\, v(\overset{\text{fixed position}}{x}, t)$$

Acceleration from spatial velocity field

[Notebook page scanned on 2014-06-20]

$$\bar{w}(\bar{c}(\mathcal{B})) = w(c(\mathcal{B}))$$

pull-back of any vector field $w$

moving $c(\mathcal{B})$, fixed $w$

$$\bar{w}(\bar{c}(\mathcal{B})) = w(c(\mathcal{B},t))$$

$$\phi(\bar{c}(\mathcal{B}),t)$$

velocity field $v(x,t)$

$$\bar{v}(\bar{c}(\mathcal{B}),t) = v(c(\mathcal{B},t),t)$$

acceleration field

$$\bar{a}(\bar{c}(\mathcal{B}),t) = \frac{d}{dt}\bar{v}(\bar{c}(\mathcal{B}),t)$$

$$= \frac{d}{dt} v(c(\mathcal{B},t),t)$$

velocity field

$$\bar{v}(\bar{p}_A,t) = \frac{d}{dt} p_A(t)$$

$$= \dot{p}_A(t) = v(p_A(t),t)$$

acceleration field

$$\bar{a}(\bar{p}_A,t) = \frac{d}{dt}\bar{v}(\bar{p}_A,t)$$

$$= \frac{d}{dt} v(p_A(t),t)$$

$\bar{v}$, $\bar{a}$ referential description (Lagrangean)

$v$, $a$ spatial description (Eulerian)

Time differentiation of the velocity field

$$\frac{d}{dt}\bar{v}(\bar{c}(\mathcal{B}),t) = \frac{d}{dt} v(c(\mathcal{B},t),t)$$

$$= \underbrace{\nabla v}_{v \text{ field at time } t}\ \underset{v(c(\mathcal{B},t),t)}{\dot{c}} + \underbrace{\frac{\partial}{\partial t} v}_{\text{at a fixed position } c(\mathcal{B},t)}$$

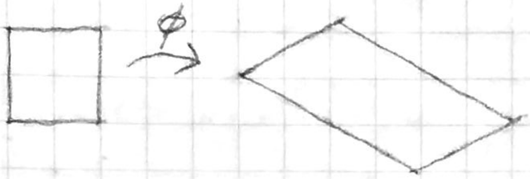
$$\bar{a}(\bar{c}(\mathcal{B}),t) = a(c(\mathcal{B},t),t) = (\nabla v)\, v + \frac{\partial}{\partial t} v$$

$$a(x,t) = (\nabla v(x,t))\, v(x,t) + \frac{\partial}{\partial t} v(x,t)$$

[Notebook page scanned on 2014-06-20]

$(23-24)_4$ Wednesday [2014-03-19]

Integrals, Jacobian, total volume, mass densities



$$\rho_0 \bar{V} \underset{\uparrow}{=} \rho V = (\rho \det F)\bar{V}$$

conservation of mass $\Rightarrow$ initial value $\rho_0 = \underbrace{\rho \det F}_{\text{constant quantity}}$

$$\frac{d}{dt}\rho_0 = 0 \Rightarrow 0 = \dot{\rho} + \rho \operatorname{div} v$$

$$\int_{\mathcal{R}} \rho\, dV = \int_{\bar{\mathcal{R}}} \rho \det F\, dV$$

possibly changing in time

mass conservation

$$\int_{\bar{\mathcal{R}}} \rho_0\, dV = \int_{\mathcal{R}} \rho\, dV$$

isochoric deformation $\det F = 1$

$$\nabla v = \dot{F} F^{-1}$$

$$\frac{d}{dt}(\det F) = (\det F) \operatorname{tr} \dot{F}F^{-1} = 0 \Rightarrow \operatorname{tr} \nabla v = 0$$

$$\operatorname{div} v = \operatorname{tr} \nabla v \qquad \operatorname{div} v = 0$$

$$\int_{\bar{\mathcal{R}}} dV = \operatorname{vol}(\bar{\mathcal{R}}) \; ; \quad \int_{\mathcal{R}} dV = \operatorname{vol}(\mathcal{R})$$

$$\int_{\mathcal{R}} dV = \int_{\bar{\mathcal{R}}} \det F\, dV$$

$$\frac{d}{dt}\int_{\mathcal{R}} dV = \frac{d}{dt}\int_{\bar{\mathcal{R}}} \det F\, dV = \int_{\bar{\mathcal{R}}} \frac{d}{dt}\det F\, dV = \int_{\bar{\mathcal{R}}} (\det F) \operatorname{tr} \nabla v\, dV$$

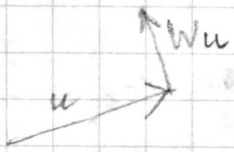
$$= \int_{\mathcal{R}} \operatorname{tr} \nabla v\, dV = \int_{\mathcal{R}} \operatorname{div} v\, dV = \int_{\partial\mathcal{R}} v \cdot n\, dA$$

[Notebook page scanned on 2014-06-20]

$$\nabla v = \dot{F}F^{-1}$$

$$\nabla v = \dot{R}R^{T} \in \mathrm{Skw} \qquad \frac{d}{dt} RR^{T} = \dot{R}R^{T} + R\dot{R}^{T} = 0$$

SPIN $\quad W := \dot{R}R^{T} \qquad Wu \cdot u = 0 \qquad Wu \cdot u = -u \cdot Wu$




$$Wa = \lambda a$$

$$Wa \cdot a = 0 \Rightarrow \lambda (a \cdot a) = 0$$

$$\Downarrow$$

$$\lambda = 0$$

$$W = -W^{T}$$

$$D \in \mathrm{Sym}$$

STRETCHING

[Notebook page scanned on 2014-06-20]

$(25\text{-}26)_5$ Monday [2014-03-24] A1,3
16:00 - 18:00

$Wa = \lambda a \Rightarrow \lambda = 0$

$\omega = a / \|a\|$

$\mathrm{vol}(e_1, e_2, e_3) = 1$

$W\omega = 0 \qquad \mathrm{vol}(\omega, u, v) = \omega \times u \cdot v \qquad \forall v$

$\mathrm{vol}(w, u, v) = (\ast w)\, u \cdot v \qquad \forall w, \forall u, \forall v$

Hodge star

$\mathrm{vol}(\omega, u, v) = W u \cdot v$

Power as a linear functional on velocity fields

Force $\qquad \mathcal{F}(v) \qquad$ representation

special form $\qquad \mathcal{F}(v) = f_A \cdot v(p_A) + f_B \cdot v(p_B) + f_C \cdot v(p_C)$

more general
$$\mathcal{F}(v) = \int_{\mathcal{R}} b(x) \cdot v(x)\, dV + \int_{\partial\mathcal{R}} t(x) \cdot v(x)\, dA$$

velocity field

$$v(c(h)) = v(c(0)) + \nabla v(c(0)) \big(r(c(h)) - r(c(0))\big) + o(h)$$

$v(x) = v_0 + \nabla v\,(x - p_0) \qquad$ affine

$\nabla v = \dot{R} R^T = W \qquad$ rigid

[Notebook page scanned on 2014-06-21]

$$W u_A \cdot f_A = \omega \times u_A \cdot f_A$$

$$= f_A \cdot \omega \times (p_A - p_o) = (p_A - p_o) \times f_A \cdot \omega$$

$$\mathcal{F}(v) = f_A \cdot v(p_A) + f_B \cdot v(p_B) + f_C \cdot v(p_C)$$

$$= f_A \cdot v_o + f_A \cdot \omega \times (p_A - p_o) + \dots$$

$$= f_A \cdot v_o + (p_A - p_o) \times f_A \cdot \omega + \dots$$

$$= \dots = f \cdot v_o + m \cdot \omega$$

$$\mathcal{F}(v) = \int_{\mathcal{R}} b(x) \cdot v(x) \, dV + \int_{\partial \mathcal{R}} t(x) \cdot v(x) \, dA$$

Bulk force (density with respect to the volume)

Traction (density with respect to the area)

per unit volume

per unit area

$$= \left( \int_{\mathcal{R}} b(x) \, dV + \int_{\partial \mathcal{R}} t(x) \, dA \right) \cdot v_o$$

$$+ \left( \int_{\mathcal{R}} (x - p_o) \times b(x) \, dV + \int_{\partial \mathcal{R}} (x - p_o) \times t(x) \, dA \right) \cdot \omega$$

$$= f \cdot v_o + m \cdot \omega$$

[Notebook page scanned on 2014-06-21]

$(27\text{-}28)_5$ Tuesday [2014-03-25]

example 1

$t(x) = 0$ , $b(x) = \rho g$  $g$ is a vector

$$f = \int_{\mathcal{R}} \rho g \, dV = \rho g \, Vol(\mathcal{R}) = \underbrace{\rho \, Vol(\mathcal{R})}_{\text{total mass}} g$$

$$m = \int_{\mathcal{R}} (x - p_0) \times (\rho g) \, dV = \int_{\mathcal{R}} (x - p_0) \, dV \times (\rho g)$$

$$\frac{1}{V} \int_{\mathcal{R}} (x - p_0) \, dV = (x_G - p_0)$$

↑ barycenter of $\mathcal{R}$

if $p_0 = x_G$ then $m = 0$

example 2

$b(x) = 0$, $t(x) = \rho g$  $\rho$ "skin" mass density

$$f = \int_{\partial\mathcal{R}} \rho g \, dA = \underbrace{\rho \, Area(\partial\mathcal{R})} g \, ; \quad m = \int_{\partial\mathcal{R}} (x - p_0) \, dA \times (\rho g)$$

$$\frac{1}{A} \int_{\partial\mathcal{R}} (x - p_0) \, dA = (x_G - p_0)$$

↑ barycenter of $\partial\mathcal{R}$

[Notebook page scanned on 2014-06-21]

Balance principle

$$\mathfrak{F}(v) = 0 \quad \forall v$$

$$v(x) = v_0 + W(x - p_0)$$

$$\Rightarrow \quad \mathfrak{F}(v) = f \cdot v_0 + m \cdot \omega$$

$$\mathfrak{F}(v) = 0 \quad \forall v \quad \Leftrightarrow \quad f = 0, \; m = 0$$

$v$ general continuous and differentiable vector field

$$\mathfrak{F}(v) = \int_{\mathcal{R}} b(x) \cdot v(x)\, dV + \int_{\partial \mathcal{R}} t(x) \cdot v(x)\, dV$$

$$\mathfrak{F}(v) = 0 \quad \forall v \quad \Rightarrow \quad b = 0, \; t = 0$$ no force is allowed

A convenient approach consists in setting

$$\mathfrak{F}(v) = \mathfrak{F}^{ext}(v) + \mathfrak{F}^{int}(v)$$

based on a suitable characterization of the two different parts, and still requiring

$$\mathfrak{F}(v) = 0 \quad \forall v$$

[Notebook page scanned on 2014-06-21]

In order to have some insight into the way of characterizing the force distribution let us consider just the extension of rigid velocity fields to affine velocity fields:

$$v(x) = v_o + L\,(x - p_o)$$

where $L$ is the velocity gradient (a tensor)

$$\mathcal{F}^{ext}(v) = f_A \cdot v_A + f_B \cdot v_B + f_C \cdot v_C$$

$$= f_A \cdot v_o + f_A \cdot L\,(p_A - p_o) + \dots$$

$$= f_A \cdot v_o + (p_A - p_o) \otimes f_A \cdot L + \dots$$

$$= (f_A + f_B + f_C) \cdot v_o$$
$$+ \big( (p_A - p_o) \otimes f_A + (p_B - p_o) \otimes f_B + (p_C - p_o) \otimes f_C \big) \cdot L$$

$$= f \cdot v_o + M \cdot L$$

[Notebook page scanned on 2014-06-21]

$(29\text{-}30)_5$ Wednesday [2014-03-26]

Gurtin

$$f_A \cdot L(p_A - p_O) = \underbrace{(p_A - p_O) \circledast f_A}_{= f_A \otimes (p_A - p_O)} \cdot L \qquad \circledast, \otimes$$

$$M \cdot L = \mathrm{tr}(M^T L)$$

$$\mathrm{tr}(AB) = \frac{\mathrm{vol}(AB e_1, e_2, e_3) + \cdots}{\mathrm{vol}(e_1, e_2, e_3)}$$

$$B e_i = b_{1i} e_1 + b_{2i} e_2 + b_{3i} e_3$$

$$A e_i = a_{1i} e_1 + a_{2i} e_2 + a_{3i} e_3$$

any basis

$$\begin{aligned} AB e_i &= b_{1i} A e_1 + b_{2i} A e_2 + b_{3i} A e_3 \\ &= b_{1i}(a_{11} e_1 + a_{21} e_2 + a_{31} e_3) \\ &\quad + b_{2i}(a_{12} e_1 + a_{22} e_2 + a_{32} e_3) \\ &\quad + b_{3i}(a_{13} e_1 + a_{23} e_2 + a_{33} e_3) \end{aligned}$$

$$\mathrm{vol}(AB e_1, e_2, e_3) = (b_{11} a_{11} + b_{21} a_{12} + b_{31} a_{13})$$

$$\mathrm{vol}(e_1, AB e_2, e_3) = (b_{12} a_{21} + b_{22} a_{22} + b_{32} a_{23})$$

$$\mathrm{vol}(e_1, e_2, AB e_3) = (b_{13} a_{31} + b_{23} a_{32} + b_{33} a_{33})$$

If we exchange A and B the sum above will be left unchanged. So $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

[Notebook page scanned on 2014-06-21]

$$M \cdot L = tr(M^T L)$$

$$tr(M^T L) = m^T_{11} v_{11} + m^T_{21} v_{12} + m^T_{31} v_{13}$$

$$+$$

$$+$$

By using an orthonormal basis we realize that

since $m^T_{21} = m_{12}$ then

$$tr(M^T L) = \sum\sum m_{ij} v_{ij}$$

$$A \cdot W = tr(A^T W) = tr(W^T A) = W \cdot A$$

$$\left.\begin{aligned} A^T &= A \\ W^T &= -W \end{aligned}\right\} \Rightarrow \quad = tr(AW) = -tr(WA) = -tr(AW)$$

$$\Rightarrow \quad A \cdot W = 0$$

Properties of "tr"

$$tr(A) = tr(A^T)$$

$$tr(AB) = tr(BA)$$

[Notebook page scanned on 2014-06-21]

$$\mathbf{M} \cdot \mathbf{L} = (\operatorname{skw} \mathbf{M} + \operatorname{sym} \mathbf{M}) \cdot (\mathbf{W} + \mathbf{D})$$

$$= \operatorname{skw} \mathbf{M} \cdot \mathbf{W} + \cancel{\operatorname{skw} \mathbf{M} \cdot \mathbf{D}}$$

$$+ \cancel{\operatorname{sym} \mathbf{M} \cdot \mathbf{W}} + \underline{\operatorname{sym} \mathbf{M} \cdot \mathbf{D}}$$

$$\mathcal{F}^{ext}(v) = f \cdot v_0 + \mathbf{M} \cdot \mathbf{L}$$

$$= f \cdot v_0 + \operatorname{skw} \mathbf{M} \cdot \mathbf{W} + \operatorname{sym} \mathbf{M} \cdot \mathbf{D}$$

$v$ affine velocity field

$$v(x) = v_0 + \mathbf{L}(x - p_0)$$

$v$ rigid velocity field ($\mathbf{L} = \mathbf{W}$)

$$\mathcal{F}^{ext}(v) = f \cdot v_0 + \underbrace{\operatorname{skw} \mathbf{M} \cdot \mathbf{W}}_{m \cdot \omega}$$

[Notebook page scanned on 2014-06-21]

$(31-32)_6$ Monday [2014-03-31]
16:00 - 18:00 A1,3

$$\mathcal{F}(v) = \mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v)$$

$\mathcal{F}(v) = 0$ balance principle

$$v(x) = v_0 + L(x - p_0)$$

$$\mathcal{F}^{ext} = f \cdot v_0 + M \cdot L$$

examples $\longrightarrow$

let us set $\mathcal{F}^{int}(v) = -(z \cdot v_0 + T \cdot L)\, V_\mathcal{R}$ $\longleftarrow$ why $V_\mathcal{R}$?

and give the following characterization

$z \cdot v_0 + T \cdot L = 0$ $\forall$ rigid velocity field

$$v(x) = v_0 + W(x - p_0)$$

$$\Rightarrow \quad z \cdot v_0 + T \cdot W = 0 \quad \forall v_0 \; \forall W$$

$$\Downarrow$$

$$z = 0, \quad \mathrm{skw}\, T = 0$$

$$\mathcal{F}(v) = (f - z V_\mathcal{R}) \cdot v_0 + (M - T V_\mathcal{R}) \cdot L$$

$$= f \cdot v_0 + \mathrm{skw}\, M \cdot W + (\mathrm{sym}\, M - T V_\mathcal{R}) \cdot D$$

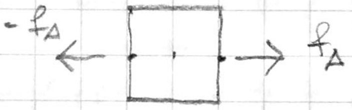
$$\mathcal{F}(v) = 0 \; \forall v \quad \Rightarrow \quad f = 0; \; \mathrm{skw}\, M = 0; \; \mathrm{sym}\, M = T V_\mathcal{R}$$

[Notebook page scanned on 2014-06-21]

arm force (Gurtin)

$$M_\Delta = (p_A - p_0) \otimes f_\Delta = f_\Delta \otimes (p_\Delta - p_0)$$

force arm




$$\frac{l_1}{2} e_1 \otimes (f e_1) = \frac{l_1}{2} f \, e_1 \otimes e_1$$


$$t_1 = t_{11} e_1 + t_{21} e_2 + t_{31} e_3$$

$$\frac{l_1}{2} e_1 \otimes t_1 \qquad \int_{\mathcal{F}_1} t \cdot v \, dA$$

$$e_1 \otimes e_2 = e_2 \otimes e_1 \qquad (e_1 \otimes e_2) e_1 = (e_1 \cdot e_1) e_2$$

$$[e_1 \otimes e_2] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (e_2 \otimes e_1) e_1 = e_2 (e_1 \cdot e_1)$$

$$(e_1 \otimes t_1) n_1 = (e_1 \cdot n_1) t_1$$

$$M_\Delta \cdot L = (p_A - p_0) \otimes f_A \cdot L = f_\Delta \cdot L (p_A - p_0) = f_A \cdot (v_A - v_0)$$
$$= f_A \otimes (p_A - p_0) \cdot L = f_\Delta \cdot L (p_A - p_0) = f_A \cdot (v_A - v_0)$$

[Notebook page scanned on 2014-06-21]


$$t_1 = t_{11}\, e_1 + t_{21}\, e_2 + t_{31}\, e_3$$

$$\int_{\mathscr{S}_1} t_1(x)\cdot v(x)\, dA = \int_{\mathscr{S}_1} t_1(x)\cdot v_0\, dA + \int_{\mathscr{S}_1} t_1(x)\cdot L(x-p_0)\, dA$$

$$= \int_{\mathscr{S}_1} t_1\, dA \cdot v_0 + \int_{\mathscr{S}_1} (x-p_0)\otimes t_1\, dA \cdot L = A_{\mathscr{S}_1}(g_{\mathscr{S}_1}-p_0)\otimes t_1 \cdot L$$

$\downarrow$

useless parameterization
$$\begin{cases} x-p_0 = \dfrac{\ell_1}{2}\, e_1 + s_2\, e_2 + s_3\, e_3 \\ -\dfrac{\ell_2}{2} \le s_2 \le \dfrac{\ell_2}{2} \qquad -\dfrac{\ell_3}{2} \le s_3 \le \dfrac{\ell_3}{2} \end{cases}$$

$$g_{\mathscr{S}_1} - p_0 = \frac{1}{A_{\mathscr{S}_1}} \int_{\mathscr{S}_1} (x-p_0)\, dA$$

$$\int_{\mathscr{S}_1 \cup \mathscr{S}_{-1}} t(x)\cdot v(x)\, dA = \int_{\mathscr{S}_1} t_1 \cdot v(x)\, dA - \int_{\mathscr{S}_{-1}} t_1 \cdot v(x)\, dA$$

$$= (t_1 A_{\mathscr{S}_1}) \otimes (g_{\mathscr{S}_1} - g_{\mathscr{S}_{-1}}) \cdot L$$

look at the general formula $\rightarrow$

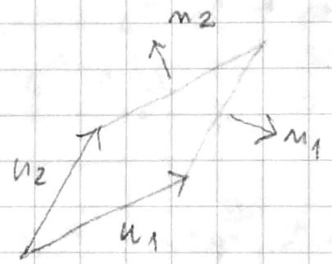
$$= (t_1\, \ell_2 \ell_3) \otimes (\ell_1 e_1)\cdot L = (t_1 \otimes e_1 \cdot L)\, V_{\mathscr{R}}$$

$\leftarrow$ back to the first Monday page

[Notebook page scanned on 2014-06-21]

$(33\text{-}34)_6$ Tuesday [2014-04-01] A1.3

9:00 – 11:00




$n_1$ orthogonal to span$\{u_2, u_3\}$

$n_2$ orthogonal to span$\{u_3, u_1\}$

$n_3$ orthogonal to span$\{u_1, u_2\}$

$\mathcal{F}_1$ for face $n_1, u_2, u_3$

$\mathcal{F}$ (<u>not</u> $\mathcal{J}$)

$$(*)\quad M = A_{\mathcal{F}_1}\, t_1 \otimes u_1 + A_{\mathcal{F}_2}\, t_2 \otimes u_2 + A_{\mathcal{F}_3}\, t_3 \otimes u_3$$

$$M n_1 = A_{\mathcal{F}_1}\, t_1 \underbrace{(u_1 \cdot n_1)}_{h_1} + A_{\mathcal{F}_2}\, t_2 \underbrace{(u_2 \cdot n_1)}_{0} + A_{\mathcal{F}_3}\, t_3 \underbrace{(u_3 \cdot n_1)}_{0}$$

$$M n_1 = (A_{\mathcal{F}_1} h_1)\, t_1 = V_{\mathcal{R}}\, t_1$$

$$(*)\quad \int_{\mathcal{F}_1} t \cdot v \, dA = \int_{\mathcal{F}_1} t_1 \cdot v_0 \, dA + \int_{\mathcal{F}_1} t_1 \cdot L(x - p_0)\, dA$$

$$= A_{\mathcal{F}_1} t_1 \cdot v_0 + \int_{\mathcal{F}_1} t_1 \otimes (x - p_0) \cdot L \, dA = A_{\mathcal{F}_1} t_1 \cdot v_0 + A_{\mathcal{F}_1} t_1 \otimes (\underbrace{g_{\mathcal{F}_1} - p_0}_{\frac{1}{2} u_1}) \cdot L$$

[...]

[Notebook page scanned on 2014-06-21]

Change of observer (frame-invariance)

Galilean transformations (Galilean group)

$$p_A^* = q_o^* + Q(p_A - p_o)$$

This transformation is defined by $q_o^*$ and $Q$

$q \equiv q_o^*$

$$\dot{p}_A^* = \dot{q}_o^* + \dot{Q}(p_A - p_o) + Q(\dot{p}_A - \dot{p}_o)$$

$$= \dot{q}_o^* + (\dot{Q} + QL)(p_A - p_o) = \dot{q}_o^* + (\dot{Q}Q^T + QLQ^T)(p_A^* - q_o^*)$$

$$v_A^* = v_o^* + L^*(p_A^* - q_o^*) \qquad p_o^* = q_o^*$$

$$\mathcal{F}^*(v^*) = -(z^* \cdot v_o^* + T^* \cdot L^*) V_{\mathcal{B}}$$

$$\mathcal{F}^*(v^*) = \mathcal{F}(v)$$

$$z^* \cdot v_o^* + T^* \cdot L^* = z \cdot v_o + T \cdot L$$

where $v_o^* = \dot{q}_o^*$ an arbitrary vector $\dot{q}$

and $L^* = \dot{Q}Q^T + QLQ^T$ with arbitrary $Q, \dot{Q}$

$$\Rightarrow \quad z^* \cdot v_o^* = z \cdot v_o \quad \forall v_o, \forall \dot{q} \quad \Rightarrow z = 0, z^* = 0$$

$$\Rightarrow \quad T^* \cdot (\dot{Q}Q^T + QLQ^T) = T \cdot L \quad \forall L, \forall \dot{Q}$$

$$\dot{Q} = 0 \quad \Rightarrow \quad T^* \cdot QLQ^T = T \cdot L$$

$$z^* \cdot (v_o + w_o^*) = z \cdot v_o \Rightarrow (z^* - z) \cdot v_o = z^* \cdot w^*$$

[Notebook page scanned on 2014-06-21]

$$Q^T T^* Q \cdot L = T \cdot L \qquad \forall L$$

$$\Rightarrow \quad Q^T T^* Q = T$$

$$T^* = Q T Q^T$$

$$\dot{Q} \neq 0 \qquad T^* \cdot \dot{Q} Q^T = 0 \qquad \forall \dot{Q} Q^T \qquad \Rightarrow \quad \text{skw}\, T^* = 0$$

$$0 = \text{skw}\, T^* = \frac{1}{2}\left(Q T Q^T - Q T^T Q^T\right) = \frac{1}{2} Q \left(T - T^T\right) Q^T$$

$$\Rightarrow \quad T = T^T$$

[Notebook page scanned on 2014-06-21]

$(35-36)_6$ Wednesday [2014-04-02] A1.3
11:00 - 13:00

Summarizing affine motions
(affine bodies: bodies undergoing affine motions)

$$v(x) = v_0 + L(x - p_0)$$

$$\mathcal{F}^{ext}(v) = f \cdot v_0 + M \cdot L$$

$$\mathcal{F}^{int}(v) = -(z \cdot v_0 + T \cdot L)V_{\mathcal{R}}$$

Balance $\quad \mathcal{F}(v) = 0 \quad \forall v$

Objectivity $\quad z = 0, \quad T = T^T$

$$\Rightarrow \quad f = 0$$
$$\operatorname{skw} M = 0$$
$$\operatorname{sym} M = T V_{\mathcal{R}}$$

with $\quad T n_1 = t_1 \quad [\ldots]$

[Notebook page scanned on 2014-06-21]

Balance equations in general

$$\mathcal{F}^{ext}(v) = \int_{\mathcal{R}} b(x) \cdot v(x)\, dV + \int_{\partial\mathcal{R}} t(x) \cdot v(x)\, dA$$

$$\mathcal{F}^{int}(v) = -\int_{\mathcal{R}} T(x) \cdot \underbrace{L(x)}_{\nabla v}\, dV$$

$$\mathcal{F}(v) = \mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = \int_{\mathcal{R}} b \cdot v\, dV + \int_{\partial\mathcal{R}} t \cdot v\, dA - \int_{\mathcal{R}} T \cdot \nabla v\, dV$$

$\operatorname{div} v = \operatorname{tr} \nabla v$ $\qquad$ $\nabla v$ already defined

$$\operatorname{div} T \cdot a = \operatorname{div}(T^T a) = \operatorname{tr} \nabla(T^T a)$$

$$\operatorname{div}(T^T v) = \operatorname{tr} \nabla(T^T v) = \operatorname{div} T \cdot v + T \cdot \nabla v$$

$$\nabla(T^T v)e = \lim_{h \to 0} \frac{T^T(x+he)v(x+he) - T^T(x)v(x)}{h}$$

$$= \lim_{h \to 0} \frac{T^T(x+he)v(x) - T^T(x)v(x)}{h} + T^T(x)\nabla v(x)e$$

$(21\text{-}22)_4 \rightarrow \quad v(c(h)) = v(c(0)) + \nabla v(c(0))\,(c(h) - c(0)) + o(h)$

$c(h) = x + he \Rightarrow v(x+he) = v(x) + \nabla v(x)(he) + o(h)$

[Notebook page scanned on 2014-06-21]

$$w(x+he) := T^T(x+he)\,v(x)$$

$$\nabla(T^T v)e = \lim_{h\to 0} \frac{w(x+he) - w(x)}{h} + T^T(x)\nabla v(x)\,e$$

$$= \nabla w(x)\,e + T^T(x)\nabla v(x)\,e$$

$$\operatorname{tr}\nabla(T^T v) = \operatorname{tr}\nabla w + \operatorname{tr}(T^T\nabla v)$$

$\operatorname{tr}\nabla w = \operatorname{div}T\cdot v$ since $\operatorname{tr}\nabla w$ is linear in $v(x)$

by the representation theorem

$$\operatorname{div}(T^T v) = \operatorname{tr}\nabla(T^T v) = \operatorname{div}T\cdot v + \operatorname{tr}(T^T\nabla v)$$

$$\operatorname{div}T\cdot v = \operatorname{div}(T^T v) - T\cdot\nabla v$$

$$-\int_{\mathcal{R}} T\cdot\nabla v\,dV = \int_{\mathcal{R}} \operatorname{div}T\cdot v\,dV - \int_{\mathcal{R}} \operatorname{div}(T^T v)\,dV$$

$$\int_{\mathcal{R}} \operatorname{div}T\cdot v\,dV - \int_{\partial\mathcal{R}} T^T v\cdot n\,dA$$

$$= \int_{\mathcal{R}} \operatorname{div}T\cdot v\,dV - \int_{\partial\mathcal{R}} Tn\cdot v\,dA$$

$$\mathcal{F}(v) = \int_{\mathcal{R}} b\cdot v\,dV + \int_{\partial\mathcal{R}} t\cdot v\,dA + \int_{\mathcal{R}} \operatorname{div}T\cdot v\,dV - \int_{\partial\mathcal{R}} Tn\cdot v\,dA = 0$$

CAUCHY BALANCE EQUATIONS
$$\begin{cases} b + \operatorname{div}T = 0 & \mathcal{R} \\ t - Tn = 0 & \partial\mathcal{R} \end{cases}$$

[Notebook page scanned on 2014-06-21]

$(37-38)_7$ Monday [2014-04-07]
16:00 - 18:00 A13

Material response

$$T = \hat{T}(F) \quad \text{elastic material}$$
$\hookleftarrow$ response function

$$p_\Delta = p_o + F(\bar{p}_\Delta - \bar{p}_o)$$

$$p_\Delta^* = q_o^* + Q(p_\Delta - p_o) \quad \text{change of observer}$$
$$p_o^* = q_o^* + Q(p_o - p_o) = q_o^*$$

$$p_\Delta^* - p_o^* = Q(p_\Delta - p_o) = QF(\bar{p}_\Delta - \bar{p}_o) = F^*(\bar{p}_\Delta - \bar{p}_o)$$

$$T^* = QTQ^T \qquad T^* = \hat{T}(F^*) \qquad F^* = QF$$

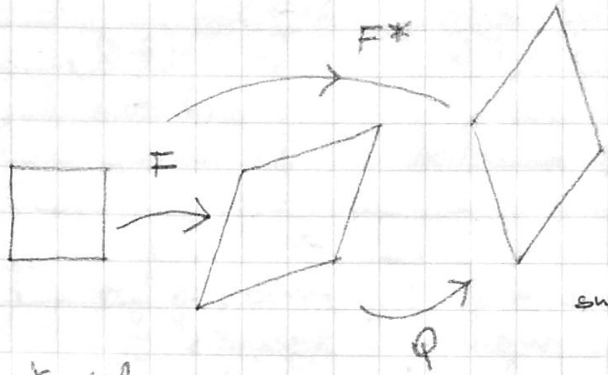
$$\hat{T}(F^*) = Q\hat{T}(F)Q^T \quad \forall Q$$
$$\hat{T}(QF) = Q\hat{T}(F)Q^T \quad \forall Q, \forall F$$
$$\hat{T}(QRU) = Q\hat{T}(RU)Q^T \quad \forall Q$$

$$Q = R^T \Rightarrow \hat{T}(U) = R^T\hat{T}(F)R$$

$$\hat{T}(F) = R\hat{T}(U)R^T$$

objective response function reduced form

[Notebook page scanned on 2014-06-21]



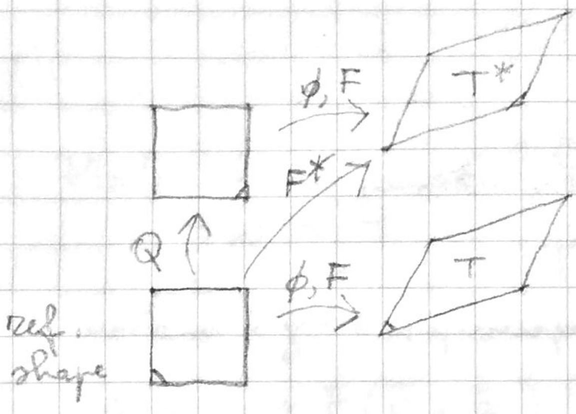

$$F^* = QF$$

superimposed rigid motion

material objectivity

$$\hat{T}(QF) = Q\hat{T}(F)Q^T$$

[Notebook page scanned on 2014-06-21]




$$F^* = FQ \Rightarrow T^* = \hat{T}(FQ)$$

$$T = \hat{T}(F)$$

material symmetry

$$\hat{T}(F) = \hat{T}(FQ) \qquad \hat{T} \text{ is } Q \text{ invariant}$$

Symmetry group: collection of such rotations $Q$

Isotropic material: symmetry group $Orth^+$

Isotropy

$$\hat{T}(F) = \hat{T}(FQ) = \hat{T}(RUQ) = \hat{T}(\overbrace{R\underbrace{QQ^T}_{I}}^{\tilde{R}}\overbrace{UQ}^{\tilde{U}}) \qquad \forall Q$$

objectivity

$$= (RQ)\hat{T}(Q^TUQ)(RQ)^T$$

$$= R\,Q\hat{T}(Q^TUQ)Q^TR^T$$

$$\hat{T}(F) = \hat{T}(RU) = R\hat{T}(U)R^T$$

$$\Rightarrow \quad Q\hat{T}(Q^TUQ)Q^T = \hat{T}(U)$$

$$\Leftrightarrow \quad \hat{T}(Q^TUQ) = Q^T\hat{T}(U)Q \qquad \forall Q$$

Isotropic response function

[Notebook page scanned on 2014-06-21]

[Notebook page scanned on 2014-06-21]

$(39\text{-}40)_7$ Tuesday [2014-04-08]
9:00–11:00

$$T \cdot L \; V_{\mathcal{R}} = T \cdot L \; V_{\bar{\mathcal{R}}} \det F$$

$$L\,(p_A - p_o) = LF\,(\bar{p}_A - \bar{p}_o)$$

$$T \cdot L \; V_{\bar{\mathcal{R}}} \det F = S \cdot LF \; V_{\bar{\mathcal{R}}}$$

$$= SF^T \cdot L \; V_{\bar{\mathcal{R}}}$$

$$T \det F = SF^T$$

$$S = T F^{-T} \det F$$

PIOLA stress

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\bar{\mathcal{R}}} (T \cdot \nabla v) \det F \, dV = \int_{\bar{\mathcal{R}}} S \cdot \nabla \bar{v} \, dV$$

$$\nabla \bar{v} = \nabla v \, F \quad \Rightarrow \quad T \cdot \nabla v \det F = T \cdot \nabla \bar{v} F^{-1} \det F$$

$$= (\det F)(T F^{-T}) \cdot \nabla \bar{v} = S \cdot \nabla \bar{v}$$

[Notebook page scanned on 2014-06-21]

$$\operatorname{div}(S^T \bar{v}) = S \cdot \nabla\bar{v} + \operatorname{div} S \cdot \bar{v}$$

fields on $\bar{\mathcal{R}}$

$$\int_{\mathcal{R}} b \cdot v \, dV = \int_{\bar{\mathcal{R}}} (b \cdot v) \det F \, dV = \int_{\bar{\mathcal{R}}} \bar{b} \cdot \bar{v} \, dV$$

$$\bar{v}(x) = v(\varphi(x)) \quad , \quad \bar{b} = b \det F$$

$$\int_{\partial\mathcal{R}} t \cdot v \, dA = \int_{\partial\bar{\mathcal{R}}} \bar{t} \cdot \bar{v} \, d\Delta \qquad \bar{t} = t \frac{\Delta_{\mathcal{F}}}{\Delta_{\bar{\mathcal{F}}}}$$

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\bar{\mathcal{R}}} S \cdot \nabla\bar{v} \, d\Delta$$

$$\int_{\bar{\mathcal{R}}} \bar{b} \cdot \bar{v} \, dV + \int_{\partial\bar{\mathcal{R}}} \bar{t} \cdot \bar{v} \, dV - \int_{\bar{\mathcal{R}}} S \cdot \nabla\bar{v} \, d\Delta = 0 \qquad \forall \bar{v}$$

[...]

$$\int_{\bar{\mathcal{R}}} \operatorname{div}(S^T \bar{v}) \, dV = \int_{\partial\bar{\mathcal{R}}} S^T \bar{v} \cdot \bar{n} \, dV$$

outward unit normal

[...]

$$\begin{cases} \bar{b} + \operatorname{div} S = 0 & \bar{\mathcal{R}} \\ \bar{t} - S\bar{n} = 0 & \partial\bar{\mathcal{R}} \end{cases}$$

compare

$$t = Tn \quad , \quad \bar{t} = S\bar{n} \qquad S = T \operatorname{cof} F$$

[Notebook page scanned on 2014-06-21]

[Notebook page scanned on 2014-06-22]

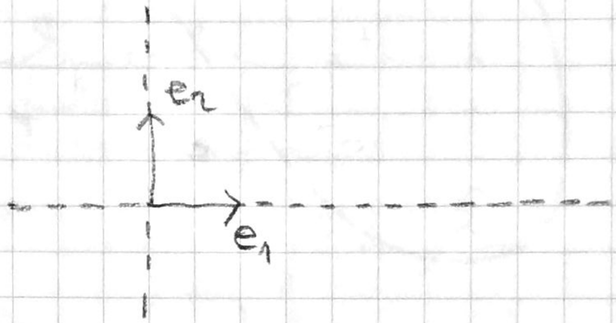
$(41-42)_7$ Wednesday [2014-04-09]

9:00 - 11:00 A1,2

Scalar form of the balance equations

$$\operatorname{div} T \cdot e_i = \operatorname{div}(T^T e_i) = \operatorname{tr} \nabla (T^T e_i)$$

$$[T] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$



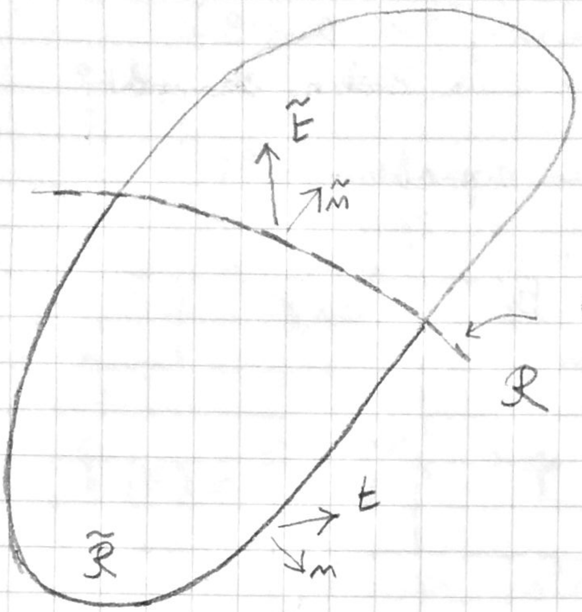

$$T^T e_1 = \sigma_{11}\, e_1 + \sigma_{12}\, e_2 + \sigma_{13}\, e_3$$
$$T^T e_2 = \sigma_{21}\, e_1 + \sigma_{22}\, e_2 + \sigma_{23}\, e_3$$
$$T^T e_3 = \sigma_{31}\, e_1 + \sigma_{32}\, e_2 + \sigma_{33}\, e_3$$

$$[\nabla(T^T e_1)] = \begin{pmatrix} \sigma_{11,1} & \sigma_{11,2} & \sigma_{11,3} \\ \sigma_{12,1} & \sigma_{12,2} & \sigma_{12,3} \\ \sigma_{13,1} & \sigma_{13,2} & \sigma_{13,3} \end{pmatrix}$$

$$\operatorname{div} T \cdot e_1 = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3}$$

$$\operatorname{div} T \cdot e_i = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3}$$

[Notebook page scanned on 2014-06-21]




$$\operatorname{div} T + b = 0$$

$$T n = t$$

an ideal splitting

$$\int_{\tilde{\mathcal{R}}} b \cdot v \, dV + \int_{\partial \tilde{\mathcal{R}}} t \cdot v \, dA - \int_{\tilde{\mathcal{R}}} T \cdot \nabla v \, dV = 0$$

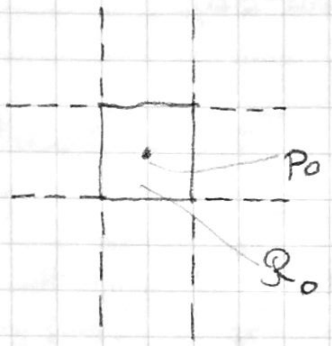
[...] $\operatorname{div} T + b = 0$

$$T n = t \,, \quad T \tilde{m} = \tilde{t}$$

$\tilde{t}$ is the traction we should apply to keep the lower part of the body balanced after removing the upper part.

[Notebook page scanned on 2014-06-22]




we can cut a sample in the shape of a cube centered at $p_o$ : $\mathcal{R}_o \subset \mathcal{R}$

By using an affine test velocity field we get

$$\int_{\mathcal{R}_o} T \cdot \nabla v \, dV = \int_{\mathcal{R}_o} T \cdot L \, dV = \left( \int_{\mathcal{R}_o} T \, dV \right) \cdot L$$

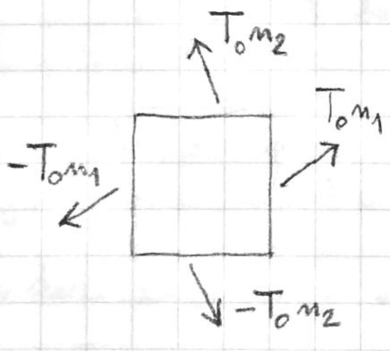
If we compare this expression with the one we wrote for an affine body we find

stress in the affine body

$$T_o \cdot L \, V_{\mathcal{R}_o} = \int_{\mathcal{R}_o} T \, dV \cdot L$$

stress field

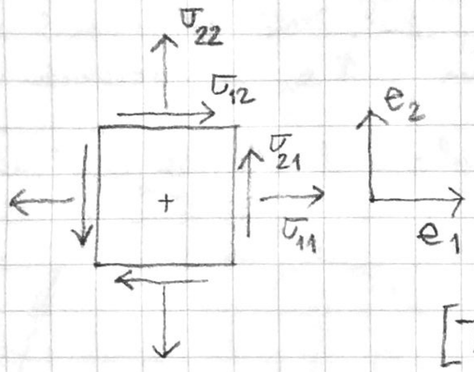
$$\Rightarrow \quad T_o = \frac{1}{V_{\mathcal{R}_o}} \int_{\mathcal{R}_o} T \, dV \qquad \text{mean stress}$$




traction on the boundary of the cube balanced by the uniform stress

$$T_o \simeq T(p_o) \quad (\text{small } V_{\mathcal{R}_o})$$

[Notebook page scanned on 2014-06-21]




$$
\begin{array}{cccc}
 & Te_1 & Te_2 & Te_3 \\
[T]_e = & \multicolumn{3}{c}{\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}}
\end{array}
$$

$T = T^T$

$\Rightarrow \quad \sigma_{21} = \sigma_{12} \quad [\ldots]$

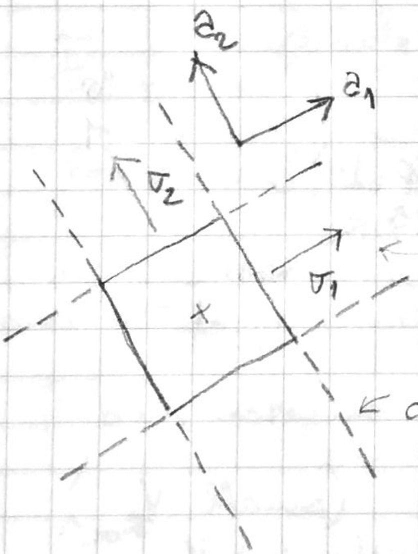
$\Downarrow$

real eigenvalues and orthogonal eigenvectors

$\sigma_1, \sigma_2, \sigma_3$ — $a_1, a_2, a_3$

principal stresses — principal directions

diagonal matrix
$$[T]_a = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}$$




← normal stress
no tangential stress

← cuts orthogonal to the principal directions

[Notebook page scanned on 2014-06-21]

$(43\text{-}44)_8$ Monday [2014-04-14] A1.3
16:00-18:00

Elastic energy

stress power density per unit current volume $\quad T\cdot\nabla v = T\cdot\dot{F}F^{-1}$

In an affine motion

$$T\cdot\dot{F}F^{-1}\,V_{\mathcal{R}} = T\cdot\dot{F}F^{-1}\,V_{\bar{\mathcal{R}}}\det F$$
$$= TF^{-T}\cdot\dot{F}\,V_{\bar{\mathcal{R}}}\det F$$
$$= \underbrace{S\cdot\dot{F}}\,V_{\bar{\mathcal{R}}}$$

power density per unit reference volume

Hyperelastic material: there exists $\varphi$ such that

$$\hat{T}(F)\cdot\dot{F}F^{-1}\,V_{\mathcal{R}} = \frac{d}{dt}\varphi(F)\,V_{\bar{\mathcal{R}}}$$

in any (affine) motion

equivalently $\quad \hat{S}(F)\cdot\dot{F} = \dfrac{d}{dt}\varphi(F)$

elastic energy per unit reference volume

Objectivity: elastic energy invariance

under superposed rigid motion: $\varphi(F) = \varphi(QF)$

$Q = R^T \Rightarrow \qquad \varphi(F) = \varphi(R^TRU) = \varphi(U)$

necessary condition

[Notebook page scanned on 2014-06-21]

Material symmetry

$$\varphi(F) = \varphi(FQ) \quad \Rightarrow Q \text{ belongs to the symmetry group}$$

Isotropic material

$$\forall Q \qquad \varphi(F) = \varphi(FQ) = \varphi(RUQ) = \varphi(\underbrace{RQ}_{\tilde{R}}\underbrace{Q^T UQ}_{\tilde{U}})$$

objectivity

$$\varphi(U) = \varphi(Q^T UQ)$$

isotropic energy function

Spectral decomposition of the stretch

$$U = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \qquad P_i = \underbrace{u_i}_{\text{unit eigenvector}} \otimes u_i$$

$$\forall Q \qquad Q^T U Q = \lambda_1 Q^T P_1 Q + \lambda_2 Q^T P_2 Q + \lambda_3 Q^T P_3 Q$$

if $\quad u = P_i u$ (eigenvector of $U$)

then $\quad U u = \lambda_i u$

and $\quad (Q^T U Q) \underbrace{Q^T u}_{\text{eigenvector of } Q^T UQ} = \lambda_i Q^T u$

isotropy $\varphi(U) = \varphi(\underbrace{Q^T U Q}_{\substack{\text{same eigenvalues} \\ \text{different eigenvectors}}})$

$\Rightarrow \varphi(U)$ is independent of the eigenvectors of $U$

[Notebook page scanned on 2014-06-21]

$(45-46)_8$ Tuesday [2014-04-15] A1.3
9:00-11:00

Isotropic elastic energy

$$\varphi(F) = \varphi(U) = \tilde{\varphi}(\lambda_1, \lambda_2, \lambda_3)$$

$$\varphi(U) = \breve{\varphi}(U^2) = \breve{\varphi}(C) = \tilde{\varphi}(\lambda_1^2, \lambda_2^2, \lambda_3^2)$$
$$= \hat{\varphi}(\iota_1, \iota_2, \iota_3)$$

principal invariants of C [iota]

coefficients of the characteristic polynomial

$$\det(C - \eta I) = \eta^3 - \iota_1 \eta^2 + \iota_2 \eta - \iota_3$$

$$\iota_1 = \operatorname{tr} C$$
$$\iota_2 = \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right)$$
$$\iota_3 = \det C$$

$$\frac{d}{dt}\varphi(F) = \frac{d}{dt}\hat{\varphi}(\iota_1, \iota_2, \iota_3)$$
$$= \hat{\varphi}_{,1}\frac{d\iota_1}{dt} + \hat{\varphi}_{,2}\frac{d\iota_2}{dt} + \hat{\varphi}_{,3}\frac{d\iota_3}{dt}$$

[Notebook page scanned on 2014-06-21]

$$\frac{d}{dt}\iota_1 = \frac{d}{dt}(F\cdot F) = 2F\cdot\dot{F} \qquad \iota_1 = \operatorname{tr}(F^TF) = F\cdot F$$

$$\frac{d}{dt}\iota_2 = 2\iota_1 F\cdot\dot{F} - \frac{1}{2}\frac{d}{dt}(C\cdot C)$$

$$= 2\iota_1 F\cdot\dot{F} - C\cdot\dot{C} = 2\iota_1 F\cdot\dot{F} - 2FC\cdot\dot{F}$$

$$C\cdot\dot{C} = C\cdot(\dot{F}^TF + F^T\dot{F}) = C\cdot((F^T\dot{F})^T + F^T\dot{F})$$

$$= 2C\cdot \operatorname{sym} F^T\dot{F} = 2C\cdot F^T\dot{F} = 2FC\cdot\dot{F}$$

$$\frac{d}{dt}\iota_3 = \frac{d}{dt}\det C = \frac{d}{dt}(\det F)^2 = 2\det F\frac{d}{dt}\det F$$

$$= 2\det F \quad \det F\operatorname{tr}(\dot{F}F^{-1}) = 2\iota_3\operatorname{tr}(\dot{F}F^{-1})$$

$$= 2\iota_3\dot{F}\cdot F^{-T} = 2\iota_3 F^{-T}\cdot\dot{F}$$

$$= 2\iota_3 FC^{-1}\cdot\dot{F}$$

isotropic elastic energy

$$\frac{d}{dt}\varphi(F) = 2F\left(\left(\frac{\partial\hat{\varphi}}{\partial\iota_1} + \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_1\right)I - \frac{\partial\hat{\varphi}}{\partial\iota_2}C + \frac{\partial\hat{\varphi}}{\partial\iota_3}\iota_3 C^{-1}\right)\cdot\dot{F}$$

$$\hat{S}(F)\cdot\dot{F} = \frac{d}{dt}\varphi(F)$$

[Notebook page scanned on 2014-06-21]

$$\hat{S}(F)\cdot\dot{F} = \left(\hat{T}(F)F^{-T}\det F\right)\cdot\dot{F} = \hat{T}(F)\cdot\dot{F}F^{-1}\det F$$

$$\hat{T}(F) = \hat{S}(F)F^{T}\frac{1}{\det F}$$

$$\hat{T}(F) = \frac{2}{\sqrt{\iota_3}}\left(\left(\frac{\partial\hat{\varphi}}{\partial\iota_1}+\frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_1\right)B - \frac{\partial\hat{\varphi}}{\partial\iota_2}B^2 + \frac{\partial\hat{\varphi}}{\partial\iota_3}\iota_3 I\right)$$

$B = FF^T$ left Cauchy-Green tensor

$$FCF^T = FF^TFF^T = B^2$$

$$FC^{-1}F^T = F(F^{-1}F^{-T})F^T = I$$

Second Piola stress tensor

$$S\cdot\dot{F} = \frac{1}{2}S_{II}\cdot\dot{C}$$

$C = C^T \Rightarrow S_{II}$ is a symmetric tensor

$$\frac{1}{2}S_{II}\cdot\dot{C} = \frac{1}{2}S_{II}\cdot\left(\dot{F}^TF + F^T\dot{F}\right) = S_{II}\cdot\frac{1}{2}\left((F^T\dot{F})^T + (F^T\dot{F})\right)$$

$$= S_{II}\cdot \operatorname{sym} F^T\dot{F} = S_{II}\cdot F^T\dot{F} = FS_{II}\cdot\dot{F}$$

$$\Rightarrow \quad S = FS_{II} \Rightarrow S_{II} = F^{-1}S = F^{-1}TF^{-T}\det I$$

[Notebook page scanned on 2014-06-21]

$(47-48)_8$ Wednesday [2014-04-16] A1.3
9:00-11:00

Force balance principle

$$\mathcal{F}^{ext}(v) + \mathcal{F}^{int}(v) = 0 \quad \forall v$$

Power expended

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dV = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

by using the Piola stress

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\mathcal{R}} T \cdot \dot{F} F^{-1} dV = \int_{\bar{\mathcal{R}}} S \cdot \dot{F} \, dV$$

Energy balance principle (dissipation inequality)

$$S \cdot \dot{F} - \frac{d}{dt} \varphi(F) \geq 0$$

$$S \cdot \dot{F} - \hat{S}(F) \cdot \dot{F} \geq 0$$

$$\overbrace{\left(S - \hat{S}(F)\right)}^{S^+} \cdot \dot{F} \geq 0 \qquad S^+ \cdot \dot{F} \geq 0$$

$$\underbrace{\left(T - \hat{T}(F)\right)}_{T^+} \cdot \dot{F} F^{-1} \geq 0 \qquad T^+ \cdot \dot{F} F^{-1} \geq 0$$

$$T = \hat{T}(F) + T^+$$

Possible choice (viscous stress)

$$T^+ := 2\mu \, \mathrm{sym} \, \dot{F} F^{-1}$$

$$\left(\mathrm{sym} \, \dot{F} F^{-1}\right) \cdot \dot{F} F^{-1} \geq 0$$

viscosity $\qquad \mu \geq 0 \Leftrightarrow T^+ \cdot \dot{F} F^{-1} \geq 0$

[Notebook page scanned on 2014-06-21]

Incompressible materials

$$\det F = 1 \quad \text{isochoric deformation (motion)}$$

$$\frac{d}{dt}\det F = (\det F)\, \mathrm{tr}(\dot{F}F^{-1}) = 0$$

$$\Rightarrow \quad \mathrm{tr}(\dot{F}F^{-1}) = 0 \Leftrightarrow \mathrm{tr}\nabla v = 0 \Leftrightarrow \mathrm{div}\, v = 0$$

$$aI \cdot \nabla v = a\, \mathrm{tr}\nabla v = 0 \qquad \forall \text{ spherical tensor } aI$$

$$\mathrm{sph}\, T := \frac{1}{3}(\mathrm{tr}T)I \Rightarrow (\mathrm{sph}\, T)\cdot \nabla v = 0$$

$$\mathrm{dev}\, T := T - \mathrm{sph}\, T \Rightarrow \mathrm{tr}(\mathrm{dev}\, T) = \mathrm{tr}T - \frac{1}{3}(\mathrm{tr}T)3 = 0$$

$$T = \underbrace{\mathrm{sph}\, T}_{\text{spherical part}} + \underbrace{\mathrm{dev}\, T}_{\text{deviatoric part}}$$

$$\underbrace{\mathrm{tr}\nabla v = 0}_{\substack{\text{isochoric} \\ \text{velocity field}}} \Rightarrow T\cdot\nabla v = (\mathrm{sph}\, T + \mathrm{dev}\, T)\cdot \nabla v = \mathrm{dev}\, T \cdot \nabla v$$

$$\det F = 1 \Rightarrow \hat{S}(F)\cdot\dot{F} = \hat{T}(F)\cdot\dot{F}F^{-1} = \mathrm{dev}\,\hat{T}(F)\cdot\dot{F}F^{-1}$$

elastic stress $\quad \mathrm{dev}\,\hat{T}(F)\cdot\dot{F}F^{-1} = \frac{d}{dt}\varphi(F)$

reactive stress $\quad \mathrm{sph}\, T = -p I$

internal pressure

[Notebook page scanned on 2014-06-21]

only the deviatoric part of the stress is determined by the energy because of the incompressibility

$$\Rightarrow \quad \underbrace{\left(\operatorname{dev} T - \hat{T}(F)\right)}_{T^{+} \text{ is a deviatoric tensor}} \cdot \dot{F} F^{-1} \geq 0$$

$$T = \operatorname{dev} T + \operatorname{sph} T = \hat{T}(F) + T^{+} - pI$$

$$T = \hat{T}(F) - pI + T^{+}$$

elastic reactive dissipative

deviatoric spherical deviatoric

[Notebook page scanned on 2014-06-21]

$(49-50)_9$ Wednesday [2014-04-23] A1.3
9:00 - 11:00

A review about

- incompressibility
- isochoric motions
- deviatoric and spherical parts of the stress
- dissipation and viscosity

A final remark about why $T^+$ should depend on $\nabla v$ in order to satisfy the assumption $T^+ \cdot \nabla v \geq 0$ in any motion.

[Notebook page scanned on 2014-06-21]

$(51-52)_{10}$ Monday [2014-04-28] A1.3
16:00-18:00

Newtonian fluids

incompressibility $\operatorname{div} v(x,t) = 0$

$\hat{T}(F) = 0$

$T^{+} = 2\mu \operatorname{sym} \nabla v$

$$\Rightarrow \qquad T = -pI + 2\mu \operatorname{sym} \nabla v$$

balance equation $\operatorname{div} T + b = 0 \qquad b = b_0 + b^{in}$

$$\operatorname{div} T = -\operatorname{div}(pI) + \mu(\operatorname{div} \nabla v + \operatorname{div} \nabla v^{T})$$

$\operatorname{div}(pI) = \nabla p$ (as a vector)

$\operatorname{div} \nabla v = \Delta v$ (Laplacian)

$$\operatorname{div} \nabla v^{T} = \nabla(\underbrace{\operatorname{div} v}_{0}) = 0$$

[Notebook page scanned on 2014-06-21]

$$\operatorname{div} p\mathbf{I} \cdot a = \operatorname{div}(p a) = \operatorname{tr}(\nabla(p a))$$
$$= \operatorname{tr}(a \otimes \nabla p) = \nabla p \cdot a$$

$$\nabla(p a) e = \lim_{h \to 0} \frac{1}{h} \big(p(x + h e) - p(x)\big) a$$
$$= (\nabla p \cdot e) a = (a \otimes \nabla p) e$$

$$\operatorname{tr}(u \otimes v) = \frac{\operatorname{vol}((u \otimes v) e_1, e_2, e_3)}{\operatorname{vol}(e_1, e_2, e_3)} + \dots + \dots$$
$$= \frac{\operatorname{vol}(u (v \cdot e_1), e_2, e_3)}{\operatorname{vol}(e_1, e_2, e_3)} + \dots + \dots$$
$$= u_1 (v \cdot e_1) + u_2 (v \cdot e_2) + u_3 (v \cdot e_3)$$
$$= v \cdot (u_1 e_1 + u_2 e_2 + u_3 e_3) = v \cdot u$$

$$\operatorname{div} \nabla v^T \cdot a = \operatorname{div}((\nabla v) a) = \operatorname{tr} \nabla w \qquad w := (\nabla v) a$$

$$(\nabla w) e = \lim_{h \to 0} \frac{1}{h} \big(w(x + h e) - w(x)\big)$$
$$= \lim_{h \to 0} \frac{1}{h} \big(\nabla v(x + h e) a - \nabla v(x) a\big)$$

[Notebook page scanned on 2014-06-21]

$$\nabla v(x+he)a = \lim_{d\to 0}\frac{1}{d}\Big(v(x+he+da)-v(x+he)\Big)$$

$$\nabla v(x)a = \lim_{d\to 0}\frac{1}{d}\Big(v(x+da)-v(x)\Big)$$

$$(\nabla w)e = \lim_{d\to 0}\frac{1}{d}\Big(\lim_{h\to 0}\frac{1}{h}\big(v(x+he+da)-v(x+da)\big)$$

$$-\lim_{h\to 0}\frac{1}{h}\big(v(x+he)-v(x)\big)\Big)$$

$$= \lim_{d\to 0}\frac{1}{d}\Big(\nabla v(x+da)e-\nabla v(x)e\Big)$$

$$\Rightarrow \quad \nabla w = \lim_{d\to 0}\frac{1}{d}\Big(\nabla v(x+da)-\nabla v(x)\Big)$$

$$\operatorname{tr}\nabla w = \lim_{d\to 0}\frac{1}{d}\Big(\underbrace{\operatorname{tr}(\nabla v(x+da))}_{\operatorname{div} v(x+da)}-\underbrace{\operatorname{tr}(\nabla v(x))}_{\operatorname{div} v(x)}\Big)$$

$$\operatorname{tr}\nabla w = \big(\nabla(\operatorname{div} v)\big)\cdot a$$

$$\Rightarrow \operatorname{div}\nabla v^T = \nabla(\operatorname{div} v)$$

$\operatorname{div} v = 0$ (incompressibility) $\Rightarrow \operatorname{div}\nabla v^T = 0$

Finally let us define $\Delta v := \operatorname{div}\nabla v$

it is a vector field ↑ Laplacian

[Notebook page scanned on 2014-06-21]

$$\operatorname{div} \nabla v \cdot e_i = \operatorname{div}(\nabla v^T e_i) = \operatorname{tr} \nabla(\nabla v^T e_i)$$

$$\nabla v^T e_1 = v_{1,1}\, e_1 + v_{1,2}\, e_2 + v_{1,3}\, e_3$$

$$\nabla v^T e_i = v_{i,1}\, e_1 + v_{i,2}\, e_2 + v_{i,3}\, e_3$$

$$\left[\nabla(\nabla v^T e_i)\right] = \begin{pmatrix} v_{i,11} & v_{i,12} & v_{i,13} \\ v_{i,21} & v_{i,22} & v_{i,23} \\ v_{i,31} & v_{i,32} & v_{i,33} \end{pmatrix}$$

$$\operatorname{tr} \nabla(\nabla v^T e_i) = v_{i,11} + v_{i,22} + v_{i,33}$$

$$\operatorname{div} \nabla v = \Delta v \quad \text{Laplacian}$$

[Notebook page scanned on 2014-06-21]

$$\operatorname{div} \nabla v^T \cdot e_i = \operatorname{div}(\nabla v\, e_i) = \operatorname{tr} \nabla(\nabla v\, e_i)$$

$$\nabla v\, e_1 = v_{1,1}\, e_1 + v_{2,1}\, e_2 + v_{3,1}\, e_3$$

$$\nabla v\, e_i = v_{1,i}\, e_1 + v_{2,i}\, e_2 + v_{3,i}\, e_3$$

$$[\nabla(\nabla v\, e_i)] = \begin{pmatrix} v_{1,i1} & v_{1,i2} & v_{1,i3} \\ v_{2,i1} & v_{2,i2} & v_{2,i3} \\ v_{3,i1} & v_{3,i2} & v_{3,i3} \end{pmatrix}$$

$$\begin{aligned} \operatorname{tr} \nabla(\nabla v\, e_i) &= v_{1,i1} + v_{2,i2} + v_{3,i3} \\ &= v_{1,1i} + v_{2,2i} + v_{3,3i} \\ &= \underbrace{(v_{1,1} + v_{2,2} + v_{3,3})}_{\operatorname{tr} \nabla v}{}_{,i} \end{aligned}$$

$$\operatorname{tr} \nabla v = 0 \Rightarrow \operatorname{div} \nabla v^T = 0$$

[Notebook page scanned on 2014-06-21]

$(53\text{-}54)_{10}$ Tuesday [2014-06-29] A1.3
9:00 - 11:00

Kinetic energy (density per unit current volume)

$$\frac{1}{2}\rho\|v\|^2 = \frac{1}{2}\rho\, v\cdot v$$

$$K := \int_{\mathcal{R}} \frac{1}{2}\rho\|v\|^2 dV = \int_{\bar{\mathcal{R}}} \frac{1}{2}\underbrace{(\det F)\rho}_{\rho_0 \text{ mass density per unit reference volume}}\|\bar{v}\|^2 dV$$ total kinetic energy

$$= \int_{\bar{\mathcal{R}}} \underbrace{\frac{1}{2}\rho_0\,\bar{v}\cdot\bar{v}}_{\text{kinetic energy density per unit reference volume}} dV$$

Inertial force (density per unit current volume)

$$\int_{\mathcal{R}} b^{in}\cdot v\, dV = -\frac{d}{dt}\int_{\mathcal{R}} \frac{1}{2}\rho\|v\|^2 dV$$

$$= -\frac{d}{dt}\int_{\bar{\mathcal{R}}} \frac{1}{2}\rho_0\|\bar{v}\|^2 dV = -\int_{\bar{\mathcal{R}}} \frac{d}{dt}\frac{1}{2}\rho_0\,\bar{v}\cdot\bar{v}\, dV$$

independent of time — costant in time

$$= -\int_{\bar{\mathcal{R}}} \rho_0\,\bar{a}\cdot\bar{v}\, dV = -\int_{\mathcal{R}} \underbrace{\rho_0 \frac{1}{\det F}}_{\rho \text{ mass density per unit current volume}} a\cdot v\, dV$$

$$\Rightarrow\quad b^{in}(x) = -\rho(x)\, a(x)$$

[Notebook page scanned on 2014-06-21]

$$\operatorname{div} T + b^{in} + b_o = 0$$

$$-\nabla p + \mu \Delta v - \rho a + b_o = 0$$

$$-\nabla p + \mu \Delta v - \rho\left((\nabla v) v + v'\right) + b_o = 0$$

(velocity and acceleration spatial description)

↳ $(21-22)_4$

Navier-Stokes equation
for Newtonian fluids

[Notebook page scanned on 2014-06-21]

$(55-56)_{10}$ Wednesday [2014-04-30] A1.3
9:00-11:00

In any motion, because of the balance principle,

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

If $b = b^{in}$ and $t \cdot v = 0$ on $\partial\mathcal{R}$ then

$$\int_{\mathcal{R}} b^{in} \cdot v \, dV = \int_{\mathcal{R}} T \cdot \nabla v \, dV$$

with $\displaystyle\int_{\mathcal{R}} b^{in} \cdot v \, dV = -\frac{d}{dt} K$

and $\displaystyle\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\mathcal{R}} (\hat{T}(F) - pI + T^+) \cdot \nabla v \, dV$

Further if the material is incompressible then

$$pI \cdot \nabla v = p \operatorname{tr} \nabla v = p \operatorname{div} v = 0$$

otherwise we set $p=0$ because there is no reason to split $\hat{T}(F)$ into a deviatoric and a spherical parts.

So whether the material is incompressible or not

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \int_{\mathcal{R}} \hat{T}(F) \cdot \nabla v \, dV + \int_{\mathcal{R}} T^+ \cdot \nabla v \, dV$$

[Notebook page scanned on 2014-06-21]

Recalling that for an hyperelastic material

$$\int_{\mathcal{R}} \hat{T}(F) \cdot \dot{F} F^{-1} dV = \int_{\bar{\mathcal{R}}} \hat{S}(F) \cdot \dot{F} \, dV$$

$$= \int_{\bar{\mathcal{R}}} \frac{d}{dt} \varphi(F) \, dV = \frac{d}{dt} \int_{\bar{\mathcal{R}}} \varphi(F) \, dV$$

denoting by

$$\Phi = \int_{\bar{\mathcal{R}}} \varphi(F) \, dV$$

the total strain energy (or elastic energy)

we get

$$\int_{\mathcal{R}} \hat{T}(F) \cdot \nabla v \, dV = \frac{d}{dt} \Phi$$

Hence

$$\int_{\mathcal{R}} T \cdot \nabla v \, dV = \frac{d}{dt} \Phi + \int_{\mathcal{R}} T^{+} \cdot \nabla v \, dV$$

By replacing these expressions in the starting power balance equation we get

$$-\frac{d}{dt} K - \frac{d}{dt} \Phi = \int_{\mathcal{R}} T^{+} \cdot \nabla v \, dV \geq 0$$

$(47\text{-}48)_8$

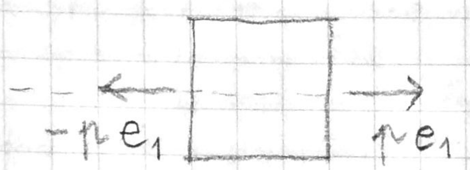
$$\Rightarrow \quad \frac{d}{dt} K + \frac{d}{dt} \Phi \leq 0$$

[Notebook page scanned on 2014-06-21]

$(57\text{-}58)_{11}$ Monday [2014-05-05] A1.3
16:00 – 18:00

A standard problem in elasticity




uniform traction

cubic reference shape

Let us consider "cylindrical deformations" only:
affine deformations with

$$F e_1 = \lambda_1 e_1 \qquad\qquad [F] = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$
$$F e_2 = \lambda_2 e_2$$
$$F e_3 = \lambda_3 e_3$$

moment tensor

$$\begin{aligned} M &= \bar{\ell} F e_1 \otimes (p e_1) A_{\mathcal{F}_1} \\ &= \bar{\ell}\, \lambda_1\, e_1 \otimes e_1\, (p A_{\mathcal{F}_1}) \\ &= \ell_1\, e_1 \otimes e_1\, (p\, \ell_2 \ell_3) \\ &= \underbrace{\ell_1 \ell_2 \ell_3}_{\text{current volume } V_{\mathcal{R}}}\, p\; e_1 \otimes e_1 \end{aligned}$$

balance equations

$$f = 0$$

$$\operatorname{skw} M = 0$$

$$\frac{M}{V_{\mathcal{R}}} = T$$

[Notebook page scanned on 2014-06-21]

from balance $\quad \mathsf{T} = p\, e_1 \otimes e_1$

material characterization

$$\mathsf{T} = \hat{\mathsf{T}}(\mathsf{F}) - p\,\mathsf{I} + \mathsf{T}^+$$

(incompressibility) (dissipation)

Let us consider a compressible material with no dissipation: $\quad \mathsf{T} = \hat{\mathsf{T}}(\mathsf{F})$

with

$$[\hat{\mathsf{T}}(\mathsf{F})] = \begin{pmatrix} \hat{\sigma}_{11}(\mathsf{F}) & & \\ & \hat{\sigma}_{22}(\mathsf{F}) & \\ & & \hat{\sigma}_{33}(\mathsf{F}) \end{pmatrix}$$

$$[\dot{\mathsf{F}}\,\mathsf{F}^{-1}] = \begin{pmatrix} \frac{\dot{\lambda}_1}{\lambda_1} & & \\ & \frac{\dot{\lambda}_2}{\lambda_2} & \\ & & \frac{\dot{\lambda}_3}{\lambda_3} \end{pmatrix}$$

[Notebook page scanned on 2014-06-21]

balance

$$\begin{pmatrix} \sigma_{11} & & \\ & \sigma_{22} & \\ & & \sigma_{33} \end{pmatrix} = \begin{pmatrix} p & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

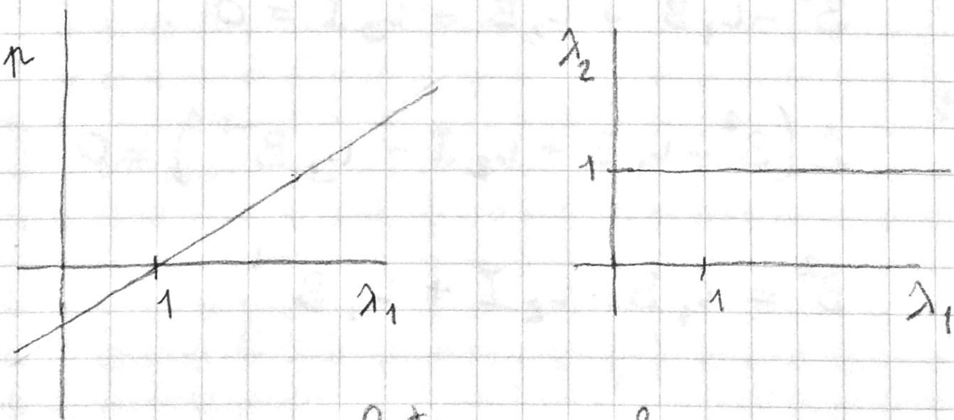
$$T = p \, e_1 \otimes e_1$$

A simple choice for $\hat{T}(F)$ is

$$\hat{\sigma}_{11}(F) = a_1(\lambda_1 - 1)$$
$$\hat{\sigma}_{22}(F) = a_2(\lambda_2 - 1)$$
$$\hat{\sigma}_{33}(F) = a_3(\lambda_3 - 1)$$

Balance and material response

$$p = a_1(\lambda_1 - 1)$$
$$0 = a_2(\lambda_2 - 1)$$
$$0 = a_3(\lambda_3 - 1)$$




solution graphs

[Notebook page scanned on 2014-06-21]

$(59\text{-}60)_{11}$ Tuesday [2014-05-06] A1.3
9:00-11:00

The solution we derived from the assumed material response is unsatisfactory because it allows $\lambda_1$ to be negative; further $\lambda_2$ and $\lambda_3$ are always equal to 1.

Let us consider the response function for a hyperelastic material $\rightarrow (45\text{-}46)_8$

$$\hat{T}(F) = \frac{2}{\sqrt{\iota_3}}\left(\left(\frac{\partial\hat{\varphi}}{\partial\iota_1} + \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_1\right)B - \frac{\partial\hat{\varphi}}{\partial\iota_2}B^2 + \frac{\partial\hat{\varphi}}{\partial\iota_3}\iota_3 I\right)$$

with $B = FF^T$ (left Cauchy-Green tensor)

We can get a new expression by using the Caley-Hamilton theorem stating that any tensor satisfies its characteristic equation:

$$B^3 - \iota_1 B^2 + \iota_2 B - \iota_3 I = 0$$

We get

$$B\left(B^2 - \iota_1 B + \iota_2 I - \iota_3 B^{-1}\right) = 0$$

$$B^2 = \iota_1 B - \iota_2 I + \iota_3 B^{-1}$$

[Notebook page scanned on 2014-06-21]

$$\hat{T}(F) = \frac{2}{\sqrt{\iota_3}}\left(\frac{\partial\hat{\varphi}}{\partial\iota_1}B + \left(\frac{\partial\hat{\varphi}}{\partial\iota_3}\iota_3 + \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_2\right)I - \frac{\partial\hat{\varphi}}{\partial\iota_2}\iota_3\, B^{-1}\right)$$

If
$$[F] = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

then
$$[B] = \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix}$$

[Notebook page scanned on 2014-06-21]

$(61-62)_{11}$ Wednesday [2014-05-07] A1.3

9:00 – 11:30

let us consider an incompressible material.

The motion will be isochoric: any deformation is characterized by $\det F = 1$.

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

A standard strain energy for so called rubber-like material is

$$\varphi_H(F) = c_1(\iota_1 - 3) \qquad \text{neo-Hookean}$$

or

$$\varphi_{MR}(F) = c_1(\iota_1 - 3) + c_2(\iota_2 - 3)$$

Mooney-Rivlin

For the assumed deformation gradient we have

$$[C] = [F^T F] = \begin{pmatrix} \lambda^2 & & \\ & \frac{1}{\lambda} & \\ & & \frac{1}{\lambda} \end{pmatrix}$$

$$\iota_1 = \operatorname{tr} C = \lambda^2 + \frac{2}{\lambda}$$

[Notebook page scanned on 2014-06-21]

$$[C^2] = \begin{pmatrix} \lambda^4 & & \\ & \frac{1}{\lambda^2} & \\ & & \frac{1}{\lambda^2} \end{pmatrix}$$

$$\iota_2 = \frac{1}{2}\left((\operatorname{tr} C)^2 - \operatorname{tr} C^2\right) = \frac{1}{2}\left(\lambda^4 + 4\lambda + \frac{4}{\lambda^2} - \lambda^4 - \frac{2}{\lambda^2}\right) = 2\lambda + \frac{1}{\lambda^2}$$

$$\iota_3 = 1$$

$$[\dot{F}] = \begin{pmatrix} \dot{\lambda} & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda} \end{pmatrix}$$

$$[\dot{F}F^{-1}] = \begin{pmatrix} \frac{\dot{\lambda}}{\lambda} & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}\lambda^{-\frac{1}{2}} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}}\dot{\lambda}\lambda^{-\frac{1}{2}} \end{pmatrix}$$

$$[\dot{F}F^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix}\frac{\dot{\lambda}}{\lambda}$$

note $\operatorname{tr} \dot{F}F^{-1} = 0$

[Notebook page scanned on 2014-06-21]

$(63-64)_{12}$ Monday [2014-05-12] A1.3
16:00 - 18:00

$$\hat{\mathrm{T}}(\mathrm{F}) \cdot \dot{\mathrm{F}}\mathrm{F}^{-1} = \left(\hat{\sigma}_1 - \frac{1}{2}\left(\hat{\sigma}_2 + \hat{\sigma}_3\right)\right)\frac{\dot{\lambda}}{\lambda}$$

for a neo-Hookean material

$$\frac{d}{dt}\varphi(\mathrm{F}) = c_1\left(2\lambda\dot{\lambda} - \frac{2}{\lambda^2}\dot{\lambda}\right) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{\dot{\lambda}}{\lambda}$$

Setting $\hat{\sigma}_0 := \hat{\sigma}_1 - \frac{1}{2}\left(\hat{\sigma}_2 + \hat{\sigma}_3\right)$

from $\hat{\mathrm{T}}(\mathrm{F}) \cdot \dot{\mathrm{F}}\mathrm{F}^{-1} = \frac{d}{dt}\varphi(\mathrm{F})$

we get $\hat{\sigma}_0 = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$

$$\left[\mathrm{dev}\,\hat{\mathrm{T}}(\mathrm{F})\right] = \begin{pmatrix} \hat{\sigma}_1^D & & \\ & \hat{\sigma}_2^D & \\ & & \hat{\sigma}_3^D \end{pmatrix}$$

$$\hat{\sigma}_1^D = \hat{\sigma}_1 - \frac{1}{3}\left(\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3\right) = \frac{2}{3}\left(\hat{\sigma}_1 - \frac{1}{2}\left(\hat{\sigma}_2 + \hat{\sigma}_3\right)\right)$$

$$\hat{\sigma}_1^D = \frac{2}{3}\hat{\sigma}_0$$

$$\begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_1^D & & \\ & \hat{\sigma}_2^D & \\ & & \hat{\sigma}_3^D \end{pmatrix} - p\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\mathrm{T} \qquad = \qquad \mathrm{dev}\,\mathrm{T} \qquad + \qquad \mathrm{sph}\,\mathrm{T}$$

[Notebook page scanned on 2014-06-21]

$$\sigma_1 = \hat{\sigma}_1^D - p \qquad \sigma_1 = \pi$$

$$\sigma_2 = \hat{\sigma}_2^D - p \qquad \sigma_2 = 0$$

$$\sigma_3 = \hat{\sigma}_3^D - p \qquad \sigma_3 = 0$$

$$T = \hat{T}(F) - pI \qquad T = \frac{M}{V_\Omega}$$

material characterization — balance equations

$$\hat{\sigma}_1^D - p = \pi$$

$$\hat{\sigma}_2^D - p = 0$$

$$\hat{\sigma}_3^D - p = 0$$

$$0 - 3p = \pi \quad \text{(taking the trace)}$$

$$\Rightarrow \quad p = -\frac{1}{3}\pi$$

$$\hat{\sigma}_1^D = \pi + p = \frac{2}{3}\pi$$

$$\hat{\sigma}_2^D = p = -\frac{1}{3}\pi$$

$$\hat{\sigma}_3^D = p = -\frac{1}{3}\pi$$

$$\hat{\sigma}_1^D = \frac{2}{3}\hat{\sigma}_0 \quad \Rightarrow \quad \frac{2}{3}\hat{\sigma}_0 = \frac{2}{3}\pi$$

$$\boxed{2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) = \pi}$$

[Notebook page scanned on 2014-06-21]

$(65\text{-}66)_{12}$ Tuesday [2014-05-13] A1.3
9:00 - 11:00

$$T = \hat{T}(F) - pI + T^{+} \qquad T^{+} \cdot \dot{F}F^{-1} \geq 0$$

dissipation principle

The standard choice for a viscous material is

$$T^{+} = 2\mu \operatorname{sym} \dot{F}F^{-1}$$

with $\mu \geq 0$ by the dissipation principle.

$$[T^{+}] = \begin{pmatrix} \sigma_1^{+} & & \\ & \sigma_2^{+} & \\ & & \sigma_3^{+} \end{pmatrix} = 2\mu \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\lambda}}{\lambda}$$

$$\sigma_1 = \hat{\sigma}_1^{D} - p + \sigma_1^{+} \qquad \sigma_1 = \pi$$

$$\sigma_2 = \hat{\sigma}_2^{D} - p + \sigma_2^{+} \qquad \sigma_2 = 0$$

$$\sigma_3 = \hat{\sigma}_3^{D} - p + \sigma_3^{+} \qquad \sigma_3 = 0$$

material characterization — balance equations

$$\hat{\sigma}_1^{D} - p + \sigma_1^{+} = \pi$$

$$\hat{\sigma}_2^{D} - p + \sigma_2^{+} = 0$$

$$\hat{\sigma}_3^{D} - p + \sigma_3^{+} = 0$$

$$0 - 3p + 0 = \pi \quad \Rightarrow \quad p = -\frac{1}{3}\pi$$

[Notebook page scanned on 2014-06-21]

$$\hat{\sigma}_1^D - p + \sigma_1^+ = \mathfrak{p}$$

$$\frac{2}{3}\hat{\sigma}_0 + \sigma_1^+ = \mathfrak{p} + p = \frac{2}{3}\mathfrak{p}$$

$$\hat{\sigma}_0 + \frac{3}{2}\sigma_1^+ = \mathfrak{p}$$

$$\boxed{2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) + 3\mu\frac{\dot{\lambda}}{\lambda} = \mathfrak{p}}$$

Find a solution for a given value $\mathfrak{p}_o$.

Let $\lambda_o$ be such that

$$2c_1\left(\lambda_o^2 - \frac{1}{\lambda_o}\right) = \mathfrak{p}_o$$

and let us look for a solution close to $\lambda_o$

$$\lambda(t) = \lambda_o + \tilde{\varepsilon}(t) \quad \text{with } \tilde{\varepsilon} \text{ a small quantity } \forall t$$

$$= 1 + \underbrace{\varepsilon_o + \tilde{\varepsilon}(t)}$$

$$= 1 + \varepsilon(t)$$

$$\Rightarrow \quad \dot{\lambda}(t) = \dot{\tilde{\varepsilon}}(t) = \dot{\varepsilon}(t)$$

We assume the motion is so slow that even $\dot{\tilde{\varepsilon}}$ is a small quantity $\forall t$.

[Notebook page scanned on 2014-06-21]

$$\lambda^2 = (\lambda_0 + \tilde{\varepsilon})^2 = \lambda_0^2 + 2\lambda_0 \tilde{\varepsilon} + o(\tilde{\varepsilon})$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} - \frac{1}{\lambda_0^2}\tilde{\varepsilon} + o(\tilde{\varepsilon})$$

$$\frac{\dot{\lambda}}{\lambda} = \left(\frac{1}{\lambda_0} - \frac{1}{\lambda_0^2}\tilde{\varepsilon}\right)\dot{\tilde{\varepsilon}} + o(\tilde{\varepsilon}) = \frac{1}{\lambda_0}\dot{\tilde{\varepsilon}} + o(\tilde{\varepsilon})$$

$$2c_1\left(\lambda_0^2 + 2\lambda_0\tilde{\varepsilon} - \frac{1}{\lambda_0} + \frac{1}{\lambda_0^2}\tilde{\varepsilon}\right) + 3\mu\frac{1}{\lambda_0}\dot{\tilde{\varepsilon}} = p_0$$

$$\underbrace{2c_1\left(\lambda_0^2 - \frac{1}{\lambda_0}\right)}_{p_0} + 2c_1\left(2\lambda_0 + \frac{1}{\lambda_0^2}\right)\tilde{\varepsilon} + 3\mu\frac{1}{\lambda_0}\dot{\tilde{\varepsilon}} = p_0$$

$$2c_1\left(2\lambda_0^2 + \frac{1}{\lambda_0}\right)\tilde{\varepsilon} + 3\mu\dot{\tilde{\varepsilon}} = 0$$

[Notebook page scanned on 2014-06-21]

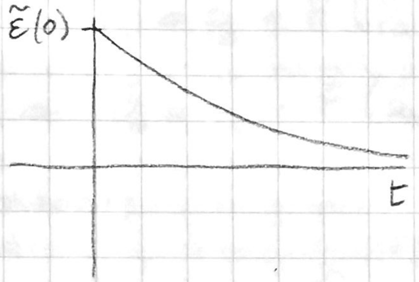
$(67-68)_{12}$ Wednesday [2014-05-14] A1.3

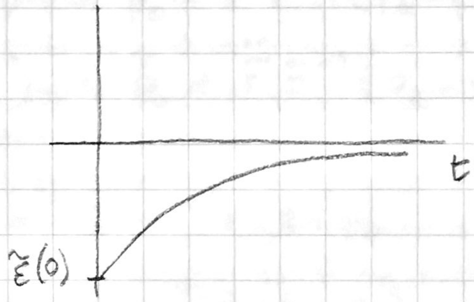
9:00-11:00

$$a\,\tilde{\varepsilon}(t) + \dot{\tilde{\varepsilon}}(t) = 0$$

$$a := \frac{2\,c_1}{3\,\mu}\left(2\,\lambda_0^2 + \frac{1}{\lambda_0}\right)$$

$$\tilde{\varepsilon}(t) = \tilde{\varepsilon}(0)\,e^{-at}$$



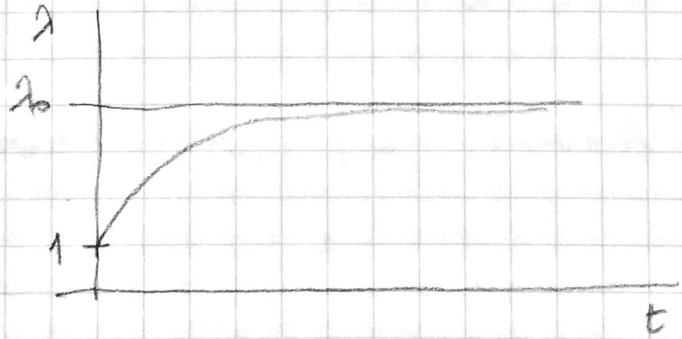




If $\varepsilon_0$ is very small we can set

$$\tilde{\varepsilon}(0) = -\varepsilon_0$$

$$\Rightarrow \quad \lambda(0) = 1 + \varepsilon_0 + \tilde{\varepsilon}(0) = 1$$

$$\lambda(t) = 1 + \varepsilon_0\left(1 - e^{-at}\right)$$




[Notebook page scanned on 2014-06-21]

The viscosity makes the deformation evolve in time. If the traction $p$ is applied suddenly at time $t = 0$ the body starts deforming and will approach the final shape with a decreasing velocity as $t \to \infty$.

[Notebook page scanned on 2014-06-21]

$(69-70)_{13}$ Monday [2014-05-19] A1.3 16:00-18:00

Inertial forces and free oscillations

$$b^{in}(x) = -\rho(x)\, a(x) \qquad \to (53-54)_{10}$$

Affine deformations

$$p_A(t) = p_0(t) + F(t)(\bar{p}_A - \bar{p}_0)$$

$$v_A = \dot{p}_A = \dot{p}_0 + \dot{F}(\bar{p}_A - \bar{p}_0) = \dot{p}_0 + \dot{F}F^{-1}(p_A - p_0)$$

$$a_A = \ddot{p}_A = \ddot{p}_0 + \ddot{F}(\bar{p}_A - \bar{p}_0) = \ddot{p}_0 + \ddot{F}F^{-1}(p_A - p_0)$$

test velocity field

$$v(x) = v_0 + \nabla v\,(x - p_0) \qquad x \in \mathcal{R}$$

$$\bar{v}(x) = v_0 + \nabla \bar{v}\,(x - p_0) \qquad x \in \bar{\mathcal{R}}$$

acceleration field

$$a(x) = a_0 + \ddot{F}F^{-1}(x - p_0) \qquad x \in \mathcal{R}$$

$$\bar{a}(x) = a_0 + \ddot{F}(x - \bar{p}_0) \qquad x \in \bar{\mathcal{R}}$$

$$\mathcal{F}^{in}(v) = \int_{\mathcal{R}} b^{in}(x) \cdot v(x)\, dV = -\int_{\bar{\mathcal{R}}} \rho_0\, \bar{a}(x) \cdot \bar{v}(x)\, dV$$

$$= -\int_{\bar{\mathcal{R}}} \rho_0 \left(a_0 + \ddot{F}(x - \bar{p}_0)\right) \cdot \left(v_0 + \nabla\bar{v}\,(x - \bar{p}_0)\right) dV$$

[Notebook page scanned on 2014-06-21]

$$\overset{in}{\mathcal{F}}(v) = -\int_{\bar{\mathcal{R}}} \rho_o\, a_o \cdot v_o \, dV - \int_{\bar{\mathcal{R}}} \rho_o\, a_o \cdot \nabla\bar{v}\,(x-\bar{p}_o)\, dV$$

$$- \int_{\bar{\mathcal{R}}} \rho_o\, \ddot{F}\,(x-\bar{p}_o)\cdot v_o\, dV - \int_{\bar{\mathcal{R}}} \rho_o\, \ddot{F}(x-\bar{p}_o)\cdot \nabla\bar{v}(x-p_o)\, dV$$

The barycenter of the reference shape is the position $x_G$ such that

$$(x_G - p_o) = \frac{1}{V_{\bar{\mathcal{R}}}} \int_{\bar{\mathcal{R}}} \rho_o (x-\bar{p}_o)\, dV$$

We choose $p_o \equiv x_G$ so that $\int_{\bar{\mathcal{R}}} \rho_o (x-\bar{p}_o)\,dV = 0$ and

$$\overset{in}{\mathcal{F}}(v) = -\int_{\bar{\mathcal{R}}} \rho_o\, dV\, a_o \cdot v_o$$

$$- \int_{\bar{\mathcal{R}}} \rho_o\, \ddot{F}(x-\bar{p}_o) \otimes (x-\bar{p}_o) \cdot \nabla\bar{v}\, dV$$

$$= -\underset{\substack{\uparrow \\ \text{total mass}}}{m}\, a_o \cdot v_o - \ddot{F}\, \underbrace{\int_{\bar{\mathcal{R}}} \rho_o (x-p_o) \otimes (x-p_o)\, dV}_{\bar{J} \text{ Euler tensor}} \cdot \nabla\bar{v}$$

[Notebook page scanned on 2014-06-21]

$$\mathcal{P}^{in}(v) = -m\, a_0 \cdot v_0 - \ddot{F}\bar{J} \cdot \nabla \bar{v}$$

$$= -m\, a_0 \cdot v_0 - \ddot{F}\bar{J} \cdot (\nabla v\, F)$$

$$= -m\, a_0 \cdot v_0 - \ddot{F}\bar{J}F^T \cdot \nabla v = f^{in} \cdot v_0 + M^{in} \cdot \nabla v$$

By using coordinates as in

$$x - \bar{p}_0 = s_1 e_1 + s_2 e_2 + s_3 e_3$$

we can compute the matrix of $\bar{J}$ for uniform $\rho_0$

$$[\bar{J}]_{ij} = \rho_0 \int_{-\frac{\bar{l}_3}{2}}^{\frac{\bar{l}_3}{2}} \int_{-\frac{\bar{l}_2}{2}}^{\frac{\bar{l}_2}{2}} \int_{-\frac{\bar{l}_1}{2}}^{\frac{\bar{l}_1}{2}} s_i s_j \, ds_1 \, ds_2 \, ds_3$$

$$[\bar{J}] = \frac{\rho_0}{12} \begin{pmatrix} \bar{l}_1^3 \bar{l}_2 \bar{l}_3 & & \\ & \bar{l}_2^3 \bar{l}_3 \bar{l}_1 & \\ & & \bar{l}_3^3 \bar{l}_1 \bar{l}_2 \end{pmatrix}$$

where $l_1, l_2, l_3$ are the lengths of the edges of the reference shape

If we set $\bar{l}_1 = \bar{l}$, $\bar{l}_2 = \bar{l}_3 = \alpha \bar{l}$ we get

$$V_{\bar{\mathcal{R}}} = \alpha^2 \bar{l}^3$$

[Notebook page scanned on 2014-06-21]

$$[\bar{J}] = \frac{\rho_0 \bar{l}^{-2}}{12} \begin{pmatrix} 1 & & \\ & \alpha^2 & \\ & & \alpha^2 \end{pmatrix} V_{\bar{\mathcal{R}}}$$

let us consider again affine deformations such that

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

with $\lambda$ depending on time, and compute time derivatives

$$[\dot{F}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}} \end{pmatrix} \dot{\lambda}$$

$$[\ddot{F}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2}\lambda^{-\frac{3}{2}} & \\ & & -\frac{1}{2}\lambda^{-\frac{3}{2}} \end{pmatrix} \ddot{\lambda} + \begin{pmatrix} 0 & & \\ & \frac{3}{4}\lambda^{-\frac{5}{2}} & \\ & & \frac{3}{4}\lambda^{-\frac{5}{2}} \end{pmatrix} \dot{\lambda}^2$$

$$[\ddot{F} F^T] = \begin{pmatrix} \lambda & & \\ & -\frac{1}{2}\lambda^{-2} & \\ & & -\frac{1}{2}\lambda^{-2} \end{pmatrix} \ddot{\lambda} + \begin{pmatrix} 0 & & \\ & \frac{3}{4}\lambda^{-3} & \\ & & \frac{3}{4}\lambda^{-3} \end{pmatrix} \dot{\lambda}^2$$

[Notebook page scanned on 2014-06-21]

$(71-72)_{13}$ Tuesday [2014-05-20] A1.3

9:00 – 11:00

To make expressions simpler let us set

$$\rho_s := \frac{1}{12}\,\rho_o\,\bar{l}^{\,2}$$

so we get

$$[\ddot{F}\bar{J}F^T] = \rho_s \begin{pmatrix} \ddot{\lambda}\lambda & 0 & 0 \\ 0 & \alpha^2\left(-\frac{1}{2}\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4}\frac{\dot{\lambda}^2}{\lambda^3}\right) & 0 \\ 0 & 0 & \alpha^2\left(-\frac{1}{2}\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4}\frac{\dot{\lambda}^2}{\lambda^3}\right) \end{pmatrix} V_{\mathcal{R}}$$

Let us apply to the body in the shape of a cylinder with a square cross section a normal traction on the opposite end faces while taking into account the inertial forces


The total force is $f = -m\,\ddot{p}_o$

The total moment tensor is

$$M = V_{\mathcal{R}}\,p\,e_1 \otimes e_1 - \ddot{F}\bar{J}F^T$$

[Notebook page scanned on 2014-06-21]

The balance equations are

$$f = 0$$

$$\operatorname{skw} M = 0$$

$$\operatorname{sym} M = T\, V_{\mathcal{R}}$$

From the first one we get $\ddot{p}_o = 0$

So the barycenter will move at a constant speed.

The moment tensor turns out to be symmetric, as we can see from its matrix. So the second of the balance equations above is fullfilled.

Since the motion is isochoric

$$V_{\mathcal{R}} = V_{\bar{\mathcal{R}}}$$

Hence, from the last balance equation,

$$T = p\, e_1 \otimes e_1 - \ddot{F} \bar{J} F^T \frac{1}{V_{\mathcal{R}}}$$

whose scalar form is as follows

[Notebook page scanned on 2014-06-21]

$$\sigma_1 = p - \rho_s \lambda \ddot{\lambda}$$

$$\sigma_2 = 0 - \rho_s \alpha^2 \left( -\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

$$\sigma_3 = 0 - \rho_s \alpha^2 \left( -\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

Let the material be incompressible and viscoelastic

$$\sigma_1 = \hat{\sigma}_1^D - p + \sigma_1^+$$

$$\sigma_2 = \hat{\sigma}_2^D - p + \sigma_2^+$$

$$\sigma_3 = \hat{\sigma}_3^D - p + \sigma_3^+ \qquad \rightarrow (65\text{-}66)_{12}$$

Substituting these expressions into the balance equations we get

$$\sigma_1^D - p + \sigma_1^+ = p - \rho_s \lambda \ddot{\lambda}$$

$$\sigma_2^D - p + \sigma_2^+ = -\rho_s \alpha^2 \left( -\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

$$\sigma_3^D - p + \sigma_3^+ = -\rho_s \alpha^2 \left( -\frac{1}{2} \frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{4} \frac{\dot{\lambda}^2}{\lambda^3} \right)$$

By adding terms on the left side and terms on the right side (like taking the trace of the corresponding tensors) we get

[Notebook page scanned on 2014-06-21]

$$-3p = \mathfrak{p} - \varrho_s \left( \lambda\ddot{\lambda} + \alpha^2 \left( -\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3} \right) \right)$$

where

$$\hat{\sigma}_1^D + \hat{\sigma}_2^D + \hat{\sigma}_3^D = 0$$

$$\sigma_1^+ + \sigma_2^+ + \sigma_3^+ = 0$$

because both $\hat{T}(F)$ and $T^+$ are deviatoric tensors.

We solve the last equation for the inner pressure $p$ and replace its expression in the first of the previous equations

$$\hat{\sigma}_1^D + \sigma_1^+ = \mathfrak{p} - \frac{1}{3}\mathfrak{p} - \varrho_s \lambda\ddot{\lambda} + \frac{1}{3}\varrho_s \left( \lambda\ddot{\lambda} + \alpha^2 \left( -\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3} \right) \right)$$

$$\hat{\sigma}_1^D + 2\mu\frac{\dot{\lambda}}{\lambda} = \frac{2}{3}\mathfrak{p} + \frac{1}{3}\varrho_s \left( -2\lambda\ddot{\lambda} + \alpha^2 \left( -\frac{\ddot{\lambda}}{\lambda^2} + \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3} \right) \right)$$

$$\hat{\sigma}_1^D = \frac{2}{3}\mathfrak{p} - 2\mu\frac{\dot{\lambda}}{\lambda} - \frac{2}{3}\varrho_s \left( \lambda\ddot{\lambda} + \frac{\alpha^2}{2} \left( \frac{\ddot{\lambda}}{\lambda^2} - \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3} \right) \right)$$

$$\hat{\sigma}_1^D = \frac{2}{3}\hat{\sigma}_0 \, ; \quad \hat{\sigma}_0 = 2c_1 \left( \lambda^2 - \frac{1}{\lambda} \right)$$ neo-Hookean material $\rightarrow (63\text{-}64)_{12}$

$$\hat{\sigma}_0 = \mathfrak{p} - 3\mu\frac{\dot{\lambda}}{\lambda} - \varrho_s \left( \lambda\ddot{\lambda} + \frac{\alpha^2}{2} \left( \frac{\ddot{\lambda}}{\lambda^2} - \frac{3}{2}\frac{\dot{\lambda}^2}{\lambda^3} \right) \right)$$

[Notebook page scanned on 2014-06-21]

The final expression for the equation of motion is

$$2c_1\left(\lambda^2-\frac{1}{\lambda}\right)=p-3\mu\frac{\dot\lambda}{\lambda}-\rho_s\left(\lambda\ddot\lambda+\frac{\alpha^2}{2}\left(\frac{\ddot\lambda}{\lambda^2}-\frac{3}{2}\frac{\dot\lambda^2}{\lambda^3}\right)\right)$$

We can linearize this equation according to the following procedure

Let us set $\lambda(t)=\lambda_0+\beta\tilde\varepsilon(t)$

$$p=p_0$$

and replace the original expression with a series expansion with respect to $\beta$ starting at $\beta=0$, up to order 1. Then collect separately terms of order 0 and terms of order 1 with respect to $\beta$. We get

$$2c_1\left(\lambda_0^2-\frac{1}{\lambda_0}\right)=p_0$$

$$4c_1(1+\lambda_0^3)\,\tilde\varepsilon+6\lambda_0\mu\,\dot{\tilde\varepsilon}+(\alpha^2+2\lambda_0^3)\,\rho_s\,\ddot{\tilde\varepsilon}=0$$

[Notebook page scanned on 2014-06-21]

$(73-74)_{13}$ Wednesday [2014-05-21] A1.3
9:00-11:00

Summary about the equation of motion and its linearization, describing small oscillations around the solution $(\lambda_0, p_0)$.

Small oscillations

Let us consider solutions in the general form

$$\tilde{\varepsilon}(t) = \tilde{\varepsilon}_0 \, e^{kt}$$

The characteristic equation turns out to be

$$4 c_1 (1 + 2\lambda_0^3) + (6\lambda_0 \mu) k + \rho_s (\alpha^2 + 2\lambda_0^3) k^2 = 0$$

$$\Rightarrow \quad k = \frac{-3\lambda_0 \mu \pm \sqrt{9\lambda_0^2 \mu^2 - 4 c_1 \rho_s (1 + 2\lambda_0^3)(\alpha^2 + 2\lambda_0^3)}}{(\alpha^2 + 2\lambda_0^3)\rho_s}$$

Critical viscosity value $\mu_0$ (depending on $\lambda_0$)

$$\mu_0^2 = \frac{2}{3} c_1 \rho_s \frac{1}{\lambda_0^2} (1 + 2\lambda_0^3)(\alpha^2 + 2\lambda_0^3)$$

$\mu < \mu_0$ damped oscillations

$\mu > \mu_0$ overdamping, no oscillations

[Notebook page scanned on 2014-06-21]

$(75-76)_{14}$ Monday [2014-05-26] A1.3
16:00 - 18:00

Linearization around the reference shape

Let us consider a one-parameter family of deformations

$$F(\beta) = R(\beta)\,U(\beta) \qquad \beta \text{ dimensionless}$$

$$R(\beta) = I + \beta \frac{d}{d\beta} R(\beta)\Big|_{\beta=0} + o(\beta)$$

$$U(\beta) = I + \beta \frac{d}{d\beta} U(\beta)\Big|_{\beta=0} + o(\beta)$$

$$R(\beta)^T R(\beta) = I \quad \Rightarrow \frac{d}{d\beta}\left(R(\beta)^T R(\beta)\right) = 0$$

$$\Rightarrow \frac{d}{d\beta} R(\beta)^T\Big|_{\beta=0} + \frac{d}{d\beta} R(\beta)\Big|_{\beta=0} = 0$$

infinitesimal rotation

$$\Theta := \frac{d}{d\beta} R(\beta)\Big|_{\beta=0} \quad \Rightarrow \quad \Theta^T + \Theta = 0$$

infinitesimal stretch

$$E := \frac{d}{d\beta} U(\beta)\Big|_{\beta=0} \quad \Rightarrow \quad E^T = E$$

$$F(\beta) = I + \beta\,\Theta + \beta E + o(\beta)$$

[Notebook page scanned on 2014-06-21]

$(77-78)_{14}$ Tuesday [2014-05-27] $\Delta$1.3
9:00 - 11:00

Following the formula about time differentiation already derived $\to (17-18)_3$

we get

$$\det F(\beta) = \det F(0) + \beta \frac{d}{d\beta} \det F(\beta)\Big|_{\beta=0} + o(\beta)$$

$$\frac{d}{d\beta} \det F(\beta)\Big|_{\beta=0} = \det F(\beta)\, \operatorname{tr}\Big(\big(\frac{d}{d\beta}F(\beta)\big)F(\beta)^{-1}\Big)\Big|_{\beta=0}$$

$$= \operatorname{tr}\big((\Theta + E)(I - \beta\Theta - \beta E)\big)\Big|_{\beta=0}$$

$$= \operatorname{tr}(\Theta + E) = \operatorname{tr} E$$

$$\det F(\beta) = 1 + \beta \operatorname{tr} E + o(\beta)$$

In view of linearization it is customary to define the displacement vector field

$$u(\bar{p}_\Delta) := \phi(\bar{p}_\Delta) - \bar{p}_\Delta \quad \forall\, \bar{p}_\Delta \in \bar{\mathcal{R}}$$

Along any curve $\bar{c}$ we get

$$u(\bar{c}(h)) = \phi(\bar{c}(h)) - \bar{c}(h)$$
$$u(\bar{c}(0)) = \phi(\bar{c}(0)) - \bar{c}(0)$$

[Notebook page scanned on 2014-06-21]

$$\frac{u(\bar{c}(h)) - u(\bar{c}(0))}{h} = \frac{\phi(\bar{c}(h)) - \phi(\bar{c}(0))}{h} - \frac{\bar{c}(h) - \bar{c}(0)}{h}$$

Taking the limit as $h \to 0$ we arrive at

$$\nabla u\, c' = \nabla \phi\, c' - c'$$

Hence

$$\nabla u = F - I$$

By using the corresponding series expansions we get

$$\nabla u(\beta) = F(\beta) - I = \beta\left(\Theta + E\right) + o(\beta)$$

$$\lim_{\beta \to 0} \frac{\nabla u(\beta)}{\beta} = \Theta + E$$

Since there is a unique decomposition of any tensor into the sum of a skewsymmetric and a symmetric tensor, by setting

$$\Theta := \operatorname{skw} \nabla u \quad , \quad E := \operatorname{sym} \nabla u$$

we get

$$\nabla u(\beta) = \beta\left(\operatorname{skw}\nabla u + \operatorname{sym}\nabla u\right) + o(\beta) = \beta \nabla u + o(\beta)$$

In the linear theory we use quantities like $\Theta$, $E$, $\nabla u$ which are derivatives with respect to $\beta$, a dimensionless "scaling factor" for some control parameters like forces.

[Notebook page scanned on 2014-06-21]

For the response function of an elastic material we write the series expansion

$$\hat{T}(F(\beta)) = \hat{T}(I) + \beta \frac{d}{d\beta}\hat{T}(F(\beta))\Big|_{\beta=0} + o(\beta)$$

where, because of the objectivity condition

$$\hat{T}(F) = R\,\hat{T}(U)\,R^T \qquad \rightarrow (37\text{-}38)_7$$

we have

$$\frac{d}{d\beta}\hat{T}(F(\beta))\Big|_{\beta=0} = \frac{d}{d\beta} R(\beta)\,\hat{T}(U(\beta))\,R(\beta)^T\Big|_{\beta=0}$$

$$= W\,\underbrace{\hat{T}(I)}_{0} + \frac{d}{d\beta}\hat{T}(U(\beta)) - \underbrace{\hat{T}(I)}_{0}\,W$$

with

$$\frac{d}{d\beta}\hat{T}(U(\beta)) = \frac{d}{d\beta}\hat{T}(I+\beta E) = \mathbb{C}\,E$$

where $\mathbb{C} : Sym(\mathcal{V}) \to Sym(\mathcal{V})$ is the gradient of of a tensor field over the space of symmetric tensors.

It is referred to as the elasticity tensor.

Notice how

$$\frac{d}{d\beta}\hat{T}(U(\beta))\Big|_{\beta=0} = \lim_{\beta\to 0}\frac{1}{\beta}\Big(\hat{T}(U(\beta)) - \hat{T}(U(0))\Big) = \mathbb{C}\,E$$

with

$$E = \frac{d}{d\beta}U(\beta)\Big|_{\beta=0}$$

looks like

$(19\text{-}20)_4 \leftarrow$
$$\lim_{h\to 0}\frac{1}{h}\Big(v(c(h)) - v(c(0))\Big) = \nabla v\, c'$$

[Notebook page scanned on 2014-06-21]

$(79\text{-}80)_{14}$ Wednesday [2014-05-28] A1.3
9:00 – 11:00

Let us recall the definition of the Piola stress

$$S = (\det F)\, T\, F^{-T} \qquad \to (39\text{-}40)_7$$

Then the first order term in the series expansion of

$$S(\beta) = (\det F(\beta))\, T(\beta)\, F(\beta)^{-T}$$

will be

$$\left.\frac{d}{d\beta} S(\beta)\right|_{\beta=0} = (\operatorname{tr} E)\, T(0)\, F(0)^{-T} + (\det F(0)) \frac{d}{d\beta} T(\beta)\, F(0)^{-T}$$
$$+ (\det F(0))\, T(0)\, (\Theta - E)$$

$$T(0) = 0 \quad \Rightarrow \quad \left.\frac{d}{d\beta} S(\beta)\right|_{\beta=0} = \frac{d}{d\beta} T(\beta)$$

$$\Rightarrow \quad S(\beta) = T(\beta) + o(\beta)$$

By this property, for a hyperelastic material the relation $\hat{S}(F(\beta)) \cdot \dot{F}(\beta) = \frac{d}{dt} \varphi(F(\beta))$

can be rewritten as

$$o(\beta) + \hat{T}(F(\beta)) \cdot \dot{F}(\beta) = \frac{d}{dt} \varphi(F(\beta))$$

and then, by the results on the previous pages,

$$o(\beta^2) + (\beta\, \mathbb{C}(E)) \cdot (\beta \dot{\Theta} + \beta \dot{E}) = \frac{d}{dt} \varphi(U(\beta))$$

where $\varphi(F(\beta)) = \varphi(U(\beta))$ because of objectivity.

[Notebook page scanned on 2014-06-21]

By the symmetry of $\mathbb{C}(E)$ we get

$$o(\beta^2) + \beta^2\,\mathbb{C}(E)\cdot\dot{E} = \frac{d}{dt}\varphi(U(\beta))$$

For a potential to exist the stress power on the left side should be such that

$$\mathbb{C}(E)\cdot\dot{E} = \mathbb{C}(\dot{E})\cdot E$$

since $\mathbb{C}: Sym(\mathcal{V}) \to Sym(\mathcal{V})$ is a linear function

That means that the elasticity tensor should be symmetric

$$\mathbb{C}E\cdot\dot{E} = E\cdot\mathbb{C}^T\dot{E} = E\cdot\mathbb{C}\dot{E}$$

Hence the strain energy is, up to second order, given by

$$\varphi(E) = \frac{1}{2}\,\mathbb{C}(E)\cdot E$$

Since $\dim(Sym(\mathcal{V}))=6$, the elasticity tensor matrix is a $6\times 6$ matrix. As a symmetric matrix (in an orthonormal tensor basis) it has only 21 independent entries $\left(\frac{1}{2}(6\times 6-6)+6\right)$, describing material properties.

It is amazing to find out that by linearizing the general response function for isotropic hyperelastic materials $\to (45\text{-}46)_8$ we get $\mathbb{C}(E) = \lambda_L(\mathrm{tr}\,E)I + 2\mu_L E$

where, by isotropy, there are only two coefficients left, $\lambda_L$ and $\mu_L$, called the Lamé constants (or moduli).

[Notebook page scanned on 2014-06-21]

The symmetry property of the elasticity tensor for a hyperelastic material can be derived through the following detailed computation.

Let

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ \cdots & & & & & \\ \cdots & & & & & \\ \cdots & & & & & \\ \cdots & & & & & \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{pmatrix}$$

be the component form of the linear elasticity response

$$T = \mathbb{C}E$$

where both $T$ and $E$ are symmetric tensors.

The stress power can be written by using the above components as

$$T \cdot \dot{E} = \mathbb{C}(E) \cdot \dot{E}$$

$$= c_{11}\,\varepsilon_{11}\,\dot{\varepsilon}_{11} + c_{12}\,\varepsilon_{12}\,\dot{\varepsilon}_{11} + c_{13}\,\varepsilon_{13}\,\dot{\varepsilon}_{11} + \ldots$$

$$+ c_{21}\,\varepsilon_{11}\,\dot{\varepsilon}_{12} + c_{22}\,\varepsilon_{12}\,\dot{\varepsilon}_{12} + c_{23}\,\varepsilon_{13}\,\dot{\varepsilon}_{12} + \ldots$$

$$+ c_{31}\,\varepsilon_{11}\,\dot{\varepsilon}_{13} + c_{32}\,\varepsilon_{12}\,\dot{\varepsilon}_{13} + c_{33}\,\varepsilon_{13}\,\dot{\varepsilon}_{13} + \ldots$$

$$+ \ldots$$

from which it is clear how $c_{12} = c_{21}$, $c_{13} = c_{31}$, $c_{23} = c_{32}$ etc are the conditions for a primitive to exist.

[Notebook page scanned on 2014-06-21]

The linearized response function for an isotropic hyperelastic material can be derived from the general expression → $(45\text{-}46)_8$ by differentiating $\hat{T}(F(\beta))$ with respect to $\beta$.

Since
$$\frac{d}{d\beta} F(\beta)\Big|_{\beta=0} = \textcircled{4} + E$$

$$\frac{d}{d\beta} B(\beta)\Big|_{\beta=0} = \frac{d}{d\beta} F(\beta) + \frac{d}{d\beta} F(\beta)^T = 2E$$

$$\frac{d}{d\beta} B^2(\beta)\Big|_{\beta=0} = 4E \quad , \quad \frac{d}{d\beta} B(\beta)^{-1}\Big|_{\beta=0} = -2E$$

$$\frac{d}{d\beta} \iota_3(\beta)\Big|_{\beta=0} = \det B(\beta)\, \mathrm{tr}\left(\frac{d}{d\beta} B(\beta)\; B(\beta)^{-1}\right)\Big|_{\beta=0} = 2\,\mathrm{tr}\,E$$

$$\frac{d}{d\beta} \iota_1(\beta)\Big|_{\beta=0} = 2\,\mathrm{tr}\,E \quad , \quad \frac{d}{d\beta} \iota_2(\beta)\Big|_{\beta=0} = 4\,\mathrm{tr}\,E$$

we get an expression as a linear combination of the two tensors $(\mathrm{tr}E)I$ and $E$

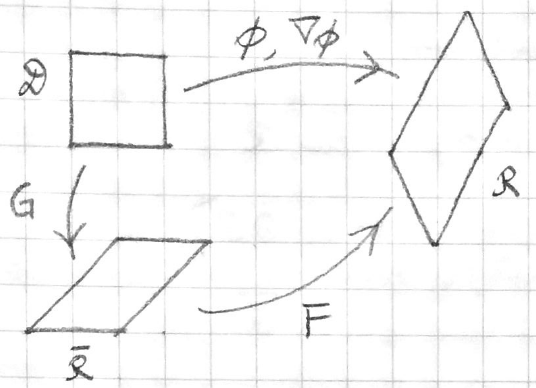
which we write

$$\mathbb{C}(E) = \lambda_L\, (\mathrm{tr}E)I + 2\mu_L\, E$$

[Notebook page scanned on 2014-06-21]

$(81-82)_{15}$ Tuesday [2014-06-03] Δ1.3
9:00 - 11:00

Remodeling




$\nabla\phi = F\,G$ Kröner-Lee decomposition

$F$ elastic distortion

$G$ remodeling distortion with $\det G > 0$

$\mathcal{D}$ reference shape

$\bar{\mathcal{R}}$ relaxed shape

$\mathcal{R}$ current shape

$v$ velocity field on $\mathcal{R}$

$\mathbb{V}$ remodeling velocity tensor field on $\bar{\mathcal{R}}$

$\varphi$ strain energy density in $\bar{\mathcal{R}}$

Basic assumption: only $F$ affects the strain energy $\varphi$.

$T$ Cauchy stress (tensor field on $\mathcal{R}$)

$S$ Piola stress (tensor field on $\bar{\mathcal{R}}$) $S = (\det F)\, T F^{-T}$

$\mathbb{S}$ Piola stress (tensor field on $\mathcal{D}$) $\mathbb{S} = (\det \nabla\phi)\, T \nabla\phi^{-T}$

$$\mathcal{F}^{ext}(v, \mathbb{V}) = \int_{\mathcal{R}} b\cdot v\, dV + \int_{\partial\mathcal{R}} t\cdot v\, dV + \int_{\bar{\mathcal{R}}} \mathbb{Q}\cdot \mathbb{V}\, dV$$

remodeling force

$$\mathcal{F}^{int}(v, \mathbb{V}) = -\int_{\mathcal{R}} T\cdot \nabla v\, dV - \int_{\bar{\mathcal{R}}} \mathbb{A}\cdot \mathbb{V}\, dV$$

remodeling stress

[Notebook page scanned on 2014-06-21]

From the force balance principle

$$\mathcal{F}^{ext}(v, \mathbb{V}) + \mathcal{F}^{int}(v, \mathbb{V}) = 0 \qquad \forall v, \forall \mathbb{V}$$

we get the balance equations

$$\operatorname{div} T + b = 0 \qquad \text{in } \mathcal{R}$$

$$Tn = t \qquad \text{in } \partial\mathcal{R}$$

$$\mathbb{Q} = A \qquad \text{in } \bar{\mathcal{R}}$$

In any motion

$$\nabla v = \nabla\dot{\phi}\, \nabla\phi^{-1}$$

$$\mathbb{V} = \dot{G}\, G^{-1}$$

and, by the balance principle above,

$$\int_{\mathcal{R}} b \cdot v \, dV + \int_{\partial\mathcal{R}} t \cdot v \, dA + \int_{\bar{\mathcal{R}}} \mathbb{Q} \cdot \dot{G} G^{-1} \, dV$$

$$= \int_{\mathcal{R}} T \cdot \nabla\dot{\phi} \nabla\phi^{-1} \, dV + \int_{\bar{\mathcal{R}}} A \cdot \dot{G} G^{-1} \, dV$$

Hence the energy balance principle (dissipation inequality) states that in any motion

$$\int_{\mathcal{R}} T \cdot \nabla\dot{\phi} \nabla\phi^{-1} \, dV + \int_{\bar{\mathcal{R}}} A \cdot \dot{G} G^{-1} \, dV - \frac{d}{dt} \int_{\bar{\mathcal{R}}} \varphi(F) \, dV \geq 0$$

[Notebook page scanned on 2014-06-21]

After transforming the integrals into integrals over the reference shape

$$\int_{\mathcal{D}} T \nabla\phi^{-T} \cdot \nabla\dot{\phi} \det\nabla\phi \, dV + \int_{\mathcal{D}} A \cdot \dot{G} G^{-1} \det G \, dV - \frac{d}{dt}\int_{\mathcal{D}} \varphi(F) \det G \, dV \geq 0$$

we replace the former statement with the localized form

$$S \cdot \nabla\dot{\phi} + A \cdot \dot{G} G^{-1} \det G - \frac{d}{dt}\varphi(F) \det G - \varphi(F) \det G \operatorname{tr}(\dot{G} G^{-1}) \geq 0$$

where the formula for the time derivative of det G has been used $\rightarrow (17\text{-}18)_3$

Now let us substitute the relation

$$\hat{S}(F) \cdot \dot{F} = \frac{d}{dt}\varphi(F)$$

together with the expressions for the Piola stress tensors and the Kröner-Lee decomposition, and get in turn

$$S \cdot (\dot{F} G + F \dot{G}) \frac{1}{\det G} + A \cdot \dot{G} G^{-1} - \hat{S}(F) \cdot \dot{F} - \varphi(F) I \cdot \dot{G} G^{-1} \geq 0$$

$$\frac{\det\nabla\phi}{\det G} T F^{-T} G^{-T} \cdot (\dot{F} G + F \dot{G}) - \hat{S}(F) \cdot \dot{F} + (A - \varphi(F) I) \cdot \dot{G} G^{-1} \geq 0$$

$$S G^{-T} \cdot (\dot{F} G + F \dot{G}) - \hat{S}(F) \cdot \dot{F} + (A - \varphi(F) I) \cdot \dot{G} G^{-1} \geq 0$$

$$S \cdot \dot{F} G G^{-1} + F^T S \cdot \dot{G} G^{-1} - \hat{S}(F) \cdot \dot{F} + (A - \varphi(F) I) \cdot \dot{G} G^{-1} \geq 0$$

[Notebook page scanned on 2014-06-21]

$(83-84)_{15}$ Wednesday [2014-06-04] A1.3
9:00 – 11:00

The final expression for the dissipation inequality is

$$\big(S - \hat{S}(F)\big)\cdot \dot{F} + \big(A - \varphi(F)I + F^T S\big)\cdot \dot{G}G^{-1} \geq 0$$

or, equivalently,

$$\det F\,\big(T - \hat{T}(F)\big)\cdot \dot{F}F^{-1} + \big(A - \underbrace{(\varphi(F)I - F^T S)}_{\text{Eshelby tensor}}\big)\cdot \dot{G}G^{-1} \geq 0$$

If we set

$$T^+ := T - \hat{T}(F), \quad A^+ := A - (\varphi(F)I - F^T S)$$

the inequality can be written as

$$(\det F)\, T^+ \cdot \dot{F}F^{-1} + A^+ \cdot \dot{G}G^{-1} \geq 0$$

We can now characterize the dissipation through suitable expressions for $T^+$ and $A^+$ consistently with the above inequality in any motion.

A consistent choice is for example

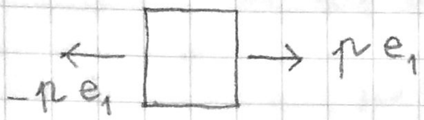
$$T^+ = 2\mu\, \mathrm{sym}(\dot{F}F^{-1})$$

$$A^+ = \eta\, \dot{G}G^{-1} \qquad (\eta \geq 0)$$

Note that while $T$ is a symmetric tensor because of the objectivity condition, no such a property has been derived for $A$.

[Notebook page scanned on 2014-06-21]

Let us consider again the body undergoing cylindrical deformations




with

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \qquad [G] = \begin{pmatrix} \gamma & & \\ & \frac{1}{\sqrt{\gamma}} & \\ & & \frac{1}{\sqrt{\gamma}} \end{pmatrix}$$

Hence

$$[\nabla\phi] = \begin{pmatrix} \lambda\gamma & & \\ & \frac{1}{\sqrt{\lambda\gamma}} & \\ & & \frac{1}{\sqrt{\lambda\gamma}} \end{pmatrix}$$

Balance equations

$$f = 0$$

$$\operatorname{skw} M = 0$$

$$\frac{M}{V_{\mathcal{R}}} = T$$

$$\mathcal{Q} = A$$

where

$$M = p\, e_1 \otimes e_1\, V_{\mathcal{R}}$$

[Notebook page scanned on 2014-06-21]

let us set the remodeling force

$$\mathcal{Q} = 0$$

From the balance equations we get

$$T = \mu\, e_1 \otimes e_1$$

$$A = 0$$

Material characterization

$$T = \hat{T}(F) - p\,I + T^+$$

$$A = \varphi(F)I - F^T S - a\,I + A^+$$

where the inner pressure $p$ is the reactive stress defined by the property that its power is zero, because $\dot{F}F^{-1}$ is deviatoric ($\mathrm{tr}\,\dot{F}F^{-1} = 0$) here.

Since $\dot{G}G^{-1}$ is deviatoric as well, because of the assumption of isochoric remodeling, we have to add the remodeling pressure $a$, whose power is $aI \cdot \dot{G}G^{-1} = 0$, because $\mathrm{tr}\,\dot{G}G^{-1} = 0$ here.

let us set now $\quad T^+ = 0 \quad (\Leftrightarrow \mu = 0)$

$$A^+ = \eta\, \dot{G}G^{-1} \quad (\eta > 0)$$

and choose a neo-Hookean strain energy

$$\varphi(F) = c_1(\iota_1 - 3)$$

[Notebook page scanned on 2014-06-21]

$(85-86)_{16}$ Monday[2014-06-09] A1.3
16:00-18:00

$$T = \hat{T}(F) - p I$$

$$A = \varphi(F) I - F^T T F^{-T} \det F - a I + \eta \dot{G} G^{-1}$$

$\det F = 1$

$$[F^T T F^{-T}] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & & \\ & \sqrt{\lambda} & \\ & & \sqrt{\lambda} \end{pmatrix} = [T]$$

$$[T] = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \qquad [A] = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$$

$$[\dot{G} G^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\gamma}}{\gamma} \qquad [\dot{F} F^{-1}] = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \frac{\dot{\lambda}}{\lambda}$$

$$[C] = [F^T F] = \begin{pmatrix} \lambda^2 & & \\ & \frac{1}{\lambda} & \\ & & \frac{1}{\lambda} \end{pmatrix}$$

$$\iota_1 = \operatorname{tr} C = \lambda^2 + \frac{2}{\lambda}$$

[Notebook page scanned on 2014-06-16]

stress characterization

$$\sigma_1 = \hat{\sigma}_1^D - p$$
$$\sigma_2 = \hat{\sigma}_2^D - p$$
$$\sigma_3 = \hat{\sigma}_3^D - p$$

balance

$$\sigma_1 = \mu$$
$$\sigma_2 = 0$$
$$\sigma_3 = 0$$

$$T = \hat{T}(F) - pI \qquad T = \frac{M}{V_R}$$

$$\hat{\sigma}_1^D - p = \mu$$
$$\hat{\sigma}_2^D - p = 0$$
$$\hat{\sigma}_3^D - p = 0$$

trace $\quad 0 - 3p = \mu \quad \Rightarrow \quad p = -\frac{1}{3}\mu$

$$\hat{\sigma}_1^D = \mu + p = \frac{2}{3}\mu$$
$$\hat{\sigma}_2^D = p = -\frac{1}{3}\mu$$
$$\hat{\sigma}_3^D = p = -\frac{1}{3}\mu$$

response function

$$\hat{T}(F) \cdot \dot{F}F^{-1} = \left(\hat{\sigma}_1 - \frac{1}{2}\left(\hat{\sigma}_2 + \hat{\sigma}_3\right)\right)\frac{\dot{\lambda}}{\lambda} \qquad \hat{\sigma}_0 := \hat{\sigma}_1 - \frac{1}{2}\left(\hat{\sigma}_2 + \hat{\sigma}_3\right)$$

for a neo-Hookean material

$$\frac{d}{dt}\varphi(F) = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{\dot{\lambda}}{\lambda} \quad \Rightarrow \quad \hat{\sigma}_0 = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

[Notebook page scanned on 2014-06-16]

Because of incompressibility only the deviatoric part of the response $\hat{T}(F)$ is delivered by the strain energy.

Hence $\hat{\sigma}_1^D = \hat{\sigma}_1 - \frac{1}{3}\left(\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3\right) = \frac{2}{3}\left(\hat{\sigma}_1 - \frac{1}{2}(\hat{\sigma}_2 + \hat{\sigma}_3)\right)$

$$\Rightarrow \quad \hat{\sigma}_1^D = \frac{2}{3}\,\hat{\sigma}_0$$

$\hat{\sigma}_1^D = \frac{2}{3}\,p$ (balance & stress characterization)

$$\frac{2}{3}\,\hat{\sigma}_0 = \frac{2}{3}\,p$$

$$\boxed{2\,c_1\left(\lambda^2 - \frac{1}{\lambda}\right) = p}$$

balance with neo-Hookean response

The deformation gradient depends on both $\lambda$ and $\gamma$.

remodeling stress

$$a_1 = \varphi(F) - \sigma_1 - a + \eta\frac{\dot{\gamma}}{\gamma}$$

$$a_2 = \varphi(F) - \sigma_2 - a - \frac{1}{2}\eta\frac{\dot{\gamma}}{\gamma}$$

$$a_3 = \varphi(F) - \sigma_3 - a - \frac{1}{2}\eta\frac{\dot{\gamma}}{\gamma}$$

balance

$$a_1 = 0$$

$$a_2 = 0$$

$$a_3 = 0$$

$$\varphi(F) - \hat{\sigma}_1^D + p - a + \eta\frac{\dot{\gamma}}{\gamma} = 0$$

$$\varphi(F) - \hat{\sigma}_2^D + p - a - \frac{1}{2}\eta\frac{\dot{\gamma}}{\gamma} = 0$$

$$\varphi(F) - \hat{\sigma}_3^D + p - a - \frac{1}{2}\eta\frac{\dot{\gamma}}{\gamma} = 0$$

$$3\varphi(F) + 3p - 3a = 0$$

[Notebook page scanned on 2014-06-16]

$$\Rightarrow \quad a = \varphi(F) + p$$

$$\cancel{\varphi(F)} - \frac{2}{3}\mu + \cancel{p} - \cancel{\varphi(F)} - \cancel{p} + \eta\frac{\dot\gamma}{\gamma} = 0$$

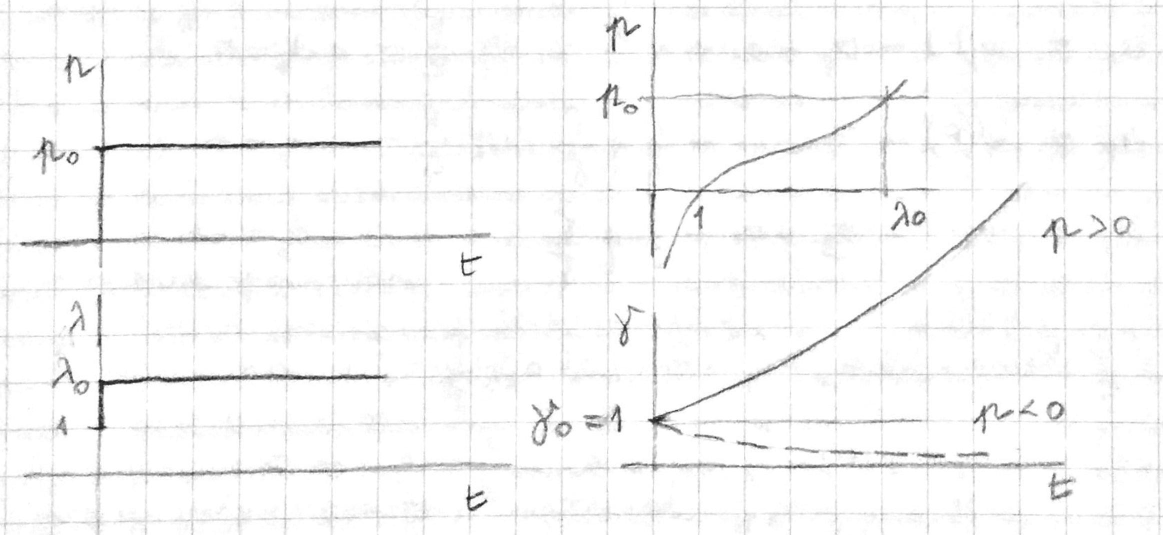
$$\cancel{\varphi(F)} - \cancel{\varphi(F)} - p - \frac{1}{2}\eta\frac{\dot\gamma}{\gamma} = 0$$

$$\cancel{\varphi(F)} - \cancel{\varphi(F)} - p - \frac{1}{2}\eta\frac{\dot\gamma}{\gamma} = 0$$

From the first equation we get

$$1 \quad \boxed{\eta\frac{\dot\gamma}{\gamma} = \frac{2}{3}\mu}$$

$$\gamma(t) = \gamma_0\, e^{kt} \qquad k = \frac{1}{\eta}\frac{2}{3}\mu$$




$$\lambda(t)\gamma(t) = \lambda_0\gamma(t) \begin{cases} \nearrow \infty & \mu > 0 \\ \searrow 0 & \mu < 0 \end{cases}$$

with $\mu(t) = \mu_0$ and $\lambda(t) = \lambda_0$

[Notebook page scanned on 2014-06-16]

We can address a different problem within the same setting.

Let the body be constrained to keep its length unchanged after stretching it at time $t=0$ by $\lambda(0)=\lambda_0$, with $\gamma(0)=1$, so that

$$\lambda(t)\gamma(t)=\lambda_0 \qquad t\geq 0$$

Let $p_0$ be the force needed to get such a stretch.

If $\lambda$ changes in time the force will change according to

$$2c_1\left(\lambda(t)^2-\frac{1}{\lambda(t)}\right)=p(t) \qquad \text{(neo-Hookean material)}$$

Therefore

$$\eta\frac{\dot\gamma(t)}{\gamma(t)}=\frac{2}{3}p(t)$$

becomes

$$\eta\frac{\dot\gamma(t)}{\gamma(t)}=\frac{4}{3}c_1\left(\lambda(t)^2-\frac{1}{\lambda(t)}\right)$$

From the constraint $\lambda(t)\gamma(t)=\lambda_0$ we get

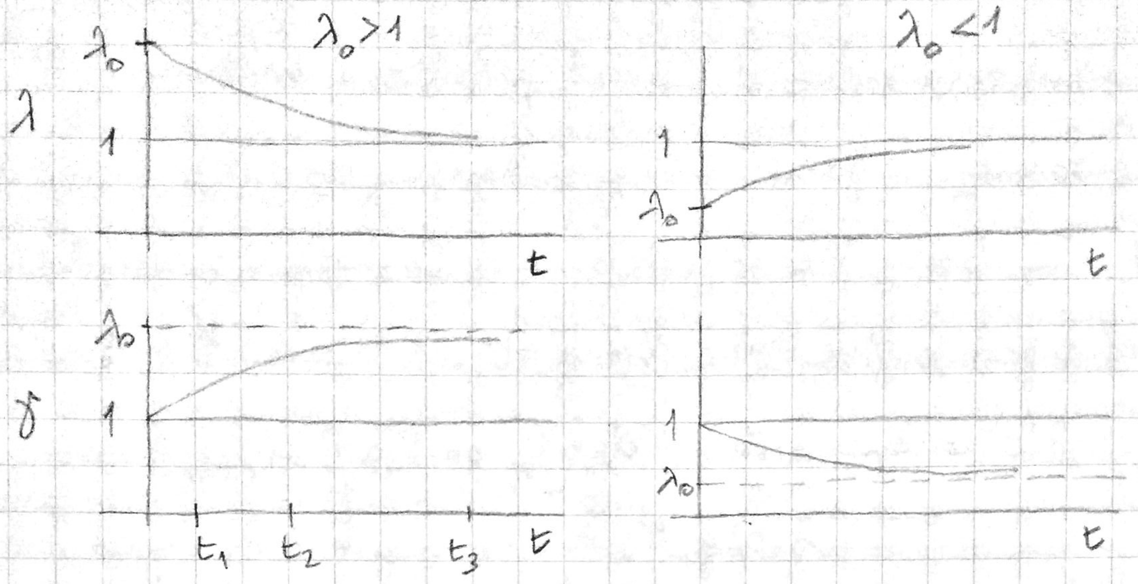
$$\dot\lambda(t)\gamma(t)+\lambda(t)\dot\gamma(t)=0$$

$$\frac{\dot\lambda(t)}{\lambda(t)}=-\frac{\dot\gamma(t)}{\gamma(t)}$$

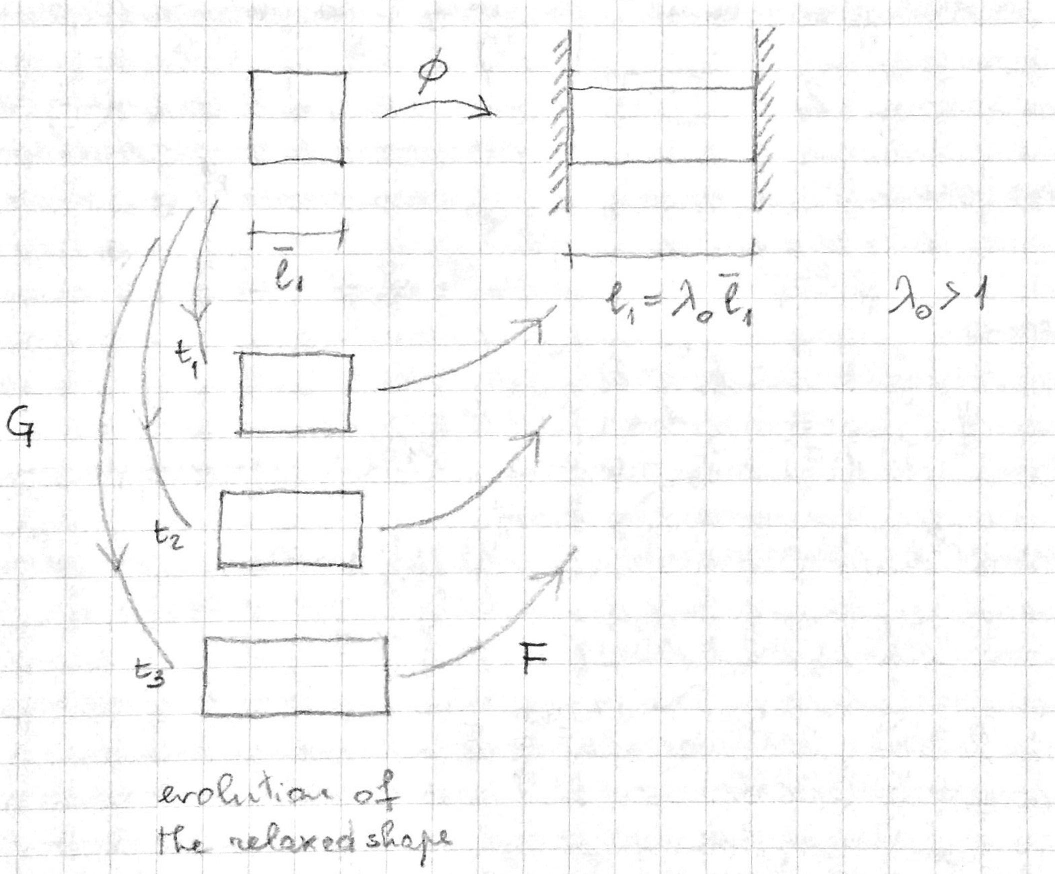
Hence the evolution of the elastic stretch is described by

$$(2)\quad \boxed{\eta\frac{\dot\lambda}{\lambda}=-\frac{4}{3}c_1\left(\lambda^2-\frac{1}{\lambda}\right)} \quad \text{with } \lambda(0)=\lambda_0$$

[Notebook page scanned on 2014-06-16]




Note that the only stationary solution is $\lambda = 1$




[Notebook page scanned on 2014-06-16]

$(87-88)_{16}$ Wednesday [2014-06-11] A1.3
9:00 - 11:00

Let us now consider a case where the remodeling force is different from zero

$$\mathcal{Q} = q\, e_1 \otimes e_1$$

From the balance equation

$$A = \mathcal{Q}$$

we get

$$a_1 = q$$

$$a_2 = 0$$

$$a_3 = 0$$

and, from the remodeling stress characterization,

$$\varphi(F) - \sigma_1^D + p - a + \eta \frac{\dot{\gamma}}{\gamma} = q$$

$$\varphi(F) - \sigma_2^D + p - a - \frac{1}{2}\eta \frac{\dot{\gamma}}{\gamma} = 0$$

$$\varphi(F) - \sigma_3^D + p - a - \frac{1}{2}\eta \frac{\dot{\gamma}}{\gamma} = 0$$

$$3\varphi(F) + 3p - 3a = q$$

$$\Rightarrow \quad a = \varphi(F) + p - \frac{1}{3} q$$

Substituting this expression into the equations above we get

$$\cancel{\varphi(F)} - \frac{2}{3}\sigma + \cancel{p} - \cancel{\varphi(F)} - \cancel{p} + \frac{1}{3} q + \eta \frac{\dot{\gamma}}{\gamma} = q$$

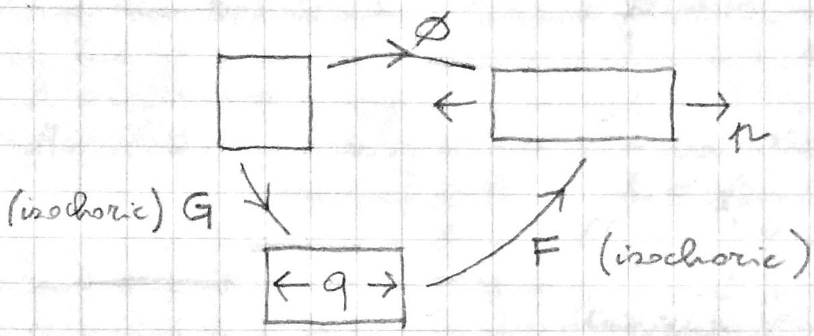
[Notebook page scanned on 2014-06-16]

which simplifies to

$$\boxed{\eta\,\frac{\dot\gamma}{\gamma} = \frac{2}{3}\left(p + q\right)} \qquad (3)$$

Note that the remodeling force $q$ can either increase or decrease the effect of $p$




Let us now consider the body constrained to keep its length unchanged while under the action of $q$ so that $\lambda(t)\,\gamma(t) = \lambda_0$, with $\gamma(0) = 1$.

Since $\dfrac{\dot\gamma}{\gamma} = -\dfrac{\dot\lambda}{\lambda}$, by using the expression for $p$ derived from a neo-Hookean strain energy we get

$$\boxed{\eta\,\frac{\dot\lambda}{\lambda} = -\frac{2}{3}\left(2c_1\left(\lambda^2 - \frac{1}{\lambda}\right) + q\right)} \qquad (4)$$

There is a unique stationary solution for each value of $q$, as shown by the graph $(\lambda, q) \equiv (\lambda, -p)$

[Notebook page scanned on 2014-06-16]

In order to describe a volumetric growth let us consider a case where $G$ is a spherical tensor

$$[G] = \begin{pmatrix} \gamma & & \\ & \gamma & \\ & & \gamma \end{pmatrix}$$

while $F$ is still isochoric

$$[F] = \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \qquad [\nabla\phi] = \begin{pmatrix} \lambda\gamma & & \\ & \frac{\gamma}{\sqrt{\lambda}} & \\ & & \frac{\gamma}{\sqrt{\lambda}} \end{pmatrix}$$

The remodeling velocity will be a spherical tensor

$$\dot{G}G^{-1} = \frac{\dot{\gamma}}{\gamma} I$$

That is why we assume the remodeling force be described by a spherical tensor as well

$$Q = q I$$

The scalar form of the remodeling stress characterization is

$$a_1 = \varphi(F) - \sigma_1 + a_0 + \eta\frac{\dot{\gamma}}{\gamma}$$

$$a_2 = \varphi(F) - \sigma_2 - \frac{1}{2}a_0 + \eta\frac{\dot{\gamma}}{\gamma}$$

$$a_3 = \varphi(F) - \sigma_3 - \frac{1}{2}a_0 + \eta\frac{\dot{\gamma}}{\gamma}$$

[Notebook page scanned on 2014-06-16]

where $a_0$ has been used to describe the deviatoric part of the remodeling stress whose power is zero since the remodeling velocity is now a spherical tensor.

Substituting the balance equations

$$\sigma_1 = p \qquad a_1 = q$$
$$\sigma_2 = 0 \qquad a_2 = q$$
$$\sigma_3 = 0 \qquad a_3 = q$$

we get

$$\varphi(F) - p + a_0 + \eta\frac{\dot\gamma}{\gamma} = q$$
$$\varphi(F) - \frac{1}{2}a_0 + \eta\frac{\dot\gamma}{\gamma} = q$$
$$\varphi(F) - \frac{1}{2}a_0 + \eta\frac{\dot\gamma}{\gamma} = q$$

---

$$3\varphi(F) - p + 3\eta\frac{\dot\gamma}{\gamma} = 3q$$

$$\Rightarrow \quad 5 \quad \boxed{\eta\frac{\dot\gamma}{\gamma} = -\left(\varphi(F) - \frac{1}{3}p\right) + q}$$

and, from any of the equations above,

$$a_0 = \frac{2}{3}p$$

[Notebook page scanned on 2014-06-16]

For a neo-Hookean material

$$\varphi(F) = c_1(\iota_1 - 3) = c_1\left(\lambda^2 + \frac{2}{\lambda} - 3\right)$$

$$p = 2c_1\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\varphi(F) - \frac{1}{3}p = \frac{1}{3}c_1\left(\lambda^2 + \frac{8}{\lambda} - 9\right)$$

Let us consider again the case where the body is constrained between two walls in such a way that

$$\lambda(t)\,\gamma(t) = \lambda_0 \qquad t \geqslant 0$$

with $\lambda(0) = \lambda_0$ and $\gamma(0) = 1$.

From the condition above we get again

$$\frac{\dot\lambda}{\lambda} = -\frac{\dot\gamma}{\gamma}$$

leading to the evolution equation for the elastic stretch

$$(6) \qquad \boxed{\eta\frac{\dot\lambda}{\lambda} = \frac{1}{3}c_1\left(\lambda^2 + \frac{8}{\lambda} - 9\right) - q}$$

[Notebook page scanned on 2014-06-16]