MECHANICS OF SOLIDS AND MATERIALS

(1-2) [2016-02-22] Monday
12:00 - 14:00 A1.3

Positions, Euclidean space, translations
In order to introduce a metric we define
a scalar product, vector norm, distance

(3-4) [2016-02-23] Tuesday
9:00 - 11:00 A1.3

Collection of small bodies
collection of particles / body points $\mathcal{B}$
Collective description of positions

motion $p : \mathcal{B} \times \mathcal{I} \to \mathcal{E}$

$p : \{A\} \times \mathcal{I} \to \mathcal{E}$ trajectory

$p : \mathcal{B} \times \{t\} \to \mathcal{E}$ shape

(5-6) [2016-02-25] Thursday
14:00 - 16:00 A1.3

Deformations
Rigid deformations

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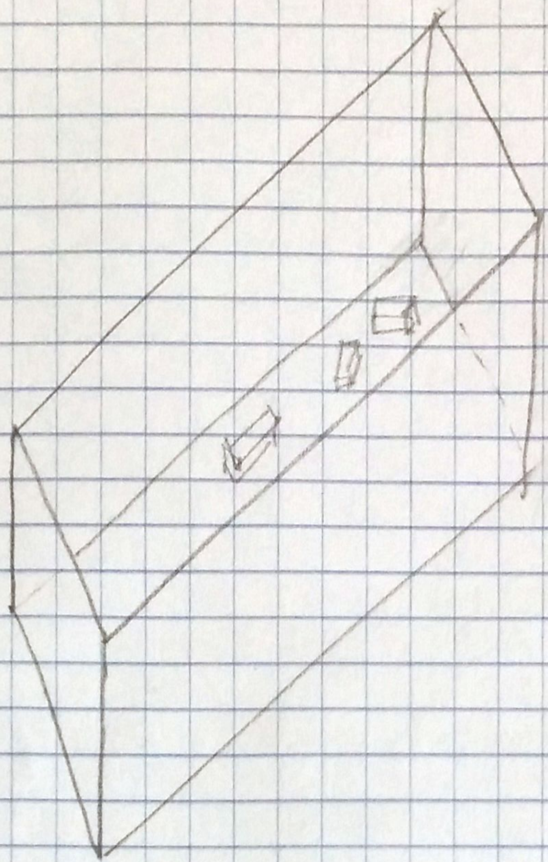

A box with "particles" inside

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(7-8) [2016-02-29] 12:00 - 14:00

Rigid deformations (continued)

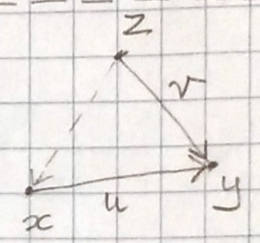
$$p_B = p_A + R(\bar{p}_B - \bar{p}_A)$$

The rotation tensor R is unique.

For if

$$y = x + u$$

$$y = z + v$$



$$x - z = v - u$$

$$\Rightarrow \quad 0 = \underbrace{(x+u)}_{\text{position}} - \underbrace{(z+v)}_{\text{position}}$$

$$(x+u) = (z+v)$$

$$(x+u) - u = (z+v) - u$$

$$x = z + (v-u) \quad \Rightarrow \quad x - z = v - u$$

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Playing with a card box

folding → unfolding → folding
(white box) (flat box) (brown box)

inside / outside

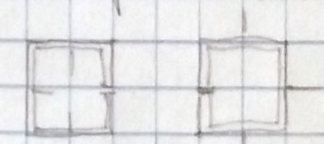
inside out

The distance is left unchanged:

$$\|u\| = \|\bar{u}\| \qquad u \cdot v = \bar{u} \cdot \bar{v}$$

There is something which is not:

the volume of the inside matter



inward pins outward pins

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How to define the volume

$$\mathrm{vol}(u_1, u_2, u_3) \in \mathbb{R}$$

$$\mathrm{vol}(u_1 + v_1, u_2, u_3) = \mathrm{vol}(u_1, u_2, u_3) + \mathrm{vol}(v_1, u_2, u_3)$$

multilinearity

$$\mathrm{vol}(e_1, e_2, e_3)$$

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3$$

$$\mathrm{vol}(u, e_2, e_3) = u_1 \mathrm{vol}(e_1, e_2, e_3)$$

$$\mathrm{vol}(e_1, v, e_3) = v_2 \mathrm{vol}(e_1, e_2, e_3)$$

$$\mathrm{vol}(u, v, e_3) =$$

$$= \mathrm{vol}(u_1 e_1, v, e_3) + \mathrm{vol}(u_2 e_2, v, e_3) + \cancel{\mathrm{vol}(u_3 e_3, v, e_3)}$$

$$= u_1 v_2 \mathrm{vol}(e_1, e_2, e_3)$$

$$+ u_2 v_1 \mathrm{vol}(e_2, e_1, e_3)$$

$$= (u_1 v_2 - u_2 v_1) \mathrm{vol}(e_1, e_2, e_3)$$

$$\mathrm{vol}(u, v, e_1) = u_2 v_3 \mathrm{vol}(e_2, e_3, e_1)$$

$$+ u_3 v_2 \mathrm{vol}(e_3, e_2, e_1)$$

$$= (u_2 v_3 - u_3 v_2) \mathrm{vol}(e_1, e_2, e_3)$$

[...]

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$\left(\frac{*}{2}\right)$

$$\frac{\mathrm{vol}(Fe_1, Fe_2, Fe_3)}{\mathrm{vol}(e_1, e_2, e_3)}$$

$$e_1 \mapsto \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

$$\frac{\mathrm{vol}(\alpha_1 Fe_1 + \alpha_2 Fe_2 + \alpha_3 Fe_3, Fe_2, Fe_3)}{\mathrm{vol}(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, e_2, e_3)}$$

$$= \frac{\alpha_1 \, \mathrm{vol}(Fe_1, Fe_2, Fe_3)}{\alpha_1 \, \mathrm{vol}(e_1, e_2, e_3)}$$

[...]

$$\frac{\cancel{(\alpha_1\beta_2 - \alpha_2\beta_1)} \, \mathrm{vol}(Fe_1, Fe_2, Fe_3)}{\cancel{(\alpha_1\beta_2 - \alpha_2\beta_1)} \, \mathrm{vol}(e_1, e_2, e_3)}$$

$$\widetilde{\mathrm{vol}}(e_1, e_2, e_3) = k \, \mathrm{vol}(e_1, e_2, e_3)$$

Cross product: it is a vector

$$\underbrace{(u \times v)}_{v} \cdot \underset{\substack{e_1 \\ e_2}}{e_3} = \mathrm{vol}(u, v, \underset{\substack{e_1 \\ e_2}}{e_3})$$

THIS IS A WAY TO RELATE SCALAR PRODUCT AND VOLUME FUNCTION

We choose first an ORIENTATION:

for a given vol function we can arrange bases in two groups according to the sign of the volume.

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(9-10) [2016-03-02]

$(\underset{1}{*})$ Let us extend

$$\phi(\bar{p}_B) = \phi(\bar{p}_A) + R\,(\bar{p}_B - \bar{p}_A)$$

to

$$\phi(\bar{p}_B) = \phi(\bar{p}_A) + F\,(\bar{p}_B - \bar{p}_A)$$

$$\bar{p}_A + h\,\bar{u} \;\mapsto\; p_A + h\,u \qquad \text{with } u = F\bar{u}$$

for any straight line

In general

$$\|F\bar{u}\| \neq \|\bar{u}\|$$

$$F\bar{u}\cdot F\bar{v} \neq \bar{u}\cdot\bar{v} \qquad \Leftrightarrow \qquad F^T F \neq I$$

Parallel straight lines $\mapsto$ parallel straight lines

Let us compare volumes

$(\leftarrow)$

$(\underset{2}{*})$

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Rotation Tensor

$\det R = \pm 1$

proper rotation $\det R = 1$

Orientation

for a given volume function vol

we can make a partition of the set

of vector bases

$\det(\qquad) > 0$

$\det(\qquad) < 0$

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The notion of "rigid deformation" is based on the metric of the Euclidean space.
How the property of leaving the distances unchanged is reflected into the volume invariance?
Let $\bar{u}_1, \bar{u}_2\, \bar{u}_3$ be an orthonormal basis, and vol a volume function such that

$$vol(\bar{u}_1, \bar{u}_2, \bar{u}_3) = 1$$

In a "rigid deformation" we get the three basis vectors, and the parallelepiped generated by them, transformed by an orthogonal Tensor R.
The volume of the new parallelepiped will be

$$vol(R\bar{u}_1, R\bar{u}_2, R\bar{u}_3)$$

which we would like to prove to be equal to ± 1.

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$\mathrm{vol}\,(u_1, u_2, u_3)$

$w_1 := u_1 - (u_1 \cdot n_1)\, n_1 \qquad n_1 \cdot u_2 = 0 \quad u_1 \cdot u_3 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad n_1 \cdot n_1 = 1$

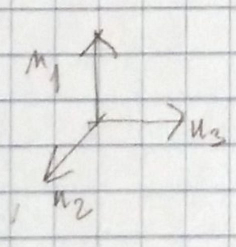
$\Rightarrow \quad w_1 \cdot n_1 = (u_1 \cdot n_1) - (u_1 \cdot n_1) = 0$

$$\mathrm{vol}\,(u_1, u_2, u_3) = \cancel{\mathrm{vol}\,(w_1, u_2, u_3)}$$
$$+ (u_1 \cdot n_1)\,\mathrm{vol}\,(n_1, u_2, u_3)$$

$w_2 := u_2 - (u_2 \cdot n_2)\, n_2 \qquad n_2 \cdot n_1 = 0, \; n_2 \cdot u_3 = 0$

$\Rightarrow \quad w_2 \cdot n_2 = (u_2 \cdot n_2) - (u_2 \cdot n_2) = 0$

$w_2 \in \mathrm{span}\,\{n_1,$



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There is a scalar product in $\mathcal{V}$ which allows us to define a norm and a distance in $\mathcal{E}$.

A volume function does not rely on the notion of distance nor on the scalar product.

Neverthelers we would like to know how the volume of a parallelepiped changes under a rigid deformation, which relies on the notion of distance.

That is why we should characterize a rigid deformation by the independent properties

- the distances are left unchanged
- the volume does not change sign (the orientation does not change)

We can then prove that the scalar product is invariant.

We should also prove that the volume is invariant as well.

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Try using the decomposition

$$\mathrm{vol}(u_1, u_2, u_3) =$$

$$(u_1 \cdot n_1)(u_2 \cdot n_2)\, \mathrm{vol}(n_1, n_2, u_3)$$

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(11-12) [2016-03-03]

$$\mathrm{vol}(\bar{u}, \bar{v}, \bar{w}) = (\bar{u}_i \bar{v}_j \bar{w}_k \varepsilon^{ijk}) \, \mathrm{vol}(e_1, e_2, e_3)$$

$\varepsilon^{ijk} = \pm 1 \quad i \neq j \neq k \qquad \varepsilon^{ijk} = 0 \quad !(i \neq j \;\&\; j \neq k \;\&\; k \neq i)$

$\varepsilon^{ijk} = 1$ if $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$

$\varepsilon^{ijk} = -1$ if $\{i, j, k\}$ is an odd permutation of $\{1, 2, 3\}$

↑ Levi-Civita index

$$\mathrm{vol}(F\bar{u}, F\bar{v}, F\bar{w}) = (\bar{u}_i \bar{v}_j \bar{w}_k \varepsilon^{ijk}) \, \mathrm{vol}(Fe_1, Fe_2, Fe_3)$$

$$\frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} = \frac{\mathrm{vol}(Fe_1, Fe_2, Fe_3)}{\mathrm{vol}(e_1, e_2, e_3)}$$

is independent of the basis

is independent of the vol function

$\Rightarrow$ it depends only on F

It is called $\det F$

$$\det AB = \ldots = \det A \det B$$

$$\det I = \det AA^{-1} = \det A \det A^{-1}$$

$$\Rightarrow \det A^{-1} = 1/\det A$$

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[2016-03-03]

Matrix of a Tensor

Entries of a matrix and the numbering of vector components

$$
\begin{array}{ccc}
A e_1 & A e_2 & A e_3 \\
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}
$$

$$\mathrm{vol}(A e_1, A e_2, A e_3) = a_{i1}\, a_{j2}\, a_{k3}\, \varepsilon^{ijk}\, \mathrm{vol}(e_1, e_2, e_3)$$

$$\det A = a_{i1}\, a_{j2}\, a_{k3}\, \varepsilon^{ijk}$$

it is independent of the basis

it is independent of the vol function

Matrix of the transposed Tensor

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(13-14) [2016-03-07]

$\forall$ orthogonal tensor Q

$$Q^T Q = I \qquad \det I = 1$$

$$\Rightarrow \quad \det Q^T \det Q = 1 \quad \Rightarrow \begin{cases} \det Q = 1 \Rightarrow \det Q^T = 1 \\ \det Q = -1 \Rightarrow \det Q^T = -1 \end{cases}$$

$\forall$ tensor A with $\det A \neq 0$

$A^T A$ is a positive definite tensor

$$\forall u \qquad u \cdot A^T A u = A u \cdot A u \geq 0$$

$\det A \neq 0 \Rightarrow$ any basis is transformed into a set of independent vectors

$$A u = 0 \Leftrightarrow u = 0$$

For any eigenvector a_i of $A^T A$

$$A^T A a_i = \mu_i a_i \qquad \underbrace{A a_i \cdot A a_i}_{>0} = \mu_i \underbrace{(a_i \cdot a_i)}_{>0}$$

$$\Rightarrow \mu_i > 0$$

Let us set $\quad U = \sum \lambda_i a_i \otimes a_i \quad$ with $\lambda_i = \sqrt{\mu_i} > 0$

$$\Rightarrow U^T = U \qquad U^2 = A^T A$$

$$Q := A U^{-1} \qquad \Rightarrow \quad A = Q U$$

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$\Rightarrow \quad Q^T Q = U^{-T} A^T A Q^{-1} = \dots = I$

Hence Q is an orthogonal tensor

Since $\qquad A = QU$

$$A^T = U^T Q^T = U Q^T$$

$$\det A = \det Q \det U$$

$$\det A^T = \det U \det Q^T$$

$$\det Q^T = \det Q \quad \Rightarrow \quad \det A^T = \det A$$

$$\det U > 0 \quad \Rightarrow \quad \begin{cases} \det A > 0 \Rightarrow \det Q > 0 \\ \det A < 0 \Rightarrow \det Q < 0 \end{cases}$$

If $\det A = 0$ then we can still compute U
but it will be a positive semidefinite tensor
because there is at least one $\lambda_i = 0$;
hence $\det U = 0$

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(15-16) [2016-03-08]

Matrix of the transposed Tensor

in general $[A^T] \neq [A]^T$

When using an orthonormal basis $e_i \cdot e_j = \delta_{ij}$

$$[A^T] = [A]^T$$

Symmetric Tensor $A = A^T$

$$(AB)^T = B^T A^T$$

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(17-18) [2016-03-10]

Polar decomposition

an exercise about the first assignment

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