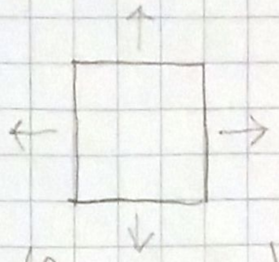
[2016-05-05]

Biaxial deformation



$t_1 = p\, e_1$

$t_2 = p\, e_2$

$$[F] = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \frac{1}{\lambda^2} \end{pmatrix}$$

$$[C] = \begin{pmatrix} \lambda^2 & & \\ & \lambda^2 & \\ & & \frac{1}{\lambda^4} \end{pmatrix} = [U^2] \qquad B = C$$

$$\hat{T}_e(F) = 2c\left(FF^T - \frac{1}{3}\,\mathrm{tr}(FF^T)\, I\right) \qquad \text{neo-Hookean}$$

$$\lambda^2 - \frac{1}{3}\left(2\lambda^2 + \frac{1}{\lambda^4}\right) = \frac{1}{3}\left(\lambda^2 - \frac{1}{\lambda^4}\right)$$

$$\frac{1}{\lambda^4} - \frac{1}{3}\left(2\lambda^2 + \frac{1}{\lambda^4}\right) = -\frac{2}{3}\left(\lambda^2 - \frac{1}{\lambda^4}\right)$$

$$[\hat{T}_e(F)] = 2c\,\frac{1}{3}\left(\lambda^2 - \frac{1}{\lambda^4}\right)\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

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$$\frac{M}{V_R} = \hat{T}_e(F) - pI$$

balance & material response

$$u_1 = \lambda \bar{u}_1$$

$$u_2 = \lambda \bar{u}_2$$

$$M = A_{\mathcal{F}_1}\, \mu\, e_1 \otimes u_1 + A_{\mathcal{F}_2}\, \mu\, e_2 \otimes u_2$$

$$[\operatorname{cof} F] = \begin{pmatrix} \frac{1}{\lambda} & & \\ & \frac{1}{\lambda} & \\ & & \lambda^2 \end{pmatrix}$$

$$A_{\mathcal{F}_1} = A_{\bar{\mathcal{F}}_1} \|(\operatorname{cof} F) e_1\| = l_2 l_3 \frac{1}{\lambda}$$

$$A_{\mathcal{F}_2} = A_{\bar{\mathcal{F}}_2} \|(\operatorname{cof} F) e_2\| = l_3 l_1 \frac{1}{\lambda}$$

$$M = \mu \left(l_2 l_3 \frac{1}{\lambda}\, e_1 \otimes (\lambda l_1 e_1) + l_3 l_1 \frac{1}{\lambda}\, e_2 \otimes (\lambda l_2 e_2) \right)$$

$$M = \mu (l_1 l_2 l_3)(e_1 \otimes e_1 + e_2 \otimes e_2)$$

$$\frac{M}{V_R} = \mu (e_1 \otimes e_1 + e_2 \otimes e_2)$$

balance & response

$$\left\{ \begin{array}{l} \operatorname{tr} \frac{M}{V_R} = -3p \quad \Rightarrow \quad 2\mu = -3p \quad \Rightarrow \quad p = -\frac{2}{3}\mu \\ \\ \operatorname{dev} \frac{M}{V_R} = \hat{T}_e(F) \quad \Rightarrow \quad \mu\left(1 - \frac{2}{3}\right) = \frac{2}{3} c \left(\lambda^2 - \frac{1}{\lambda^4}\right) \end{array} \right.$$

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(biaxial deformation)

$$\boxed{\begin{aligned} p &= -\frac{2}{3}\,\mathfrak{p} \\ \mathfrak{p} &= 2c\left(\lambda^2 - \frac{1}{\lambda^4}\right) \end{aligned}}$$

Notice that if we add a traction on $\mathcal{F}_3$

$$t_3 = \mathfrak{p}\, e_3$$

we get

$$\frac{M}{V_2} = \mathfrak{p}\left(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3\right) = \mathfrak{p}\, I$$

As a consequence

$$p = -\frac{3}{3}\,\mathfrak{p} \qquad \Rightarrow \qquad p = -\mathfrak{p}$$

$$\mathfrak{p}\left(1 - \frac{3}{3}\right) = 2c\left(\lambda^2 - \frac{1}{\lambda^4}\right) \qquad \Rightarrow \qquad \lambda = 1$$

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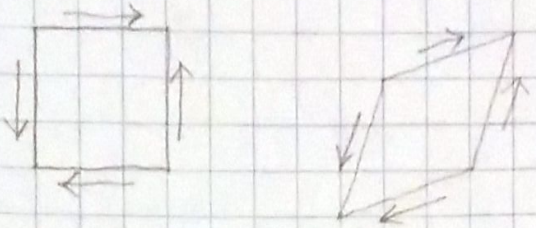
[2016-05-06]

Shear deformation



$$t_1 = p \frac{u_2}{\|u_2\|}$$

$$t_2 = p \frac{u_1}{\|u_1\|}$$

$$[F] = \begin{pmatrix} 1 & \gamma/\lambda & \\ \gamma/\lambda & 1 & \\ & & \dfrac{\lambda^2}{\lambda^2-\gamma^2} \end{pmatrix} \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \dfrac{1}{\lambda^2} \end{pmatrix} = \begin{pmatrix} \lambda & \gamma & \\ \gamma & \lambda & \\ & & \dfrac{1}{\lambda^2-\gamma^2} \end{pmatrix}$$

shear deformation   biaxial deformation

$$[C] = \begin{pmatrix} \gamma^2+\lambda^2 & 2\gamma\lambda & \\ 2\gamma\lambda & \gamma^2+\lambda^2 & \\ & & \dfrac{1}{(\lambda^2-\gamma^2)^2} \end{pmatrix}$$

$F^T = F$

$\Downarrow$

$= [B]$

eigenvalues $\left\{ (\lambda+\gamma)^2, (\lambda-\gamma)^2, \dfrac{1}{(\lambda^2-\gamma^2)^2} \right\}$

principal stretches $\left\{ (\lambda+\gamma), (\lambda-\gamma), \dfrac{1}{\lambda^2-\gamma^2} \right\}$

$\lambda > \gamma$

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$$\alpha := \lambda + \gamma$$

$$\beta := \lambda - \gamma$$

eigenvectors
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_1 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad a_2 = \frac{1}{\sqrt{2}}(-e_1 + e_2)$$

$$U = (\lambda+\gamma)\, a_1 \otimes a_1 + (\lambda-\gamma)\, a_2 \otimes a_2 \qquad (\lambda > \gamma)$$

$$\Rightarrow \quad U = F$$

$$t_1 = p\,(\gamma e_1 + \lambda e_2)\frac{1}{\sqrt{\gamma^2+\lambda^2}}\,; \quad t_2 = p\,(\lambda e_1 + \gamma e_2)\frac{1}{\sqrt{\lambda^2+\gamma^2}}$$

$$M = A_{\mathcal{F}_1}\, t_1 \otimes u_1 + A_{\mathcal{F}_2}\, t_2 \otimes u_2$$

$$M_1 = A_{\mathcal{F}_1}\, t_1 \otimes u_1 = A_{\mathcal{F}_1}\, p\, \frac{u_2}{\|u_2\|} \otimes u_1 = A_{\mathcal{F}_1}\, p\, \frac{F e_2}{\|F e_2\|} \otimes (\ell_1 F e_1)$$

$$= p\, A_{\mathcal{F}_1} \frac{\ell_1}{\sqrt{\gamma^2+\lambda^2}}\, (\gamma e_1 + \lambda e_2) \otimes (\lambda e_1 + \gamma e_2)$$

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(shear def)

$$A_{f_1} = \ell_2 \ell_3 \frac{\sqrt{\lambda^2+\gamma^2}}{\lambda^2-\gamma^2}$$

$$[\operatorname{cof} F] = \begin{pmatrix} \frac{\lambda}{\lambda^2-\gamma^2} & \frac{-\gamma}{\lambda^2-\gamma^2} & 0 \\ \frac{-\gamma}{\lambda^2-\gamma^2} & \frac{\lambda}{\lambda^2-\gamma^2} & 0 \\ 0 & 0 & (\lambda^2-\gamma^2) \end{pmatrix}$$

$$A_{f_2} = \ell_3 \ell_1 \frac{\sqrt{\lambda^2+\gamma^2}}{\lambda^2-\gamma^2}$$

$$V_{\mathcal{R}} = \ell_1 \ell_2 \ell_3 \det F = \ell_1 \ell_2 \ell_3$$

$$u_1 = \ell_1 (\lambda e_1 + \gamma e_2)$$

$$u_2 = \ell_2 (\gamma e_1 + \lambda e_2)$$

$$M_1 = \mu \frac{\ell_2 \ell_3 \ell_1}{\lambda^2-\gamma^2} (\gamma e_1 + \lambda e_2) \otimes (\lambda e_1 + \gamma e_2)$$

$$M_2 = \mu \frac{\ell_3 \ell_1 \ell_2}{\lambda^2-\gamma^2} (\lambda e_1 + \gamma e_2) \otimes (\gamma e_1 + \lambda e_2)$$

$$\left[\frac{M}{V_{\mathcal{R}}}\right] = \frac{\mu}{\lambda^2-\gamma^2} \begin{pmatrix} 2\gamma\lambda & \gamma^2+\lambda^2 & 0 \\ \lambda^2+\gamma^2 & 2\lambda\gamma & 0 \\ 0 & & 0 \end{pmatrix}$$

✓

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balance &
material response

$$\frac{M}{V_R} = \hat{T}_e(F) - pI$$

$$\begin{cases} \mathrm{tr}\,\dfrac{M}{V_R} = -3p \\[2ex] \mathrm{dev}\,\dfrac{M}{V_R} = \hat{T}_e(F) \end{cases}$$

$$\boxed{\mu \frac{4\gamma\lambda}{\lambda^2-\gamma^2} = -3p}$$

[1,1] $$\mu \frac{1}{\lambda^2-\gamma^2}\left(2\gamma\lambda - \frac{1}{3}4\gamma\lambda\right) = \\ = 2c\left((\gamma^2+\lambda^2) - \frac{1}{3}\left(2(\lambda^2+\gamma^2) + \frac{1}{(\lambda^2-\gamma^2)^2}\right)\right)$$

[1,1] $$\frac{2}{3}\mu\frac{\gamma\lambda}{\lambda^2-\gamma^2} = \frac{2}{3}c\left((\lambda^2+\gamma^2) - \frac{1}{(\lambda^2-\gamma^2)^2}\right)$$

$$\boxed{\begin{aligned} &[1,1] \quad \mu\,\gamma\lambda = c\,\frac{(\lambda^2+\gamma^2)(\lambda^2-\gamma^2)^2 - 1}{\lambda^2-\gamma^2} \\ &[1,2] \quad \mu\,\frac{\lambda^2+\gamma^2}{\lambda^2-\gamma^2} = 4c\,\lambda\gamma \end{aligned}}$$

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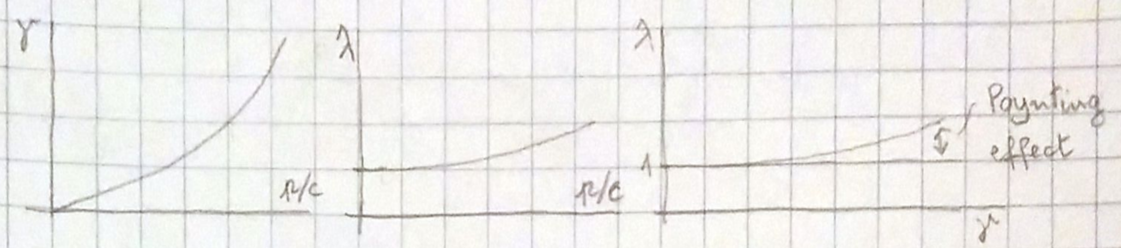
(shear def)



We get two equations

$$(\lambda,\gamma)\quad\begin{cases}\dfrac{r}{c} = \dfrac{(\lambda^2+\gamma^2)(\lambda^2-\gamma^2)^2-1}{\gamma\lambda(\lambda^2-\gamma^2)}\\[2ex] \dfrac{r}{c} = 4\lambda\gamma\,\dfrac{\lambda^2-\gamma^2}{\lambda^2+\gamma^2}\end{cases}$$

$$r = \underbrace{(2c)}_{\text{shear modulus}}\underbrace{(2\gamma)}_{\text{shear strain}}\lambda\,\frac{\lambda^2-\gamma^2}{\lambda^2+\gamma^2}$$

$$\lambda \to \frac{\alpha+\beta}{2} \qquad \alpha = \lambda+\gamma$$

principal stretches

$$\gamma \to \frac{\alpha-\beta}{2} \qquad \beta = \lambda-\gamma$$

$$\alpha^2+\beta^2 = \lambda^2+2\lambda\gamma+\gamma^2+\lambda^2-2\lambda\gamma+\gamma^2 = 2(\lambda^2+\gamma^2)$$

$$\alpha\beta = \lambda^2-\gamma^2$$

$$(\alpha,\beta)\quad\begin{cases}\dfrac{r}{c} = \dfrac{\frac{1}{2}(\alpha^2+\beta^2)(\alpha\beta)^2-1}{\frac{1}{4}(\alpha^2-\beta^2)\alpha\beta} = 2\,\dfrac{(\alpha^2+\beta^2)\alpha^2\beta^2-2}{(\alpha^2-\beta^2)\alpha\beta} \quad ✓\\[2ex] \dfrac{r}{c} = (\alpha^2-\beta^2)\,\dfrac{\alpha\beta}{\frac{1}{2}(\alpha^2+\beta^2)} = 2\,\dfrac{(\alpha^2-\beta^2)\alpha\beta}{(\alpha^2+\beta^2)} \quad ✓\end{cases}$$

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$$\alpha^2 + \beta^2 = 2\alpha^4\beta^4$$

$$x + y = 2x^2y^2$$

$$y = \frac{1 \pm \sqrt{1+8x^3}}{4x^2}$$

$$\alpha^2 = \frac{1 \pm \sqrt{1+8\beta^6}}{4\beta^4}$$

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Shear modulus (2c)

rubber $\sim 3 \times 10^{-4}$ GPa $=$ 0.3 MPa

steel ~ 80 GPa

wood ~ 10 GPa

aluminium ~ 25 GPa

silicon ~ 80 GPa

Kilo- 10^3

Mega- 10^6

Giga- 10^9

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