

[2016-05-09]

Compressible materials

$$\varphi(F) = k_I (\tilde{I}_1 - 3) + k_V (J - 1)^2$$

Deformation gradient decomposition

$$F = F_V F_I$$

$$F_V := (\det F)^{1/3} I$$

$$F_I = (\det F)^{-1/3} F$$

$$\Rightarrow \begin{cases} \det F_V = \det F \\ \det F_I = 1 \end{cases}$$

$$J := \det F$$

$$\dot{F} F^{-1} = \dots = \dot{F}_V F_V^{-1} + \dot{F}_I F_I^{-1}$$

$$F^T F = F_I^T F_V^T F_V F_I = J^{2/3} F_I^T F_I$$

$$F F^T = F_V F_I F_I^T F_V = J^{2/3} F_I F_I^T$$

$$b_1 F F^T = J^{2/3} b_2 F_I F_I^T$$

$$l_1 = J^{2/3} \tilde{l}_1 \quad \tilde{l}_1 = J^{-2/3} l_1$$

$$\text{cof } F = (\det F) F^{-T} = J F_V^{-T} F_I^{-T} = J J^{-1/3} F_I^{-T} = J^{2/3} \text{cof } F_I$$

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$$\frac{d}{dt} \varphi(F) = k_I \frac{d}{dt} \bar{L}_1 + 2k_V (J-1) \frac{d}{dt} J$$

$$\frac{d}{dt} \bar{L}_1 = -\frac{2}{3} J^{-\frac{5}{3}} \left( \frac{d}{dt} J \right) L_1 + J^{-\frac{2}{3}} \frac{d}{dt} L_1$$

$$= -\frac{2}{3} J^{-\frac{5}{3}} J \text{tr}(\dot{F}F^{-1}) L_1 + 2 J^{-\frac{2}{3}} F \cdot \dot{F}$$

$$= -\frac{2}{3} J^{-\frac{2}{3}} L_1 F^{-T} \cdot \dot{F} + 2 J^{-\frac{2}{3}} F \cdot \dot{F}$$

$$= 2 J^{-\frac{2}{3}} \left( F - \frac{1}{3} L_1 F^{-T} \right) \cdot \dot{F}$$

$$\frac{d}{dt} J = J F^{-T} \cdot \dot{F}$$

$$\hat{S}(F) \cdot \dot{F} = \frac{d}{dt} \varphi(F)$$

$$\hat{S}_e(F) = 2k_I J^{-\frac{2}{3}} \left( F - \frac{1}{3} L_1 F^{-T} \right) + 2k_V J(J-1) F^{-T}$$

$$S = J T F^{-T} \Rightarrow T = J^{-1} S F^T$$

$$\hat{T}_e(F) = 2k_I J^{-\frac{5}{3}} \left( F F^T - \frac{1}{3} L_1 I \right) + \overbrace{2k_V (J-1) I}^{\text{bulk modulus}}$$

denotonic

spherical

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Uniaxial deformation (compressible material)

$$-p e_1 \leftarrow \square \rightarrow p e_1$$

$$\frac{M}{V_0} = \hat{T}_e(F) \quad \text{balance \& material response}$$

(notice that there is no inner pressure)

$$[F] = \begin{pmatrix} \lambda_a & & \\ & \lambda_r & \\ & & \lambda_r \end{pmatrix} = [F_V][F_I] = J^{\frac{1}{3}} \begin{pmatrix} \lambda & & \\ & \frac{1}{\sqrt{\lambda}} & \\ & & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$$

$$J := \det F = \lambda_a \lambda_r^2$$

$$[C] = \begin{pmatrix} \lambda_a^2 & & \\ & \lambda_r^2 & \\ & & \lambda_r^2 \end{pmatrix} = J^{\frac{2}{3}} \begin{pmatrix} \lambda^2 & & \\ & \frac{1}{\lambda} & \\ & & \frac{1}{\lambda} \end{pmatrix}$$

$$I_1 = \text{tr} C = \lambda_a^2 + 2\lambda_r^2 = J^{\frac{2}{3}} \left( \lambda^2 + \frac{2}{\lambda} \right)$$

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$$[\text{cof } F] = \begin{pmatrix} \lambda_2^2 & & \\ & \lambda_1 \lambda_2 & \\ & & \lambda_1 \lambda_2 \end{pmatrix} = J^{\frac{2}{3}} \begin{pmatrix} \frac{1}{\lambda} & & \\ & \sqrt{\lambda} & \\ & & \sqrt{\lambda} \end{pmatrix}$$

$$M = A_{J_1} \rho e_1 \otimes (\lambda_1 l_1 e_1) = l_2 l_3 \lambda_2^2 l_1 \lambda_1 \rho (e_1 \otimes e_1)$$

$$\frac{M}{V_R} = \rho e_1 \otimes e_1$$

$$\frac{M}{V_R} = \hat{T}_e(F) \quad \text{neo-Hookean}$$

$$\begin{array}{l} \text{balance} \\ \& \\ \text{material} \\ \text{response} \end{array} \left\{ \begin{array}{l} \text{div} \frac{M}{V_R} = 2k_I J^{-\frac{5}{3}} (FF^T - \frac{1}{3} l_1 I) \\ \text{tr} \frac{M}{V_R} = 6k_V (J-1) \end{array} \right.$$

$$\Rightarrow \quad \rho = 6k_V (J-1)$$

$$\Rightarrow \quad J = 1 + \frac{1}{6} \frac{\rho}{k_V}$$

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$$\left\{ \begin{array}{l} \left(1 - \frac{1}{3}\right) \mu = 2k_I J^{-\frac{5}{3}} \left( \lambda^2 - \frac{1}{3} (\lambda^2 + 2\lambda^2) \right) \\ \lambda_1 \lambda_2^2 = 1 + \frac{1}{6} \frac{\mu}{k_V} \end{array} \right.$$

$$\left\{ \begin{array}{l} \left(1 - \frac{1}{3}\right) \mu = 2k_I J^{-\frac{5}{3}} J^{\frac{2}{3}} \left( \lambda^2 - \frac{1}{3} \left( \lambda^2 + \frac{2}{\lambda} \right) \right) \\ J = 1 + \frac{1}{6} \frac{\mu}{k_V} \end{array} \right.$$

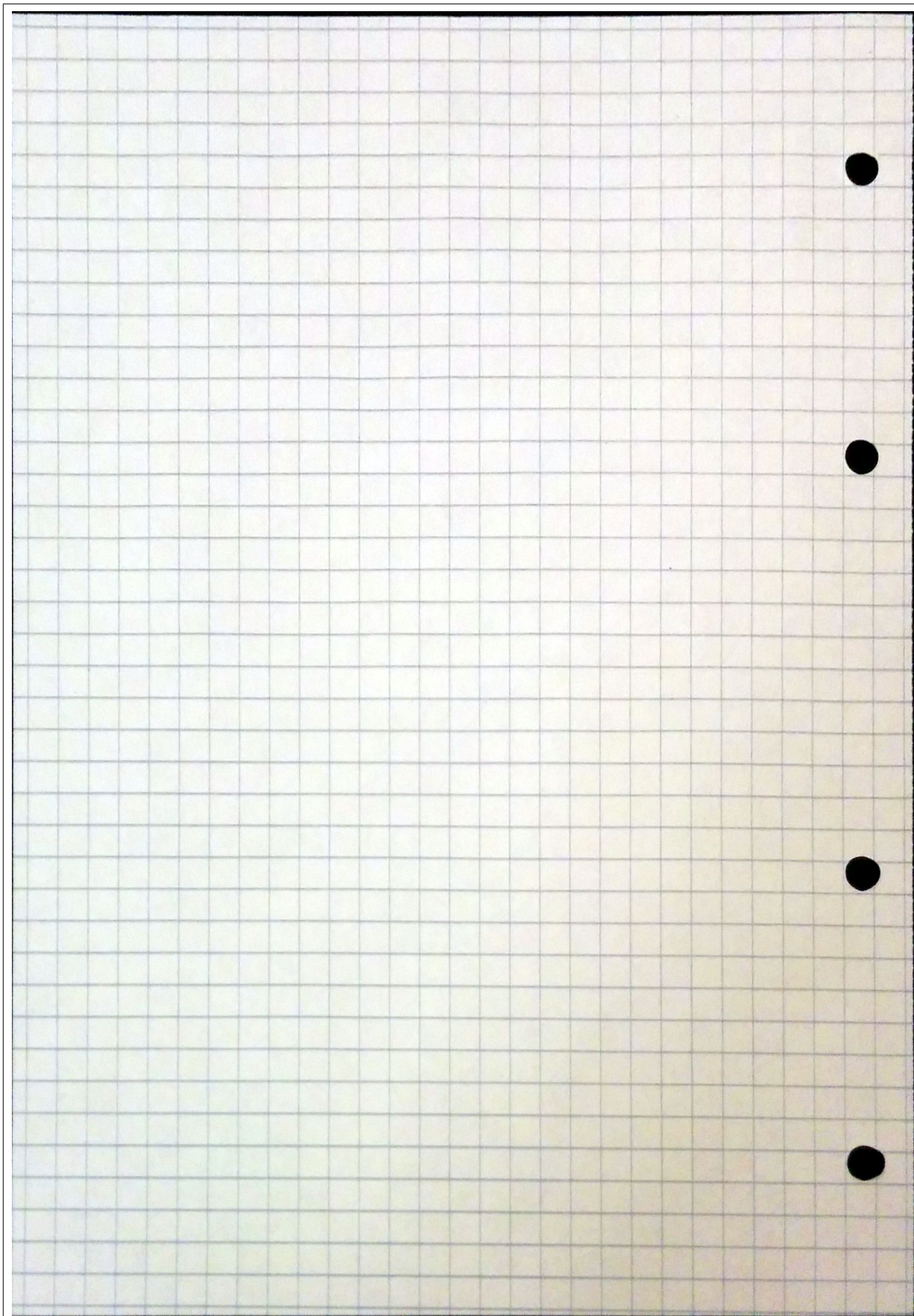
$$\left\{ \begin{array}{l} \frac{2}{3} \mu = \frac{4}{3} k_I J \left( \lambda^2 - \frac{1}{\lambda} \right) \\ J = 1 + \frac{1}{6} \frac{\mu}{k_V} \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\mu}{k_I} = 2J \left( \lambda^2 - \frac{1}{\lambda} \right) \\ J = 1 + \frac{1}{6} \frac{\mu}{k_V} \end{array} \right.$$

$k_V \rightarrow \infty$  we recover the incompressible case

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Triaxial deformation (compressible material)

$$t_1 = p e_1$$

$$t_2 = p e_2$$

$$t_3 = p e_3$$

$$[F] = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} = J^{1/3} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$F = F_V F_I$$

$$F_V = (\det F)^{1/3} I$$

$$F_I = I$$

$$F^T F = (\det F)^{2/3} I$$

$$J := \det F$$

$$k_2 F^T F = 3 J^{2/3}$$

$$k_1 = 3 \lambda^2 = 3 J^{2/3}$$

$$\bar{L}_1 = k_2 F_I^T F_I = 3$$

$$F F^T = J^{2/3} I$$

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$$\hat{T}_e(F) = 2k_I J^{-5/3} \left( \underbrace{J^{2/3} I - \frac{1}{3} 3J^{2/3}}_0 \right) + 2k_V (J-1) I$$

$$M = A_{F_1} \rho e_1 \otimes (\lambda l_1 e_1) \\ + A_{F_2} \rho e_2 \otimes (\lambda l_2 e_2) \\ + A_{F_3} \rho e_3 \otimes (\lambda l_3 e_3)$$

$$[\text{col } F] = \begin{pmatrix} J^{2/3} & & \\ & J^{2/3} & \\ & & J^{2/3} \end{pmatrix}$$

$$A_{F_1} = l_2 l_3 \lambda J^{2/3} l_1 = l_1 l_2 l_3 J^{1/3} J^{2/3} = \underbrace{l_1 l_2 l_3}_V J$$

$$\frac{M}{V_R} = \rho (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) = \rho I$$

$$\frac{M}{V_R} = \hat{T}_e(F)$$

neo-Hookean

$$\text{tr} \frac{M}{V_R} = 6k_V (J-1)$$

$$3\rho = 6k_V (J-1)$$

$$\rho = \underbrace{2k_V}_{\text{bulk modulus}} (J-1)$$

bulk modulus

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