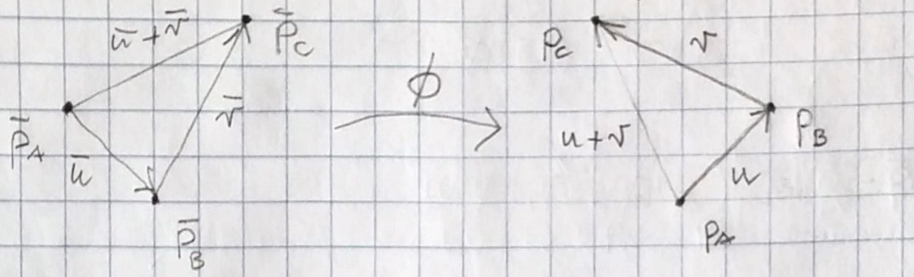
A rigid deformation is a deformation leaving distances unchanged



$$\bar{u} = \bar{p}_B - \bar{p}_A \qquad u = \phi(\bar{p}_B) - \phi(\bar{p}_A)$$

$$\bar{v} = \bar{p}_C - \bar{p}_B \qquad v = \phi(\bar{p}_C) - \phi(\bar{p}_B)$$

$$\bar{u}+\bar{v} = \bar{p}_C - \bar{p}_A \qquad u+v = \phi(\bar{p}_C) - \phi(\bar{p}_A)$$

$$\|u\| = \|\bar{u}\|, \quad \|v\| = \|\bar{v}\|, \quad \|u+v\| = \|\bar{u}+\bar{v}\|$$

$$(u+v)\cdot(u+v) = (\bar{u}+\bar{v})\cdot(\bar{u}+\bar{v})$$

$$u\cdot u + 2\,u\cdot v + v\cdot v = \bar{u}\cdot\bar{u} + 2\,\bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{v}$$

$$\|u\|^2 + 2\,u\cdot v + \|v\|^2 = \|\bar{u}\|^2 + 2\,\bar{u}\cdot\bar{v} + \|\bar{v}\|^2 \Rightarrow u\cdot v = \bar{u}\cdot\bar{v}$$

In general (linearity)

$$\bar{u} \mapsto u, \quad \bar{v} \mapsto v \ ; \quad \underbrace{\{\bar{a}_1, \bar{a}_2, \bar{a}_3\} \mapsto \{a_1, a_2, a_3\}}_{\text{bases}} \quad (*) \rightarrow$$

$$(\alpha\bar{u} + \beta\bar{v}) \mapsto \alpha u + \beta v + \varepsilon$$

$$(\alpha\bar{u} + \beta\bar{v})\cdot\bar{a}_i = (\alpha u + \beta v + \varepsilon)\cdot a_i$$

$$\alpha(\bar{u}\cdot\bar{a}_i) + \beta(\bar{v}\cdot\bar{a}_i) = \alpha(u\cdot a_i) + \beta(v\cdot a_i) + \varepsilon\cdot a_i$$

$$\Rightarrow \quad \varepsilon\cdot a_i = 0 \quad i = 1,2,3 \quad \Rightarrow \quad \varepsilon = 0$$

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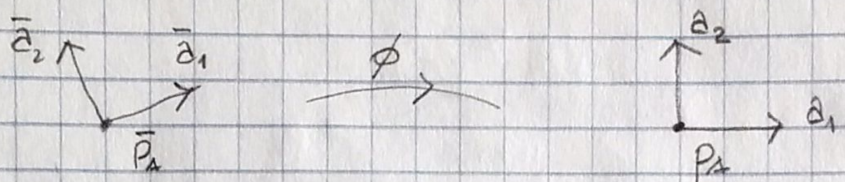
$$\{\bar{a}_1, \bar{a}_2, \bar{a}_3\} \mapsto \{a_1, a_2, a_3\}$$



Since norms and scalar products are left unchanged, an orthonormal basis is transformed into an orthonormal basis.

(*) In general if $\{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$ are linearly independent vectors then the corresponding vectors $\{a_1, a_2, a_3\}$ are linearly independent vectors.

For if

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$$

then

$$(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) \cdot a_i = 0 \qquad i = 1, 2, 3$$

$$\alpha_1 a_1 \cdot a_i + \alpha_2 a_2 \cdot a_i + \alpha_3 a_3 \cdot a_i = 0$$

$$\Rightarrow \quad \alpha_1 \bar{a}_1 \cdot \bar{a}_i + \alpha_2 \bar{a}_2 \cdot \bar{a}_i + \alpha_3 \bar{a}_3 \cdot \bar{a}_i = 0$$

$$(\alpha_1 \bar{a}_1 + \alpha_2 \bar{a}_2 + \alpha_3 \bar{a}_3) \cdot \bar{a}_i = 0 \qquad i = 1, 2, 3$$

$$\Rightarrow \quad \alpha_1 \bar{a}_1 + \alpha_2 \bar{a}_2 + \alpha_3 \bar{a}_3 = 0$$

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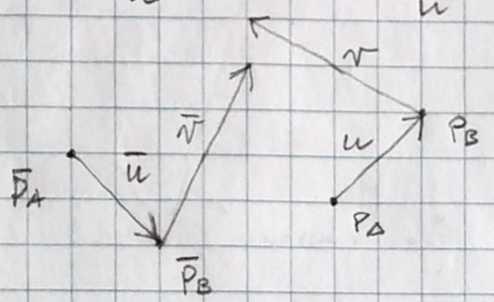
$\bar{u} \overset{R}{\mapsto} u$ Denoting by R this function,

$\bar{v} \overset{R}{\mapsto} v$ it turns out to be linear

$$\alpha\bar{u}+\beta\bar{v} \overset{R}{\mapsto} \alpha u + \beta v$$

$u = R\bar{u}$ we call R the rotation tensor

$$\underbrace{p_B - p_A}_{u} = R(\underbrace{\bar{p}_B-\bar{p}_A}_{\bar{u}})$$



$$p_B = p_A + u$$

$$p_B = p_A + R(\bar{p}_B - \bar{p}_A)$$

$$u\cdot v = R\bar{u} \cdot v = \bar{u}\cdot R^T v \qquad R^T v = \bar{v}$$

$$\Rightarrow$$

$$a_i\cdot a_j = R\bar{a}_i \cdot a_j = \bar{a}_i \cdot R^T a_j \qquad R^T a_j = \bar{a}_j$$

$$R^T = R^{-1}$$

$$u\cdot v = R\bar{u}\cdot R\bar{v} = \bar{u}\cdot R^T R\bar{v}$$

$$a_i\cdot a_j = R\bar{a}_i\cdot R\bar{a}_j = \bar{a}_i \cdot R^T R\bar{a}_j$$

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$$a_i \cdot u = 0 \qquad i = 1, 2, 3 \quad \Rightarrow \quad u = 0$$

$$u = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

$$\alpha_1 a_1 \cdot u + \alpha_2 a_2 \cdot u + \alpha_3 a_3 \cdot u = 0$$

$$\Rightarrow \quad \underbrace{(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3)}_{u} \cdot u = 0$$

$$u \cdot u = 0 \quad \Rightarrow \quad u = 0$$

$$\bar{a}_i \cdot \bar{a}_j = a_i \cdot a_j = R\,\bar{a}_i \cdot a_j = \bar{a}_i \cdot R^T a_j$$

$$\bar{a}_i \cdot \left(\bar{a}_j - R^T a_j\right) = 0$$

$$\Rightarrow \quad \bar{a}_j - R^T a_j = 0$$

$$R^T = R^{-1}$$

$$R^T R = I \,, \qquad R R^T = I$$

ORTHOGONAL TENSOR

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