Affine deformations

- Replacing the rotation tensor R by a general tensor F we get an affine (or homogeneous) deformation

$$\phi(\bar{p}_1) = \phi(\bar{p}_0) + F(\bar{p}_1 - \bar{p}_0)$$

(extension of the rigid deformation representation)

- The matrix of a tensor, with the usual numbering of entries, and the corresponding numbering of components

$$A e_j = a_{ij} e_i$$

$$\begin{array}{cccc} & Ae_1 & Ae_2 & Ae_3 \\ & \downarrow & \downarrow & \downarrow \\ [A] = & \left(\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}\right. & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \left.\begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix}\right) \end{array}$$

- The matrix of the transposed tensor in a basis made up of unitary vectors orthogonal to each other is the transposed matrix.

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An affine deformation transforms straight lines into straight lines

$$\bar{c}_A(h) = \bar{p}_A + h\,\bar{u}$$

$$\phi(\bar{c}_A(h)) = \phi(\bar{p}_A) + h\,u \qquad u = F\bar{u}$$

$$\bar{c}_B(h) = \bar{p}_B + h\,\bar{u}$$

$$\phi(\bar{c}_B(h)) = \phi(\bar{p}_B) + h\,u$$

Hence parallel lines are transformed into parallel lines, parallelograms are transformed into parallelograms, parallelepipeds are transformed into parallelepipeds.

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Polar decomposition of the deformation gradient F

Theorem: any tensor $F : \mathcal{V} \to \mathcal{V}$ with $\det F > 0$, can be decomposed into the product of a rotation R and a stretch U

$$F = RU$$

where $R \in Orth^+$ is a proper orthogonal tensor

$$R^T R = I \qquad \det R = 1$$

and $U \in Psym$ is a positive definite symmetric tensor:

$$U^T = U \qquad \begin{cases} Ua \cdot a > 0 \quad \forall a \neq 0 \\ Ua \cdot a = 0 \Leftrightarrow a = 0 \end{cases}$$

Proof:

$$C := F^T F \qquad \text{Cauchy-Green tensor}$$

$$C^T = C, \quad Cu \cdot u = F^T F u \cdot u = Fu \cdot Fu > 0 \quad \forall u \neq 0$$

$$\det F > 0 \Rightarrow \{ Fu = 0 \Leftrightarrow u = 0 \}$$

$C \in Psym \Rightarrow$ its eigenvalues are positive numbers and its eigenspaces are orthogonal to each other:

$$C = \lambda_1^2 \, a_1 \otimes a_1 + \lambda_2^2 \, a_2 \otimes a_2 + \lambda_3^2 \, a_3 \otimes a_3$$

with $\{a_1, a_2, a_3\}$ unit eigenvectors orthogonal to each other

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$(u \otimes v) w = u (v \cdot w)$ Tensor product

$(a_1 \otimes a_1) a_1 = a_1$

$(a_1 \otimes a_1) a_2 = a_1 (a_1 \cdot a_2) = 0$ [...]

$C a_1 = \lambda_1^2 a_1$ $\lambda_1^2 > 0$

$C a_2 = \lambda_2^2 a_2$ $\lambda_2^2 > 0$

$C a_3 = \lambda_3^2 a_3$ $\lambda_3^2 > 0$

Matrix of $(a_1 \otimes a_1)$

$(a_1 \otimes a_1) e_1 = a_1 (a_1 \cdot e_1)$

$(a_1 \otimes a_1) e_2 = a_1 (a_1 \cdot e_2)$

$(a_1 \otimes a_1) e_3 = a_1 (a_1 \cdot e_3)$

$a_1 = a_{11} e_1 + a_{21} e_2 + a_{31} e_3$ $e_i \cdot e_j = \delta_{ij}$

$$[a_1 \otimes a_1] = \begin{pmatrix} a_{11}^2 & a_{11} a_{21} & a_{11} a_{31} \\ a_{21} a_{11} & a_{21}^2 & a_{21} a_{31} \\ a_{31} a_{11} & a_{31} a_{21} & a_{31}^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \end{pmatrix}$$

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$$C = \lambda_1^2\, a_1 \otimes a_1 + \lambda_2^2\, a_2 \otimes a_2 + \lambda_3^2\, a_3 \otimes a_3$$

$$[C] = \lambda_1^2 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \end{pmatrix} + \lambda_2^2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \begin{pmatrix} a_{12} & a_{22} & a_{32} \end{pmatrix} + \lambda_3^2 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \begin{pmatrix} a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$[C] = [A] \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} [A]^T$$

$$U := \lambda_1\, a_1 \otimes a_1 + \lambda_2\, a_2 \otimes a_2 + \lambda_3\, a_3 \otimes a_3 \qquad \lambda_i > 0$$

$$\Rightarrow U^2 = U U = (\qquad\qquad)(\qquad\qquad) = C$$

$$(a_1 \otimes a_1)(a_1 \otimes a_1) u = (a_1 \otimes a_1) a_1 (a_1 \cdot u) = a_1 (a_1 \cdot u)$$
$$= (a_1 \otimes a_1) u$$

$$(a_1 \otimes a_1)(a_2 \otimes a_2) u = (a_1 \otimes a_1) a_2 (a_2 \cdot u) = 0$$

[...]

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$$U^{-1} = \frac{1}{\lambda_1} a_1 \otimes a_1 + \frac{1}{\lambda_2} a_2 \otimes a_2 + \frac{1}{\lambda_3} a_3 \otimes a_3$$

$$U U^{-1} = a_1 \otimes a_1 + a_2 \otimes a_2 + a_3 \otimes a_3$$

$$(U U^{-1}) u = (a_1 \cdot u) a_1 + (a_2 \cdot u) a_2 + (a_3 \cdot u) a_3 = u$$

$$U U^{-1} = I$$

$$R = F U^{-1} = \frac{1}{\lambda_1} F (a_1 \otimes a_1) + \dots$$

$$R^T R = \frac{1}{\lambda_1^2} (a_1 \otimes a_1) \underbrace{F^T F}_{C} (a_1 \otimes a_1) + \dots = I$$

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