[2018-03-15]

Volume function $\quad vol : \mathcal{V}\times\mathcal{V}\times\mathcal{V} \to \mathbb{R}$

$$vol(u_1, u_2, u_3) \in \mathbb{R}$$

such that

$$\begin{cases} vol(u_1+v, u_2, u_3) = vol(u_1, u_2, u_3) + vol(v, u_2, u_3) \\ vol(\alpha u_1, u_2, u_3) = \alpha\, vol(u_1, u_2, u_3) \\ vol(u_2, u_1, u_3) = -vol(u_1, u_2, u_3) \\ vol(u_1, u_3, u_2) = -vol(u_1, u_2, u_3) \end{cases}$$

As a consequence

$$vol(0, u_2, u_3) = vol(u_1-u_1, u_2, u_3) = \dots = 0$$

$$vol(u_1, u_1, u_3) = -vol(u_1, u_1, u_3) \Rightarrow vol(u_1, u_1, u_3) = 0$$

$$vol(\alpha_2 u_2 + \alpha_3 u_3, u_2, u_3) = \alpha_2\, vol(u_2, u_2, u_3) + \alpha_3\, vol(u_3, u_2, u_3) = 0$$

$$vol(\alpha_{11} e_1 + \alpha_{21} e_2 + \alpha_{31} e_3,\ \alpha_{12} e_1 + \alpha_{22} e_2 + \alpha_{32} e_3,\ e_3)$$
$$= \dots = (\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})\, vol(e_1, e_2, e_3)$$

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Hence a volume function is completely defined by its value on a basis.

If a volume function is zero on a basis then it is zero on any basis.

As a consequence, if it happens that

$$\mathrm{vol}(v_1, v_2, v_3) = 0$$

then either the vol function is the zero function or the three vectors $\{v_1, v_2, v_3\}$ are not linearly independent vectors.

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Determinant of a tensor F

$$\frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

$$\frac{\mathrm{vol}(F(\alpha_{11}\bar{u}_1+\alpha_{21}\bar{u}_2+\alpha_{31}\bar{u}_3), F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\alpha_{11}\bar{u}_1+\alpha_{21}\bar{u}_2+\alpha_{31}\bar{u}_3, \bar{u}_2, \bar{u}_3)}$$

$$= \frac{\alpha_{11}\,\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\alpha_{11}\,\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)} = \frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

[...]

$$\frac{\mathrm{vol}(F\bar{v}_1, F\bar{v}_2, F\bar{v}_3)}{\mathrm{vol}(\bar{v}_1, \bar{v}_2, \bar{v}_3)} = \frac{\mathrm{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3)}{\mathrm{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

Definition

$$\det F = \frac{\mathrm{vol}(Fe_1, Fe_2, Fe_3)}{\mathrm{vol}(e_1, e_2, e_3)}$$

where $\{e_1, e_2, e_3\}$ is any basis

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$$\operatorname{tr} A = \frac{\operatorname{vol}(Ae_1, e_2, e_3) + \operatorname{vol}(e_1, Ae_2, e_3) + \operatorname{vol}(e_1, e_2, Ae_3)}{\operatorname{vol}(e_1, e_2, e_3)}$$

As an application, let us compute in a motion

$$\frac{d}{dt}(\det F)$$

$$\begin{aligned}\frac{d}{dt}\operatorname{vol}(F\bar{u}_1, F\bar{u}_2, F\bar{u}_3) = {} & \operatorname{vol}(\dot{F}\bar{u}_1, F\bar{u}_2, F\bar{u}_3) \\ & + \operatorname{vol}(F\bar{u}_1, \dot{F}\bar{u}_2, F\bar{u}_3) \\ & + \operatorname{vol}(F\bar{u}_1, F\bar{u}_2, \dot{F}\bar{u}_3)\end{aligned}$$

$$\begin{aligned} &= \operatorname{vol}(\dot{F}F^{-1}u_1, u_2, u_3) \\ &\quad + \operatorname{vol}(u_1, \dot{F}F^{-1}u_2, u_3) \\ &\quad + \operatorname{vol}(u_1, u_2, \dot{F}F^{-1}u_3)\end{aligned}$$

$$\begin{aligned} &= \operatorname{tr}(\dot{F}F^{-1}) \operatorname{vol}(u_1, u_2, u_3) \\ &= \operatorname{tr}(\dot{F}F^{-1}) \det F \operatorname{vol}(\bar{u}_1, \bar{u}_2, \bar{u}_3)\end{aligned}$$

$$\Rightarrow \frac{d}{dt}(\det F) = \operatorname{tr}(\dot{F}F^{-1}) \det F$$

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